# Idempotent and multivariate copulas with fractal support

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**Abstract**

Using special Iterated Function systems (IFS) Fredricks et al. (2005) constructed two-dimensional copulas with fractal supports and showed that for every $s \in (1,2)$ there exists a copula $A$ whose support has Hausdorff dimension $s$. In the current paper we present a stronger version and prove that the same result holds for the subclass of idempotent copulas. Additionally we show that every doubly stochastic idempotent matrix $N$ (neither having minimum nor maximum rank) induces a family of idempotent copulas such that, firstly, the corresponding Markov kernels transform according to $N$ and, secondly, the set of Hausdorff dimensions of the supports of elements of the family covers $(1,2)$. Furthermore we generalize the IFS approach to arbitrary dimensions $d \geq 2$ and show that for every $s \in (1,d)$ we can find a $d$-dimensional copula whose support has Hausdorff dimension $s$.

*Keywords:* Copula, Idempotence, Markov Kernel, Iterated Function System, Fractal

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## 1. Introduction

In [15] Fredricks et al. showed how the theory of Iterated Function Systems (IFS) can be used to construct two-dimensional copulas with fractal

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support. Their approach has turned out to be useful not only in the context of counterexamples (see [2] and [28]) but, for instance, also in the construction of mutually singular copulas $A, B$ having the same fractal set as support (see [7]). For copulas with fractal support expressed in terms of measure-preserving transformations see [6]. In the current paper the IFS construction of copulas will be generalized to arbitrary dimensions $d \geq 2$ and the $d$-dimensional version of the main result in [15] will be shown, i.e. that for every fixed dimension $d \geq 2$ and every $s \in (1, d)$ we can find a $d$-dimensional copula whose support has Hausdorff dimension $s$. Using well known results from the theory of IFSs together with the fact that the Hutchison metric $h$ is a metrization of weak convergence of probability measures on any compact metric space $(X, \rho)$ allows to skip some of the steps mentioned in [15] and directly deduce the above mentioned results.
More importantly, afterwards the IFS construction will be used to construct two-dimensional copulas with fractal support which are, at the same time, idempotent with respect to the so-called star product of copulas. Since its introduction by Darsow et el. in 1992 (see [4]) the star product has been studied in various papers. In 1996 Olsen et al. showed that the space $(\mathcal{C}, *)$ of (two-dimensional) copulas with the star product as binary operation and the space $(\mathcal{M}, \circ)$ of Markov operators with the composition as binary operation are isomorphic (see [21] and Section 2) and that every copula $A \in \mathcal{C}$ can be written in the form $A = B^t * C$ whereby $B, C$ are so-called completely dependent (or, equivalently, left invertible) copulas (see [21]) and $B^t$ denotes the transpose of $B$. Using the above mentioned isomorphism Sempi (see [24]) showed in 2002 that there is a one-to-one correspondence between the class of idempotent copulas $\mathcal{C}^{ip}$ (i.e. copulas with $A * A = A$) and the subclass of $\mathcal{M}$ consisting of conditional expectations. In 2010 Darsow et al. (see [5]) answered the question posed in [4] whether idempotent copulas are necessarily symmetric and gave a complete characterization of $\mathcal{C}^{ip}$.
In the present paper it will be shown that the main result in [15] also holds if one only considers the class of idempotent copulas, i.e. that for every $s \in (1, 2)$ there exists an idempotent $A \in \mathcal{C}^{ip}$ such that the Hausdorff dimension of the support of $A$ is $s$. To do so the fact that the IFS construction also converges w.r.t. to the metric $D_1$ introduced in [27] (which is a metrization of the strong operator topology on $\mathcal{M}$) together with the fact that the star product is (jointly) continuous w.r.t. $D_1$ will be used. Additionally the just mentioned main result will be generalized and it will be shown that every doubly stochastic idempotent matrix $N$ (having neither minimum nor

maximum rank) induces a family of idempotent copulas $(A_r)_{r\in I_N}$ (whose corresponding Markov kernels transform according to $N$) such that the set of Hausdorff dimensions of the supports of $A_r$ is $(1,2)$.
The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations that will be used throughout the paper. Section 3 contains the $d$-dimensional IFS construction of copulas with fractal support and an example of a three-dimensional copula whose support is a Menger-sponge-like set. The just mentioned results on idempotent copulas with fractal support, together with two concrete examples, are the main content of Section 4.

## 2. Notation and preliminaries

As already mentioned before $\mathcal{C}$ will denote the family of all *two-dimensional copulas*, $\mathcal{C}_d$ will denote the class of all $d$-dimensional copulas for $d \geq 3$, $\Pi$ the product copula (in every dimension). For properties of copulas see [10], [20], [25]. For every metric space $(\Omega,\rho)$ $\mathcal{K}(\Omega)$ denotes the family of all non-empty compact subsets of $\Omega$, $\delta_H$ the Hausdorff metric on $\mathcal{K}(\Omega)$ (see, for instance, [19]), and $\mathcal{B}(\Omega)$ the Borel $\sigma$-field. $\mathcal{P}(\Omega)$ denotes the family of all probability measures on $(\Omega,\mathcal{B}(\Omega))$ and, in case of $\Omega=[0,1]^d$, $d\geq 2$, $\mathcal{P}_\mathcal{C}(\Omega)$ the class of all probability measure for which the corresponding distribution function is a copula (i.e. probability measures for which all one-dimensional marginals coincide with the Lebesgue measure $\lambda$ on $[0,1]$.) For very $A\in\mathcal{C}_d$, $\mu_A$ will denote the corresponding element in $\mathcal{P}_\mathcal{C}([0,1]^d)$. $\lambda_d=\mu_\Pi$ will denote the $d$-dimensional Lebesgue measure on $[0,1]^d$.
A *Markov kernel* from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$ is a mapping $K:\mathbb{R}\times\mathcal{B}(\mathbb{R})\to[0,1]$ such that $x\mapsto K(x,B)$ is measurable for every fixed $B\in\mathcal{B}(\mathbb{R})$ and $B\mapsto K(x,B)$ is a probability measure for every fixed $x\in\mathbb{R}$. Suppose that $X,Y$ are real-valued random variables on a probability space $(\Omega,\mathcal{A},\mathcal{P})$, then a Markov kernel $K:\mathbb{R}\times\mathcal{B}(\mathbb{R})\to[0,1]$ is called *regular conditional distribution of* $Y$ *given* $X$ if for every $B\in\mathcal{B}(\mathbb{R})$

$$K(X(\omega),B)=\mathbb{E}(\mathbf{1}_B\circ Y|X)(\omega) \tag{1}$$

holds $\mathcal{P}$-a.s. It is well known that for each pair $(X,Y)$ of real-valued random variables a regular conditional distribution $K(\cdot,\cdot)$ of $Y$ given $X$ exists, that $K(\cdot,\cdot)$ is unique $\mathcal{P}^X$-a.s. (i.e. unique for $\mathcal{P}^X$-almost all $x\in\mathbb{R}$) and that $K(\cdot,\cdot)$ only depends on $\mathcal{P}^{X\otimes Y}$. Hence, given $A\in\mathcal{C}$ we will denote (a version of) the regular conditional distribution of $Y$ given $X$ by $K_A(\cdot,\cdot)$ and refer to $K_A(\cdot,\cdot)$ simply as *regular conditional distribution of* $A$ or as *the Markov kernel of* $A$. Note that for every $A \in \mathcal{C}$, its conditional regular distribution $K_A(\cdot,\cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have

$$\int_{[0,1]} K_A(x, G_x)\, d\lambda(x) = \mu_A(G), \tag{2}$$

so in particular

$$\int_{[0,1]} K_A(x, F)\, d\lambda(x) = \lambda(F) \tag{3}$$

for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (3) induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (2). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see [16] and [17].

A linear operator $T$ on $L^1([0,1], \mathcal{B}([0,1]), \lambda)$ is called *Markov operator* (see [4] and [21]) if it fulfills the following three properties:

1. $T$ is positive, i.e. $T(f) \geq 0$ whenever $f \geq 0$
2. $T(\mathbf{1}_{[0,1]}) = \mathbf{1}_{[0,1]}$
3. $\int_{[0,1]} (Tf)(x) d\lambda(x) = \int_{[0,1]} f(x) d\lambda(x)$

As mentioned in the introduction $\mathcal{M}$ will denote the class of all Markov operators on $L^1([0,1]) := L^1([0,1], \mathcal{B}([0,1]), \lambda)$. It is straightforward to see that the operator norm of $T$ is one, i.e. $\|T\| := \sup\{\|Tf\|_1 : \|f\|_1 \leq 1\} = 1$ holds. According to [4] and [21] *there is a one-to-one correspondence between* $\mathcal{C}$ *and* $\mathcal{M}$ - in fact, the mappings $\Phi : \mathcal{C} \to \mathcal{M}$ and $\Psi : \mathcal{M} \to \mathcal{C}$, defined by

$$\begin{aligned} \Phi(A)(f)(x) \ &: = (T_A f)(x) := \frac{d}{dx} \int_{[0,1]} A_{,2}(x,t) f(t) d\lambda(t), \\ \Psi(T)(x,y) \ &: = A_T(x,y) := \int_{[0,x]} (T\mathbf{1}_{[0,y]})(t) d\lambda(t) \end{aligned} \tag{4}$$

for every $f \in L^1([0,1])$ and $(x,y) \in [0,1]^2$ ($A_{,2}$ denoting the partial derivative w.r.t. $y$), fulfill $\Psi \circ \Phi = id_{\mathcal{C}}$ and $\Phi \circ \Psi = id_{\mathcal{M}}$. Note that in case of $f := \mathbf{1}_{[0,y]}$ we have $(T_A \mathbf{1}_{[0,y]})(x) = A_{,1}(x,y)$ $\lambda$-a.s. According to [27] the first equality in (4) can be simplified to

$$(T_A f)(x) = \mathbb{E}(f \circ Y | X = x) = \int_{[0,1]} f(y) K_A(x, dy) \qquad \lambda\text{-a.s.} \tag{5}$$

Expressing copulas in terms of their corresponding regular conditional distributions the metric $D_1$ on $\mathcal{C}$ can be defined as follows:

$$D_1(A,B) := \int_{[0,1]} \int_{[0,1]} \big|K_A(x,[0,y]) - K_B(x,[0,y])\big| d\lambda(x)\, d\lambda(y) \qquad (6)$$

It can be shown that $(\mathcal{C}, D_1)$ is a complete metric space and that, given copulas $A, A_1, A_2 \ldots$ and their corresponding Markov operators $T_A, T_{A_1}, T_{A_2} \ldots$, the following two conditions are equivalent:

(a) $\lim_{n\to\infty} D_1(A_n, A) = 0$

(b) $\lim_{n\to\infty} \|T_{A_n} f - T_A f\|_1 = 0$ for every $f \in L^1([0,1])$,

i.e. $D_1$ is a metrization of the strong operator topology on $\mathcal{M}$ (see [27]).
Given $A, B \in \mathcal{C}$ the *star product* $A * B \in \mathcal{C}$ is defined by (see [4], [9])

$$(A * B)(x,y) := \int_{[0,1]} A_{,2}(x,t) B_{,1}(t,y) d\lambda(t) \qquad (7)$$

and fulfills

$$T_{A*B} = \Phi_{A*B} = \Phi(A) \circ \Phi(B) = T_A \circ T_B, \qquad (8)$$

so the mapping $\Phi$ in (4) actually is an isomorphism (see [21]). $A \in \mathcal{C}$ is called *idempotent* if $A * A = A$, the family of all idempotent copulas will be denoted by $\mathcal{C}^{ip}$.
Finally we recall the definition of an Iterated Function System (IFS) and some main results about IFSs (for more details see [1] and [11]). Suppose for the following that $(\Omega, \rho)$ is a compact metric space. A mapping $w : \Omega \to \Omega$ is called *contraction* if there exists a constant $L < 1$ such that $\rho(w(x), w(y)) \leq L\rho(x,y)$ holds for all $x, y \in \Omega$. A family $(w_l)_{l=1}^n$ of $n \geq 2$ contractions on $\Omega$ is called *Iterated Function System* (IFS for short) and will be denoted by $\{\Omega, (w_l)_{l=1}^n\}$. An IFS together with a vector $(p_l)_{l=1}^n \in [0,1]^n$ fulfilling $\sum_{l=1}^n p_l = 1$ is called *Iterated Function System with probabilities* (IFSP for short). We will denote IFSPs by $\{\Omega, (w_l)_{l=1}^n, (p_l)_{l=1}^n\}$. Every IFSP induces the so-called *Hutchinson operator* $\mathcal{H} : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$, defined by

$$\mathcal{H}(Z) := \bigcup_{i \leq n:\, p_i > 0} w_i(Z). \qquad (9)$$

It can be shown (see [1]) that $\mathcal{H}$ is a contraction on the compact metric space $(\mathcal{K}(\Omega), \delta_H)$, so Banach's Fixed Point theorem implies the existence of a unique, globally attractive fixed point $Z^\star$ of $\mathcal{H}$, i.e. for every $R \in \mathcal{K}(\Omega)$ we have

$$\lim_{n\to\infty} \delta_H\big(\mathcal{H}^n(R), Z^\star\big) = 0.$$

On the other hand every IFSP also induces an operator $\mathcal{V} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, defined by

$$\mathcal{V}(\mu) := \sum_{i=1}^{n} p_i \, \mu^{w_i}. \tag{10}$$

The so-called *Hutchison metric* $h$ (sometimes also called Kantorovich or Wasserstein metric) on $\mathcal{P}(\Omega)$ is defined by

$$h(\mu,\nu) := \sup\left\{ \int_\Omega f \, d\mu - \int_\Omega f \, d\nu : \; f \in Lip_1(\Omega,\mathbb{R}) \right\}, \tag{11}$$

whereby $Lip_1(X,\mathbb{R})$ is the class of all non-expanding functions $f : \Omega \to \mathbb{R}$, i.e. functions fulfilling $|f(x) - f(y)| \leq \rho(x,y)$ for all $x, y \in \Omega$. It is not difficult to show that $\mathcal{V}$ is a contraction on $(\mathcal{P}(\Omega), h)$, that $h$ is a metrization of the topology of weak convergence on $\mathcal{P}(\Omega)$ and that $(\mathcal{P}(\Omega), h)$ is a compact metric space (see [1], [8], [26]). Consequently, again by Banach's Fixed Point theorem, it follows that there is a unique, globally attractive fixed point $\mu^\star \in \mathcal{P}(\Omega)$ of $\mathcal{V}$, i.e. for every $\nu \in \mathcal{P}(\Omega)$ we have

$$\lim_{n\to\infty} h\big(\mathcal{V}^n(\nu), \mu^\star\big) = 0.$$

Furthermore $Z^\star$ is the support of $\mu^\star$ (again see [1]).

## 3. Copulas with fractal support for arbitrary dimension $d \geq 2$

Before generalizing the IFS construction given in [15] to arbitrary dimension $d \geq 2$ we start with a small lemma that will be helpful afterwards:

**Lemma 1.** *$\mathcal{P}_\mathcal{C}([0,1]^d)$ is closed in the metric space $(\mathcal{P}([0,1]^d), h)$ for every $d \geq 2$.*

**Proof:** Since $[0,1]^d$ is compact, $h$ is a metrization of the topology of weak convergence on $\mathcal{P}([0,1]^d)$. If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{P}_\mathcal{C}([0,1]^d)$ that converges to $\mu \in \mathcal{P}([0,1]^d)$ w.r.t. $h$, then $\mu_n \to \mu$ weakly, so (see [3]) the corresponding distribution functions $(A_n)_{n\in\mathbb{N}}$ converge to the distribution function

$A_\mu$ of $\mu$ at every continuity point $x$ of $A_\mu$. Since the family $(A_n)_{n\in\mathbb{N}}$ is uniformly bounded and equicontinuous the theorem of Ascoli-Arzelá (see [22]) implies the existence of a subsequence $(A_{n_k})_{k\in\mathbb{N}}$ that converges uniformly to a (continuous) distribution functions $\tilde{A}$. Weak limits are unique so $\tilde{A} = A_\mu$ and $\mu \in \mathcal{P}_\mathcal{C}([0,1]^d)$ follows. ■

**Definition 2 ([15]).** A $n \times m$- matrix $M = (t_{ij})_{i=1\ldots n,\, j=1\ldots m}$ is called *transformation matrix* if it fulfills the following four conditions: (i) $\max(n,m) \geq 2$, (ii), all entries are non-negative, (iii) $\sum_{i,j} t_{ij} = 1$, and (iv) no row or column has all entries 0.

In other words, a transformation matrix is a probability distribution $\tau$ on $(\mathcal{I}, 2^\mathcal{I})$ with $\mathcal{I} = I_1 \times I_2$, $I_1 = \{1,\ldots,n\}$ and $I_2 = \{1,\ldots,m\}$, such that $\tau(\{i\} \times I_2) > 0$ for every $i \in I_1$ and $\tau(I_1 \times \{j\}) > 0$ for every $j \in I_2$. Therefore it seems natural to extend the definition to arbitrary dimensions as follows: Fix $d \geq 2$, $m_1,\ldots,m_d \in \mathbb{N}$ and set

$$\mathcal{I}_d := \times_{i=1}^d I_i, \quad \text{whereby } I_i = \{1,\ldots,m_i\} \text{ for every } i \in \{1,\ldots,d\}. \tag{12}$$

We will denote elements in $\mathcal{I}_d$ in the form $\mathbf{i} = (i_1,\ldots,i_d)$, and, for every probability distribution $\tau$ on $(\mathcal{I}_d, 2^{\mathcal{I}_d})$ write $\tau(\mathbf{i}) := \tau(\{\mathbf{i}\})$ for the point mass in $\mathbf{i}$. The following generalizes Definition 2:

**Definition 3.** Suppose that $d \geq 2$, that $m_1,\ldots,m_d \in \mathbb{N}$, $\max_j m_j \geq 2$, and let $\mathcal{I}_d$ be defined according to (12). A probability distribution $\tau$ on $(\mathcal{I}_d, 2^{\mathcal{I}_d})$ is called *generalized transformation matrix* if for every $j \in \{1,\ldots,d\}$

$$\sum_{\mathbf{i}\in\mathcal{I}_d:\ i_j=k} \tau(\mathbf{i}) > 0$$

holds for every $k \in I_j$. The class of all generalized transformation matrices for fixed $d \geq 2$ will be denoted by $\mathcal{T}_d$.

Every $\tau \in \mathcal{T}_d$ induces a partition of $[0,1]^d$ in the following way: For each $j \in \{1,\ldots,d\}$ define $a_0^j := 0$,

$$a_k^j := \sum_{\mathbf{i}\in\mathcal{I}_d:\ i_j\leq k} \tau(\mathbf{i}),$$

and $E_k^j := [a_{k-1}^j, a_k^j]$ for every for every $k \in I_j$. Then $\bigcup_{k\in I_j} E_k^j = [0,1]$ and $E_{k_1}^j \cap E_{k_2}^j$ is empty or consists of exactly one point whenever $k_1 \neq k_2$. Setting

$R_{\mathbf{i}} := \times_{j=1}^{d} E^j_{i_j}$ for every $\mathbf{i} \in \mathcal{I}_d$ therefore yields a family of compact rectangles $(R_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_d}$ whose union is $[0,1]^d$ and which additionally fulfills that $R_{\mathbf{i}_1} \cap R_{\mathbf{i}_1}$ is empty or a set of $\lambda_d$-measure zero whenever $\mathbf{i}_1 \neq \mathbf{i}_2$.
To complete the construction of the IFSP induced by $\tau \in \mathcal{T}_d$ define affine contractions $w_{\mathbf{i}} : [0,1]^d \to R_{\mathbf{i}}$ by

$$w_{\mathbf{i}}(x_1,\ldots,x_d) = \begin{pmatrix} a^1_{i_1-1} \\ a^2_{i_2-1} \\ \vdots \\ a^d_{i_d-1} \end{pmatrix} + \begin{pmatrix} (a^1_{i_1} - a^1_{i_1-1})\, x_1 \\ (a^2_{i_2} - a^2_{i_2-1})\, x_2 \\ \vdots \\ (a^d_{i_d} - a^d_{i_d-1})\, x_d \end{pmatrix}.$$

Since the $j$-th coordinate of $w_{\mathbf{i}}(x_1,\ldots,x_d)$ only depends on $i_j$ and $x_j$ we will also denote it by $w^j_{i_j}$, i.e. $w^j_{i_j} : [0,1] \to E^j_{i_j}$, $w^j_{i_j}(x_j) := a^j_{i_j-1} + (a^j_{i_j} - a^j_{i_j-1})\, x_j$.
It follows directly from the construction that

$$\Big([0,1]^d, (w_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_d}, \tau(\mathbf{i})_{\mathbf{i}\in\mathcal{I}_d}\Big) \tag{13}$$

is an IFSP. The only thing left to show before directly applying the results about IFSPs is that the operator $\mathcal{V}_\tau$ maps $\mathcal{P}_{\mathcal{C}}([0,1]^d)$ into $\mathcal{P}_{\mathcal{C}}([0,1]^d)$.

**Lemma 4.** *Suppose that $\tau \in \mathcal{T}_d$, then $\mathcal{V}_\tau(\mathcal{P}_{\mathcal{C}}([0,1]^d)) \subseteq \mathcal{P}_{\mathcal{C}}([0,1]^d)$.*

**Proof:** Fix $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^d)$. We only need to show that $\mathcal{V}_\tau(\mu)$ has uniform one-dimensional marginals, which can be done as follows: For every $F \in \mathcal{B}([0,1])$ consider the rectangle $R := \times_{k=1}^{d} G_k$ with $G_j = F$ and $G_k = [0,1]$ for every $k \neq j$, then

$$\mu(w_{\mathbf{i}}^{-1}(R)) = \lambda\big((w^j_{i_j})^{-1}(F)\big) = \lambda^{w^j_{i_j}}(F),$$

and therefore

$$\begin{aligned}
(\mathcal{V}_\tau\mu)(R) &= \sum_{\mathbf{i}\in\mathcal{I}_d} \tau(\mathbf{i})\mu^{w_{\mathbf{i}}}(R) = \sum_{k=1}^{m_j} \sum_{\mathbf{i}\in\mathcal{I}_d :\, i_j=k} \tau(\mathbf{i})\lambda^{w^j_k}(F) \\
&= \sum_{k=1}^{m_j} \lambda^{w^j_k}(F) \sum_{\mathbf{i}\in\mathcal{I}_d :\, i_j=k} \tau(\mathbf{i}) = \sum_{k=1}^{m_j} \lambda^{w^j_k}(F)(a^j_k - a^j_{k-1}) \\
&= \lambda(F).
\end{aligned}$$

This completes the proof. ■

As a consequence we will also write $\mathcal{V}_\tau(A)$ for every $A \in \mathcal{C}_d$. Applying the results on IFSPs mentioned in the introduction yields the following:

**Theorem 5.** *Suppose that $\tau \in \mathcal{T}_d$, consider the corresponding IFSP (13), and let the Hutchinson operator $\mathcal{H}$ and the operator $\mathcal{V}_\tau$ be defined according to (9) and (10) respectively. Then there exists a unique compact set $Z^\star \in \mathcal{K}([0,1]^d)$ (called* attractor*) and a copula $A^\star \in \mathcal{C}_d$ such that the support of $A^\star$ is $Z^\star$, and such that for every $Z \in \mathcal{K}([0,1]^d)$ and $A \in \mathcal{C}_d$*

$$\lim_{n\to\infty} \delta_H\big(\mathcal{H}^n(Z), Z^\star\big) = 0 \quad \textit{and} \quad \lim_{n\to\infty} h\big(\mathcal{V}_\tau^n(A), A^\star\big) = 0$$

*holds.*

**Remark 6.** Analogously to the two-dimensional case in [15] it is straightforward to see that the attractor $Z^\star$ has $\lambda_d$-measure zero if and only if there is at least one $\mathbf{i} \in \mathcal{I}_d$ such that $\tau(\mathbf{i}) = 0$. Hence the limit copula $A^\star \in \mathcal{C}_d$ is singular (w.r.t. the Lebesgue measure $\lambda_d$) if and only if $\tau(\mathbf{i}) = 0$ for at least one $\mathbf{i} \in \mathcal{I}_d$.

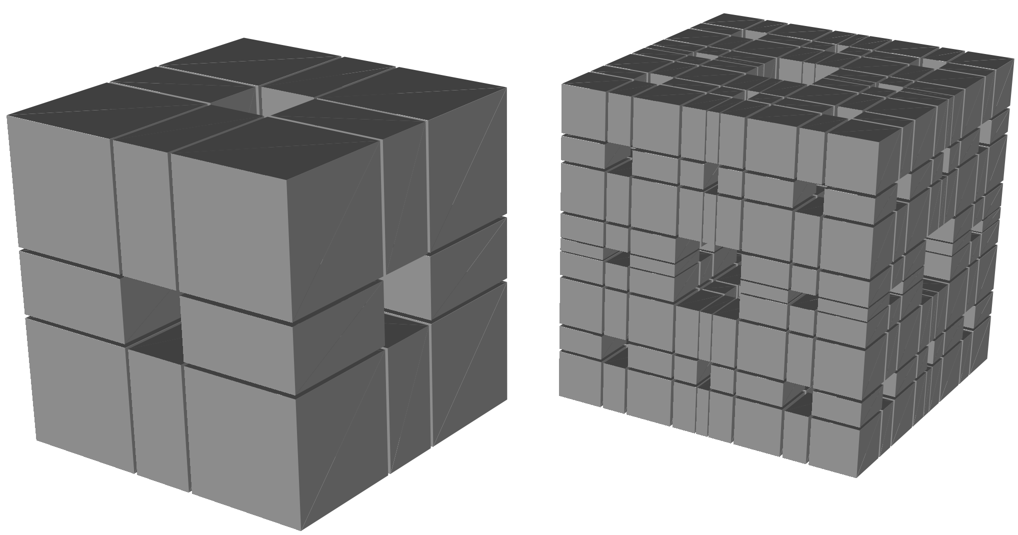

Figure 1: Support of $\mathcal{V}_\tau^1(\Pi)$ and $\mathcal{V}_\tau^2(\Pi)$, $\tau$ according to Example 1.

**Example 1.** Using the IFS approach we can easily construct a three-dimensional copula whose support is a *Menger-sponge-like* set: For every $j \in \{1,2,3\}$ set $I_j := \{1,2,3\}$ and define $\tau \in \mathcal{T}_3$ by

$$\tau_r(\mathbf{i}) = \begin{cases} \frac{1}{20} & \text{if at most one coordinate of } \mathbf{i} \text{ is } 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then we get $a_1^1 = a_1^2 = a_1^3 = 2/5$ and $a_2^1 = a_2^2 = a_2^3 = 3/5$, hence the corresponding IFS consists of 27 contractions (only some being similarities), 20 of which have positive probability $\tau(\mathbf{i}) > 0$. Obviously the support of the corresponding limit copula $A^\star$ is a Menger-sponge-like set. The support of $\mathcal{V}_\tau(\Pi)$ and $\mathcal{V}_\tau^2(\Pi)$ is depicted in Figure 1.

Having Theorem 5 we can easily prove the following generalization of Theorem 1 in [15]:

**Theorem 7.** *For every $d \geq 2$ and every $s \in (1,d)$ there exists a copula $A \in \mathcal{C}_d$ whose support has Hausdorff dimension $s$.*

**Proof:** Set $I_j = \{1,2,3\}$ for every $j \in \{1,\ldots,d\}$ and define $\tau_r \in \mathcal{T}_d$ for $r \in (0,1/2)$ by

$$\tau_r(\mathbf{i}) = \begin{cases} \frac{r}{2^{d-1}} & \text{if } \mathbf{i} \in \{1,3\}^d \\ 1-2r & \text{if } \mathbf{i} = (2,2,\ldots,2) \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then the IFSP induced by $\tau_r$ consists of similarities having contraction factor $r$ or $1-2r$ and Morgan's open set condition (see [1]) obviously is fulfilled. Hence, using Theorem 5 and [1], the Hausdorff dimension $s_r$ of support of the limit copula $A_r^\star$ is the unique solution of the equality

$$2^d r^s + (1-2r)^s = 1. \tag{15}$$

in $(1,d)$. Set $f(r,s) := 2^d r^s + (1-2r)^s$. Using monotonicity arguments (analogous to [15]) it is straightforward to see that for each $s \in (1,d)$ there exists a unique $r \in (0,1/2)$ such that $f(r,s) = 1$ holds. On the other hand, for any fixed $r \in (0,1/2)$ the partial derivative $\frac{\partial f(r,s)}{\partial s}$ is negative, so, using the fact that $f(r,1) > 1$ and $f(r,d) < 1$, it follows that there exists a unique $s_r \in (1,d)$ with $f(r,s_r) = 1$. This completes the proof. ■

## 4. Idempotent copulas with fractal support

We will start with two useful small lemmas and then construct a first idempotent copula whose support has Hausdorff dimension $\log(5)/\log(3)$.

**Lemma 8.** *The family $\mathcal{C}^{ip}$ of idempotent copulas is closed in $(\mathcal{C}, D_1)$.*

**Proof:** First of all is easy to show that the star product is (jointly) continuous w.r.t. $D_1$: Suppose that $A, A_1, A_2, \ldots$ and $B, B_1, B_2, \ldots$ are copulas with $\lim_{n\to\infty} D_1(A_n, A) = \lim_{n\to\infty} D_1(B_n, B) = 0$. Then for every $f \in L^1([0,1])$, using the triangle inequality and the fact that Markov operators have operator norm 1, we get

$$\|T_{A_n} \circ T_{B_n} f - T_A \circ T_B f\|_1 \;\; \leq \;\; \|T_{B_n} f - T_B f\|_1 + \|T_{A_n} \circ T_B f - T_A \circ T_B f\|_1,$$

hence $\|T_{A_n} \circ T_{B_n} f - T_A \circ T_B f\|_1 \longrightarrow 0$ and $\lim_{n\to\infty} D_1(A_n * B_n, A * B) = 0$. Consequently, if $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}^{ip}$ converging to $A \in \mathcal{C}$ w.r.t. $D_1$, then $(A_n)_{n\in\mathbb{N}}$ converges both to $A$ and to $A * A$, so $A = A * A$ and $A \in \mathcal{C}^{ip}$. ■

**Lemma 9.** *Suppose that $A, B \in \mathcal{C}$ and let $K_A, K_B$ denote regular conditional distributions of $A$ and $B$. Then the Markov kernel $K_A \circ K_B$, defined by*

$$(K_A \circ K_B)(x, F) := \int_{[0,1]} K_B(y, F) K_A(x, dy) \tag{16}$$

*is a regular conditional distribution of $A * B$.*

**Proof:** It is well known that the right hand side of (16) is a Markov kernel (see [16], [17]). Suppose now that $f(x) = \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}(x)$ is a non-negative simple function with $(E_i)_{i=1}^n$ being a measurable partition of $[0,1]$, then

$$\begin{aligned}
\int_{[0,1]} \int_{[0,1]} f(y)\, K_A(x, dy)\, d\lambda(x) &= \int_{[0,1]} \int_{[0,1]} \sum_{i=1}^n \alpha_i \mathbf{1}_{E_i}(y)\, K_A(x, dy)\, d\lambda(x) \\
&= \int_{[0,1]} \sum_{i=1}^n \alpha_i K_A(x, E_i)\, d\lambda(x) \\
&= \sum_{i=1}^n \alpha_i \lambda(E_i) = \int_{[0,1]} f\, d\lambda.
\end{aligned}$$

Since the class of simple functions is dense in $L^1([0,1],\mathcal{B}([0,1]),\lambda)$ we get

$$\int_{[0,1]}\int_{[0,1]} K_B(y,F)\,K_A(x,dy)\,d\lambda(x) = \int_{[0,1]} K_B(x,F)\,d\lambda(x) = \lambda(F)$$

which shows that $K_A \circ K_B$ fulfills (3). Finally, if $E \in \mathcal{B}([0,1])$, then, using (4) and (5), for $\lambda$-almost every $x \in [0,1]$ it follows that

$$\begin{aligned} K_{A*B}(x,E) = (T_A \circ T_B)(\mathbf{1}_E)(x) &= T_A(K_B(\cdot\,,E))(x) \\ &= \int_{[0,1]} K_B(y,E)K_A(x,dy), \end{aligned}$$

which completes the proof. ■

**Remark 10.** Lemma 9 implies that the Markov kernel of $A * B$ is just the standard composition of the Markov kernels of $A$ and $B$. Hence, in terms of conditional distributions, the star product can be seen as natural generalization of the multiplication of stochastic matrices in the discrete Markov chain setting and studying the star product means studying the composition of Markov kernels $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (3) and vice versa.

**Example 2.** Consider the transformation matrix $M = (t_{ij})_{i,j=1}^3 \in \mathcal{T}_2$ defined by

$$M = \begin{pmatrix} \frac{1}{6} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}. \tag{17}$$

The densities of the first four iterates of the corresponding operator $\mathcal{V}_M$ applied to the product copula $\Pi$ are depicted in Figure 2.

According to [27] there exists a unique copula $A^\star$ such that for every starting copula $B \in \mathcal{C}$ we have $\lim_{n\to\infty} D_1(\mathcal{V}_M^n B, A^\star) = 0$. Furthermore (see [15]) the support of $A^\star$ has Hausdorff dimension $\ln(5)/\ln(3)$. It remains to prove that $A^\star$ is also idempotent which can be done in three steps:

**Step 1:** We explicitly show how, for a given copula $B \in \mathcal{C}$, the kernel $K_{\mathcal{V}_M B}$ can directly be calculated from $K_B$. For every $i \in \{1,2,3\}$ define functions $h_i : \mathbb{R} \to \mathbb{R}$ by $h_i(x) := 3x - (i-1)$ and extend the definition of the kernel $K_B$ to whole $[0,1] \times \mathcal{B}(\mathbb{R})$ by setting $K_B(x,E) = 0$ whenever

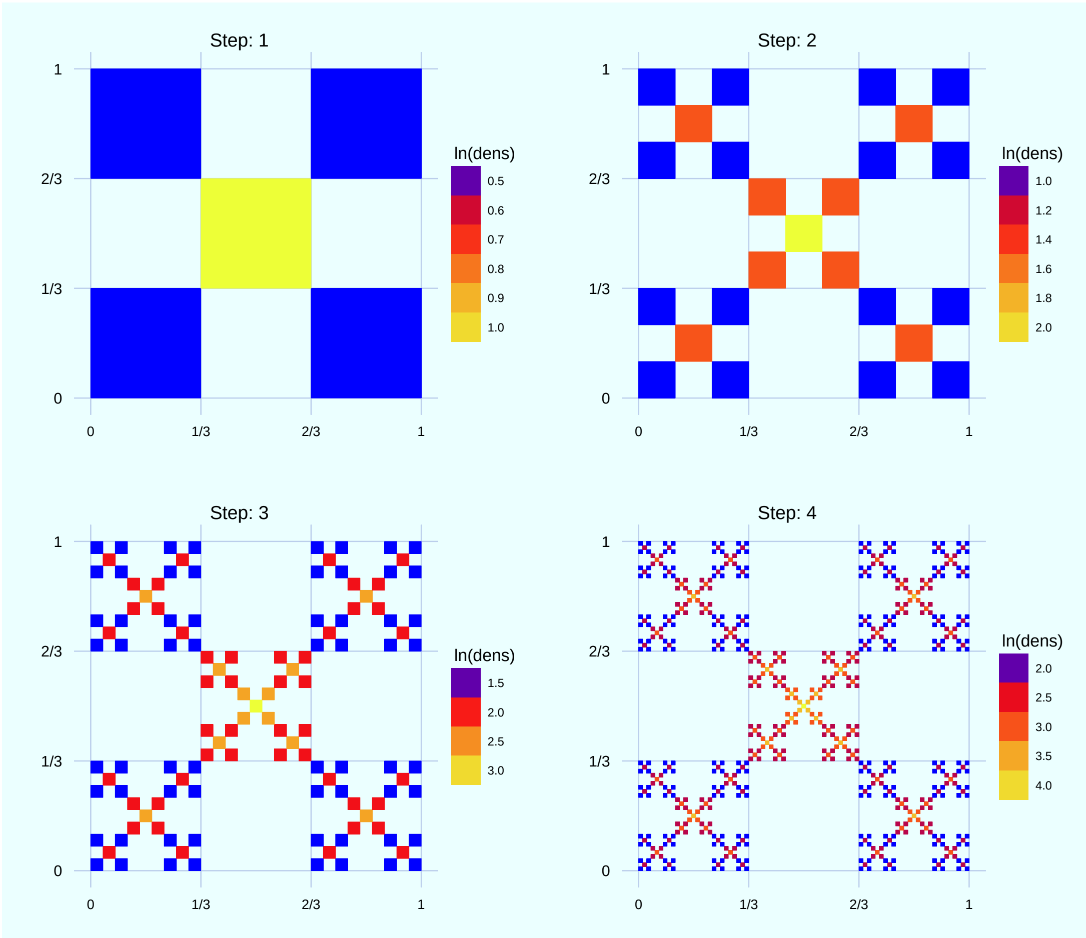


Figure 2: Image plot of the (natural) logarithm of the density of $\mathcal{V}_M^n(\Pi)$ for $n \in \{1, 2, 3, 4\}$, $M$ according to (17) in Example 2.

$E \cap [0, 1] = \emptyset$. Fix $E \in \mathcal{B}([0, 1])$ then we get

$$K_{\mathcal{V}_M B}(x, E) = \begin{cases} \frac{1}{2}\, K_B\big(h_1(x), h_1(E)\big) + \frac{1}{2}\, K_B\big(h_1(x), h_3(E)\big) & \text{if } x \in \left[0, \frac{1}{3}\right] \\ K_B\big(h_2(x), h_2(E)\big) & \text{if } x \in \left(\frac{1}{3}, \frac{2}{3}\right) \\ \frac{1}{2}\, K_B\big(h_3(x), h_1(E)\big) + \frac{1}{2}\, K_B\big(h_3(x), h_3(E)\big) & \text{if } x \in \left[\frac{2}{3}, 1\right] \end{cases}$$

which implies

$$\begin{pmatrix} K_{\mathcal{V}_M B}\big(h_1^{-1}(x), E\big) \\ K_{\mathcal{V}_M B}\big(h_2^{-1}(x), E\big) \\ K_{\mathcal{V}_M B}\big(h_3^{-1}(x), E\big) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}_{:=N} \begin{pmatrix} K_B\big(x, h_1(E)\big) \\ K_B\big(x, h_2(E)\big) \\ K_B\big(x, h_3(E)\big) \end{pmatrix} \tag{18}$$

for every $x \in [0,1]$. Consequently, since the family of simple functions is dense in $L^1([0,1])$, it follows that for every $f \in L^1([0,1])$, every $i \in \{1,2,3\}$ and every $x \in [0,1]$ the following transformation formula holds:

$$\int_{[0,1]} f(z)\, K_{\mathcal{V}_M B}(h_i^{-1}x, dz) = \int_{[0,1]} \sum_{j=1}^{3} n_{ij} f(h_j^{-1}(z))\, K_B(x, dz) \tag{19}$$

**Step 2:** We show that for every $B \in \mathcal{C}^{ip}$ the copula $\mathcal{V}_M(B)$ is idempotent again: Equalities (18) and (19) together with the idempotence of $N$ and $B \in \mathcal{C}^{ip}$ imply that for every $i \in \{1,2,3\}$, every $x \in [0,1]$ and every $E \in \mathcal{B}([0,1])$ we have

$$\begin{aligned} K_{\mathcal{V}_M B * \mathcal{V}_M B}(h_i^{-1}(x), E) &= \int_{[0,1]} K_{\mathcal{V}_M B}(z, E) K_{\mathcal{V}_M B}(h_i^{-1}(x), dz) \\ &= \int_{[0,1]} \sum_{j=1}^{3} n_{ij} K_{\mathcal{V}_M B}(h_j^{-1}(z), E)\, K_B(x, dz) \\ &= \int_{[0,1]} (n_{i1}, n_{i2}, n_{i3})\, N \begin{pmatrix} K_B\big(z, h_1(E)\big) \\ K_B\big(z, h_2(E)\big) \\ K_B\big(z, h_3(E)\big) \end{pmatrix} K_B(x, dz) \\ &\overset{N \text{ ip}}{=} \int_{[0,1]} \sum_{j=1}^{3} n_{ij} K_B(z, h_j(E))\, K_B(x, dz) \\ &\overset{B \in \mathcal{C}^{ip}}{=} \sum_{j=1}^{3} n_{ij} K_B(x, h_j(E)) = K_{\mathcal{V}_M B}(h_i^{-1}(x), E) \end{aligned}$$

which shows that $\mathcal{V}_M B \in \mathcal{C}^{ip}$.

**Step 3:** Since we have $\lim_{n\to\infty} D_1(\mathcal{V}_M^n B, A^\star) = 0$ for *every* $B \in \mathcal{C}$ we can choose $B = \Pi \in \mathcal{C}^{ip}$ to construct a sequence $(\mathcal{V}_M^n \Pi)_{n\in\mathbb{N}}$ of elements in $\mathcal{C}^{ip}$ that converges to $A^\star$. Consequently, according to Lemma 8, $A^\star$ has to be idempotent too.

The crucial point in proving that the limit $A^\star$ in Example 2 is idempotent was that the doubly stochastic matrix $N$ in equation (18) describing the transformation of the kernels was idempotent. Fix $r \in (0, 1/2)$ and consider the transformation matrix $M_r \in \mathcal{T}_2$, defined by

$$M_r = \begin{pmatrix} \frac{r}{2} & 0 & \frac{r}{2} \\ 0 & 1-2r & 0 \\ \frac{r}{2} & 0 & \frac{r}{2} \end{pmatrix}. \tag{20}$$

Then, according to [27] there exists a unique copula $A^\star$ such that for every starting copula $B \in \mathcal{C}$ we have $\lim_{n\to\infty} D_1(V_{M_r}^n B, A^\star) = 0$. Furthermore, according to Theorem 7 the support of $A^\star$ has Hausdorff dimension $s_r$ whereby $s_r$ is the unique solution of the equation

$$4r^s + (1-2r)^s = 1$$

in the interval $(1, 2)$. Define functions $h_1, h_2, h_3 : \mathbb{R} \to \mathbb{R}$ by

$$h_1(x) := \frac{x}{r},\ h_2(x) := \frac{x-r}{1-2r},\ h_3(x) := \frac{x-(1-r)}{r}$$

then it is straightforward to verify that equation (18) also holds for $\mathcal{V}_{M_r}$ with the same matrix $N$, i.e. we have

$$\begin{pmatrix} K_{\mathcal{V}_{M_r}B}\big(h_1^{-1}(x), E\big) \\ K_{\mathcal{V}_{M_r}B}\big(h_2^{-1}(x), E\big) \\ K_{\mathcal{V}_{M_r}B}\big(h_3^{-1}(x), E\big) \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}}_{:=N} \begin{pmatrix} K_B\big(x, h_1(E)\big) \\ K_B\big(x, h_2(E)\big) \\ K_B\big(x, h_3(E)\big) \end{pmatrix} \tag{21}$$

for every $x \in [0, 1]$ and every $B \in \mathcal{C}$. From this the corresponding version of (19) and Step 2 and Step 3 follow in completely the same manner, hence $A^\star \in \mathcal{C}^{ip}$ and we have constructed an idempotent copula whose support has Hausdorff dimension $s_r$. Since $r \in (0, 1/2)$ was arbitrary, using Theorem 7 again, we have the following result:

**Theorem 11.** *For every $s \in (1, 2)$ there exists an idempotent copula $A \in \mathcal{C}^{ip}$ whose support has Hausdorff dimension $s$.*

**Remark 12.** An alternative way to prove Theorem 11 would be the following: One could consider the so-called *lifting* (see [9]) $\blacktriangle : \mathcal{C} \times \mathcal{C} \to \mathcal{C}_3$, defined by

$$A \blacktriangle B(x, y, z) := \int_{[0,y]} A_{,2}(x, t) B_{,1}(t, z)\, d\lambda(t)$$

for all $A, B \in \mathcal{C}$ and $x, z \in [0,1]$, and verify that the copulas (14) in the proof of Theorem 7 fulfill

$$A^\star_{r,3} = A^\star_{r,2} \blacktriangle A^\star_{r,2}$$

for every $r \in (0, 1/2)$, whereby $A^\star_{r,d}$ denotes the corresponding limit copula in dimension $d \in \{2,3\}$. Having this idempotence of $A^\star_{r,2}$ directly follows from the fact that $A^\star_{r,3}(x,1,z) = A^\star_{r,2}(x,z)$ as well as $A^\star_{r,3}(x,1,z) = A^\star_{r,2} * A^\star_{r,2}(x,1,z)$ holds for all $x, z \in [0,1]$.

As final step we will take a closer look to more general matrices $N$ describing the interrelation (21) between the kernel $K_{\mathcal{V}B}$ and the original kernel $K_B$ for every $B \in \mathcal{C}$ and use these matrices to prove a generalization of Theorem 11. We will consider the class $\hat{\mathcal{T}}_2 \subseteq \mathcal{T}_2$ consisting of transformation matrices $M = (t_{ij}) \in \mathcal{T}_2$ fulfilling that for each non-zero entry $t_{ij} > 0$ the row and column sums through that entry are equal. It is straightforward to see that each $M \in \hat{\mathcal{T}}_2$ is quadratic and that, using the notation of the previous section, $E^1_k = [a^1_{k-1}, a^1_k] = E^2_k = [a^2_{k-1}, a^2_k]$ for every $k \in \{1, \ldots, m\}$ and, for every $(i,j)$ with $t_{ij} > 0$, $R_{ij}$ is a square and $w_{ij}$ is a similarity (also see [15]). Define affine expansions $h_i : E^1_i \to [0,1]$, $i \in \{1, \ldots, m\}$, by

$$h_i(x) := \frac{x - a^1_i}{a^1_i - a^1_{i-1}},$$

then we get $w_{ij}(x_1, x_2) = (h_i^{-1}(x_1)\,,\, h_j^{-1}(x_2))$. Set $L_i := a^1_i - a^1_{i-1}$ for every $i \in \{1, \ldots, m\}$ and define a new matrix $N_M = (n_{ij})^m_{i,j=1}$ by

$$n_{ij} := \frac{1}{L_i}\, t_{ij} = \frac{1}{\sum_{j=1}^m t_{ij}}\, t_{ij} \tag{22}$$

for all $i, j \in \{1, \ldots, m\}$. Then $N_M$ is stochastic and

$$\begin{aligned}\sum_{i=1}^m n_{ij} &= \sum_{i=1}^m \frac{1}{L_i} t_{ij} = \sum_{i\in\{1,\ldots,m\}:t_{ij}>0} \frac{1}{L_i} t_{ij} = \sum_{i\in\{1,\ldots,m\}:t_{ij}>0} \frac{1}{L_j} t_{ij} \\ &= \frac{1}{L_j} \sum_{i=1}^m t_{ij} = 1,\end{aligned}$$

for every $j \in \{1, \ldots, m\}$, so $N_M$ is doubly stochastic. Furthermore it follows directly from the IFS construction that for every $x \in [0,1]$, every

$E \in \mathcal{B}([0,1])$ and every $B \in \mathcal{C}$ the following equality holds:

$$\begin{pmatrix} K_{\mathcal{V}_M B}\big(h_1^{-1}(x), E\big) \\ K_{\mathcal{V}_M B}\big(h_2^{-1}(x), E\big) \\ \vdots \\ K_{\mathcal{V}_M B}\big(h_m^{-1}(x), E\big) \end{pmatrix} = \underbrace{\begin{pmatrix} n_{11} & n_{12} & \dots & n_{1m} \\ n_{21} & n_{22} & \dots & n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ n_{m1} & n_{m2} & \dots & n_{mm} \end{pmatrix}}_{:=N_M} \begin{pmatrix} K_B\big(x, h_1(E)\big) \\ K_B\big(x, h_2(E)\big) \\ \vdots \\ K_B\big(x, h_m(E)\big) \end{pmatrix} \tag{23}$$

Consequently, since the family of simple functions is dense in $L^1([0,1])$, the following transformation formula holds for every $x \in [0,1]$, every $f \in L^1([0,1])$, every $B \in \mathcal{C}$, and every $i \in \{1, \dots, m\}$:

$$\int_{[0,1]} f(z)\, K_{\mathcal{V}_M B}(h_i^{-1} x, dz) = \int_{[0,1]} \sum_{j=1}^{d} n_{ij} f(h_j^{-1}(z))\, K_B(x, dz) \tag{24}$$

For the following $\varphi$ we will denote the function assigning each $M \in \hat{\mathcal{T}}_2$ its corresponding doubly stochastic matrix $N_M$ according to (22), i.e. $\varphi(M) := N_M$. Using this notation we can state the following result:

**Lemma 13.** *Suppose that $M = (t_{ij})_{i,j=1}^m \in \hat{\mathcal{T}}_2$, $m \geq 2$, and let $N := \varphi(M)$. Then the unique fixed point $A^\star \in \mathcal{C}$ of $\mathcal{V}_M$ is idempotent if and only if $N$ is idempotent.*

**Proof:** (i) Suppose that $N$ is idempotent. If $B \in \mathcal{C}^{ip}$ then it follows completely analogous to Step 2 in Example 2 that

$$K_{\mathcal{V}_M B * \mathcal{V}_M B}(h_i^{-1}(x), E) = K_{\mathcal{V}_M B}(h_i^{-1}(x), E)$$

holds for every $i \in \{1, \dots, m\}$, every $x \in [0,1]$ and every $E \in \mathcal{B}([0,1])$. Therefore $\mathcal{V}_M B \in \mathcal{C}^{ip}$, so $\mathcal{V}_M^n B \in \mathcal{C}^{ip}$ for every $n \in \mathbb{N}$. Consequently, using the fact that $\lim_{n\to\infty} D_1(\mathcal{V}_M^n B, A^\star) = 0$ and applying Lemma 8, shows that $A^\star$ is idempotent, proving one implication.
(ii) If, on the other hand, $A^\star$ is idempotent, then we can proceed as follows: Set $\tilde{N} := N^2$, then for every fixed $E \in \mathcal{B}([0,1])$, every $i \in \{1, \dots, m\}$ and

every $x \in [0,1]$, using (23) and (24), we get

$$\begin{aligned}
K_{\mathcal{V}_M A^\star}(h_i^{-1}(x), E) &= K_{\mathcal{V}_M A^\star * \mathcal{V}_M A^\star}(h_i^{-1}(x), E) \\
&= \int_{[0,1]} K_{\mathcal{V}_M A^\star}(z, E) K_{\mathcal{V}_M A^\star}(h_i^{-1}(x), dz) \\
&= \int_{[0,1]} \sum_{j=1}^{m} n_{ij} K_{\mathcal{V}_M A^\star}(h_j^{-1}(z), E)\, K_{A^\star}(x, dz) \\
&= \int_{[0,1]} (n_{i1}, n_{i2}, \ldots, n_{im})\, N \begin{pmatrix} K_{A^\star}\big(z, h_1(E)\big) \\ K_{A^\star}\big(z, h_2(E)\big) \\ \vdots \\ K_{A^\star}\big(z, h_m(E)\big) \end{pmatrix} K_{A^\star}(x, dz) \\
&= \int_{[0,1]} \sum_{j=1}^{d} \tilde{n}_{ij} K_{A^\star}(z, h_j(E))\, K_{A^\star}(x, dz) \\
&\overset{B \in \mathcal{C}^{ip}}{=} \sum_{j=1}^{d} \tilde{n}_{ij} K_{A^\star}(x, h_j(E)).
\end{aligned}$$

Since the matrix $N$ in (23) is unique it follows that $\tilde{N} = \varphi(M) = N$, so $N$ is idempotent. ■

Lemma 13 directly leads to the following generalization of Theorem 11:

**Theorem 14.** *Suppose that $N$ is a $m$-dimensional idempotent doubly stochastic matrix fulfilling the condition $1 < rank(N) < m$. Then there exists a family $(M_r)_{r \in I_N} \subseteq \hat{\mathcal{T}}_2$ of transformation matrices such that:*

*(a) $N = \varphi(M_r)$ for every $r \in I_N$.*

*(b) For each $r \in I_N$ the unique fixed point $A_r^\star$ of $\mathcal{V}_{M_r}$ is idempotent.*

*(c) For each $s \in (1,2)$ there exist a unique $r_s \in I_N$ such that the Hausdorff dimension of the support of $A_{r_s}^\star$ is $s$.*

**Proof:** Suppose that $m \geq 3$ and that $1 < rank(N) < m$. Then (see [13] and [23]) there exists a permutation matrix $W$ such that $N' = WNW^{-1}$ has the form

$$N' = \begin{pmatrix} Q_1 & 0 & \ldots & 0 \\ 0 & Q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q_u \end{pmatrix},$$

whereby each $Q_i$ is a $q_i \times q_i$-matrix with all elements equal to $1/q_i$, $u = rank(N)$, and $q_1 \geq q_2 \geq \ldots \geq q_u$. Obviously $N'$ is idempotent and doubly stochastic too. Set $I_N := (0, \frac{1}{q_1})$, and, for every $r \in I_N$ define

$$r' := \frac{1 - q_1 r}{\sum_{j>1} q_j} = \frac{1 - q_1 r}{m - q_1}, \quad M'_r = \begin{pmatrix} r\,Q_1 & 0 & \ldots & 0 \\ 0 & r'\,Q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r'\,Q_u \end{pmatrix} \in \hat{\mathcal{T}}_2,$$

and set $M_r := W^{-1} M'_r W$. Then obviously $M_r \in \hat{\mathcal{T}}_2$ as well as $\varphi(M_r) = N$ holds for every $r \in I_N$, which completes the proof of point (a). Point (b) is a direct consequence of Lemma 13 since $N$ is idempotent. It remains to prove point (c), which can be done as follows: For every $r \in I_N$ let $A^\star_r \in \mathcal{C}^{ip}$ denote the unique fixed point of $\mathcal{V}_{M_r}$. Define $f : [0, \frac{1}{q_1}] \times [1, 2] \to \mathbb{R}$ by

$$f(r, s) = q_1^2 r^s + \sum_{j>1} q_j^2\,(r')^s, \tag{25}$$

then for every $r \in I_N$ the Hausdorff dimension of the support of $A^\star_r$ is the unique solution $s_r$ of $f(r, s) = 1$ in the interval $[0, 2]$ (see [1]). (i) Fix an arbitrary $s \in (1, 2)$, then obviously $f(0, s) \leq 1$ (equality if and only if $u = 2$ and $q_2 = 1$) and $f(1/q_1) > 1$ holds. Furthermore, using basic calculus, it is straightforward to verify that $r \mapsto f(r, s)$ is strictly decreasing on the interval $(0, r_0)$ and strictly increasing on $(r_0, 1/q_1)$, whereby

$$r_0 = \frac{\left(\frac{\sum_{j>1} q_j^2}{\sum_{j>1} q_j}\right)^{\frac{1}{s-1}} \frac{1}{\sum_{j>1} q_j}}{q_1^{\frac{1}{s-1}} + q_1 \left(\frac{\sum_{j>1} q_j^2}{\sum_{j>1} q_j}\right)^{\frac{1}{s-1}} \frac{1}{\sum_{j>1} q_j}} < \frac{1}{q_1},$$

so there exists exactly one $r_s \in (0, 1/q_1)$ such that $f(r_s, s) = 1$. (ii) On the other hand, for any fixed $r \in (0, 1/q_1)$ we have

$$\begin{aligned} f(r, 1) &= q_1^2\, r + \frac{1 - q_1 r}{\sum_{j>1} q_j} \sum_{j>1} q_j^2 \;\geq\; q_1^2\, r + 1 - q_1\, r = 1 + r q_1 (q_1 - 1) > 1 \\ f(r, 2) &= q_1^2\, r^2 + \left(\frac{1 - q_1 r}{\sum_{j>1} q_j}\right)^2 \sum_{j>1} q_j^2 \;\leq\; q_1^2\, r^2 + (1 - q_1\, r)^2 \\ &< q_1 r + 1 - q_1 r = 1. \end{aligned}$$

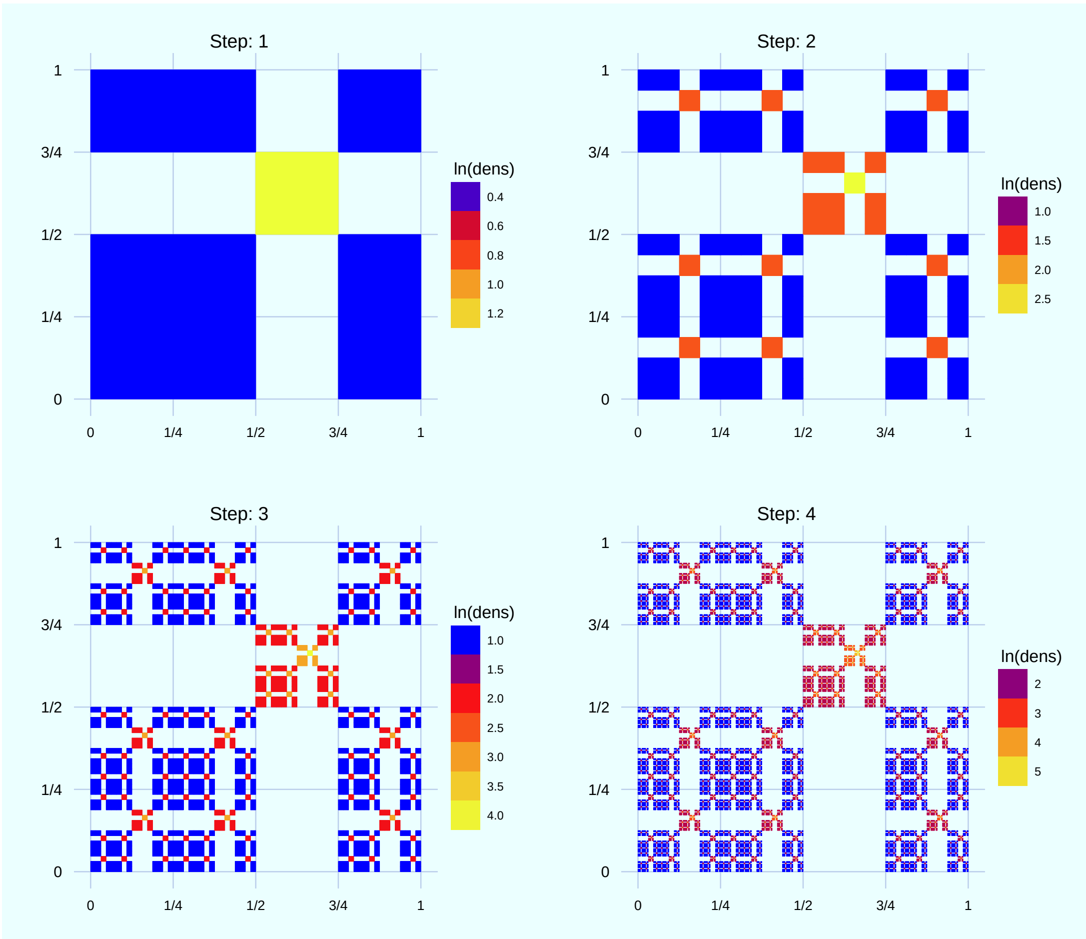


Figure 3: Image plot of the (natural) logarithm of the density of $\mathcal{V}_M^n(\Pi)$ for $n \in \{1, 2, 3, 4\}$, $M$ according to (26) in Example 3.

Using the fact that $\frac{\partial f(r,s)}{\partial s} < 0$ for every $s \in (1, 2)$ this implies the existence of a unique $s_r \in (1, 2)$ such that $f(r, s_r) = 1$. ■

**Example 3.** Figure 3 depicts the densities of the first four iterates of another operator $\mathcal{V}_M$ applied to the product copula whereby in this case $M$ is the transformation matrix

$$M = \begin{pmatrix} \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} \\ 0 & 0 & \frac{1}{4} & 0 \\ \frac{1}{12} & \frac{1}{12} & 0 & \frac{1}{12} \end{pmatrix}. \tag{26}$$

The corresponding limit copula $A^\star$ is idempotent (the matrix $N$ in this case is $4M$) and the Hausdorff dimension of its support is $\ln(10)/\ln(4)$.

**Remark 15.** Figure 2 and Figure 3 have been produced by direct calculation of the Kronecker (or tensor) product of $M$ (see the remark before Example 3 in [15]) and by using the *ggplot2* package in R for plotting. Since the rows and columns of the Kronecker product of $M$ grow exponentially the densities of higher iterates $\mathcal{V}_M^n(\Pi)$ can't be calculated this way. For higher iterates (and for better approximations of the limit copula $A^\star$) one can make use of the fact that the so-called *Chaos game* (a Markov process induced by the IFSP, see [1], [12], [14], [18]) is ergodic, simulate stochastic orbits (paths of the Chaos game) of sufficient length, and then calculate and plot two-dimensional histograms.


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