Some results on homeomorphisms between fractal supports of copulas

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Abstract

We consider parametric classes $(T_r)_{r\in(0,1/2)}$ of so-called transformation matrices and their induced families $(A_r)_{r\in(0,1/2)}$ and $(\mu_r)_{r\in(0,1/2)}$ of two-dimensional copulas and doubly stochastic measures with fractal support respectively. By using tools from Symbolic Dynamics we show that for each pair $r, r' \in (0, 1/2)$ with $r \neq r'$ there exists a homemorphism $H_{rr'}$ between the supports of μ_r and $\mu_{r'}$ mapping a Borel set of μ_r-measure one to a set of $\mu_{r'}$-measure zero. Differentiability properties of these homeomorphisms are studied and Hausdorff dimensions of related sets are calculated. Several examples and graphics illustrate the main results.

Keywords: Copula, Doubly stochastic measure, Homeomorphism, Hausdorff Dimension, Symbolic Dynamical System, Fractal

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1. Introduction

The importance of copulas in Probability Theory and Statistics stems from Sklar's well-known theorem (see [3, 23, 27]), stating that every joint distribution function can be decomposed into its marginals and a copula. In case of continuous marginals the copula is unique. Capturing all scale-invariant dependences of continuous random vectors copulas also play a crucial role in many applications. For more information about copulas and some of their applications see [20, 23, 26].

Working with special iterated function systems (IFS) Fredricks et al. [17] constructed families $(A_r)_{r\in(0,1/2)}$ of two-dimensional copulas with fractal supports fulfilling that for every $d \in (1,2)$ there exists $r_d \in (0,1/2)$ such that the Hausdorff dimension of the support S_{r_d} of A_{r_d} is d. Using the fact that the same IFS-construction also works with respect to the strong metric D_1 (a metrization of the strong operator topology of the corresponding Markov operators, see [28]) on the space $\mathcal{C}$ of two-dimensional copulas Trutschnig and Fernández-Sánchez [29] showed that the same result holds for the subclass of idempotent copulas (idempotent with respect to the star-product introduced by Darsow et al., see [10]). Families $(A_r)_{r\in(0,1/2)}$ of copulas with fractal support were also studied by the first three authors of the present paper in [1] and, more recently, in [2]. In the latter paper, using techniques from Probability and Ergodic Theory, the authors also discussed properties of subsets of the corresponding fractal supports and constructed mutually singular copulas having the same fractal set as support. Moments of these copulas were discussed in [4].

Charpentier and Juri [9, Remark 3.3] employed families $(A_r)_{r\in(0,1/2)}$ of the above-mentioned type to study lower tail-dependence copulas (LTDC). Given a copula $A \in \mathcal{C}$ and $u, v \in (0,1]$ the LTCD-copula $\Phi\left(A, u, v\right)$ relative to A is the copula relating the conditional distribution function $(x, y) \mapsto \frac{A(x,y)}{A(u,v)}$ with $0 < x \le u \le 1$ and $0 < y \le v \le 1$ with the corresponding marginal conditional distribution functions $x \mapsto \frac{A(x,v)}{A(u,v)}$ and $y \mapsto \frac{A(u,y)}{A(u,v)}$ respectively. In case $r = 0.1$, they showed that

$$\Phi\left(A_{0.1}, 0.2^k, 0.2^k\right) = A_{0.1}$$

for any $k \in \mathbb{N}$, a result easily generalizable to

$$\Phi\left(A_r, (2r)^k, (2r)^k\right) = A_r$$

for every $r \in (0, 1/2)$.

In the current paper we consider similar classes of transformation matrices $(T_r)_{r\in(0,1/2)}$ and the induced families $(A_r)_{r\in(0,1/2)}$ and $(\mu_r)_{r\in(0,1/2)}$ of copulas and doubly stochastic measures with fractal supports respectively. We study homeomorphisms $H_{rr'}$ between the corresponding supports S_r and $S_{r'}$ $(r \neq r')$ and characterize $H_{rr'}$ by a system of functional equations. More importantly, we show that $H_{rr'}$ maps a Borel set $\Lambda \subset S_r$ fulfilling $\mu_r(\Lambda) = 1$ to a set of $\mu_{r'}$-measure zero, implying that $\mu_{r'}$ and the push-forward $\mu_r^{H_{rr'}}$ of μ_r under $H_{rr'}$ are singular with respect to each other and that we can't find a function $\varphi : \mathbb{I}^2 \to \mathbb{R}$ such that the equality

$$\mu_{r'}\left(H_{rr'}(E)\right) = \int_E \varphi d\mu_r$$

holds for each Borel set E in $\mathbb{I}^2 := [0,1]^2$. As main tool for proving the above mentioned results the strong interrelation between attractors of IFSs and Code Spaces (Symbolic Dynamics) established by the well-known address map (and its inverse in the totally disconnected setting), see [6, 22], is used. Hausdorff dimensions of related sets are calculated and an Eggleston-Besicovitch-type result studying subsets of S_r with prescribed asymptotic frequencies in their 'addresses' is proved.

The rest of the paper is organized as follows: Section 2 gathers some notation and preliminaries that will be used in the sequel. Section 3 contains the construction of the homeomorphism $H_{rr'}$ mentioned before (both in the case that the IFS induced by the transformation matrix T_r is just touching and in the case that the IFS is totally disconnected) as well as the main results concerning singularity of $\mu_r^{H_{rr'}}$ with respect to $\mu_{r'}$. Section 4 gathers some calculations of the Hausdorff dimensions of related sets. Various graphics illustrate the main results.

2. Notation and preliminaries

$\mathbb{I}$ will denote the closed unit interval $[0,1]$, $\mathcal{B}(\mathbb{I}^2)$ the Borel σ-field in $\mathbb{I}^2$ and λ_2 the Lebesgue measure on $\mathcal{B}(\mathbb{I}^2)$. A *two-dimensional copula* (*copula*, for short) is a function $A : \mathbb{I}^2 \to \mathbb{I}$ satisfying (i) $A(x,0) = A(0,x) = 0$ and $A(x,1) = A(1,x) = x$ for all $x \in \mathbb{I}$ as well as (ii) $A(x_2,y_2) - A(x_1,y_2) - A(x_2,y_1) + A(x_1,y_1) \geq 0$ for x_1, x_2, y_1, y_2 in $\mathbb{I}$ fulfilling $x_1 \leq x_2$ and $y_1 \leq y_2$. Equivalently, a copula is the restriction to $\mathbb{I}^2$ of a bivariate distribution function having uniformly distributed marginals on $\mathbb{I}$. The family of all copulas will be denoted by $\mathcal{C}$. Π will denote the product copula, M the minimum copula and W the copula defined by $W(x,y) = \max\{x+y-1, 0\}$. Each copula $A \in \mathcal{C}$ induces a *doubly stochastic measure* μ_A by setting $\mu_A(R) = V_A(R) := A(x_2, y_2) - A(x_1, y_2) - A(x_2, y_1) + A(x_1, y_1)$ for every rectangle $R = [x_1, x_2] \times [y_1, y_2] \subseteq \mathbb{I}^2$ and extending μ_A in the standard measure-theoretic way from the semi-ring of all rectangles to full $\mathcal{B}(\mathbb{I}^2)$. Doubly stochastic measures may be regarded as natural generalization of doubly stochastic matrices. The family of all doubly stochastic measures on $\mathbb{I}^2$ will be denoted by $\mathcal{P}_{\mathcal{C}}$. The *support* of $A \in \mathcal{C}$ is the complement of the union of all open subsets of $\mathbb{I}^2$ with μ_A-measure zero, i.e. the smallest closed set having full μ_A-measure. d_∞ will denote the uniform distance on $\mathcal{C}$. For further information on copulas we refer to [12, 23, 26].

Before sketching the construction of copulas with fractal support via so-called transformation matrices we recall the definition of an Iterated Function System (IFS) and some main results about IFSs (for more details see [6, 14, 22]). Suppose for the following that (Ω, ρ) is a compact metric space, let $\mathcal{K}(\Omega)$ denote the family of all non-empty compact subsets of Ω, δ_H the Hausdorff metric on $\mathcal{K}(\Omega)$ and $\mathcal{P}(\Omega)$ the family of all probability measures on the Borel σ-field $\mathcal{B}(\Omega)$. A mapping $w : \Omega \to \Omega$ is called *contraction* if there exists a constant $L < 1$ such that $\rho(w(x), w(y)) \leq L\rho(x, y)$ holds for all $x, y \in \Omega$. A family $(w_l)_{l=1}^n$ of $n \geq 2$ contractions on Ω is called *Iterated Function System* (IFS for short) and will be denoted by $\{\Omega, (w_l)_{l=1}^n\}$. An IFS together with a vector $(p_l)_{l=1}^n \in (0, 1]^n$ fulfilling $\sum_{l=1}^n p_l = 1$ is called *Iterated Function System with probabilities* (IFSP for short). We will denote IFSPs by $\{\Omega, (w_l)_{l=1}^n, (p_l)_{l=1}^n\}$. Every IFSP induces the so-called *Hutchinson operator* $\mathcal{H} : \mathcal{K}(\Omega) \to \mathcal{K}(\Omega)$, defined by

$$\mathcal{H}(Z) := \bigcup_{l=1}^n w_i(Z). \tag{1}$$

It can be shown (see [6, 22]) that $\mathcal{H}$ is a contraction on the compact metric space $(\mathcal{K}(\Omega), \delta_H)$, so Banach's Fixed Point theorem implies the existence of a unique, globally attractive fixed point $Z^\star$ of $\mathcal{H}$. Hence, for every $R \in \mathcal{K}(\Omega)$, we have

$$\lim_{n\to\infty} \delta_H\big(\mathcal{H}^n(R), Z^\star\big) = 0.$$

The *attractor* $Z^\star$ will be called *self-similar* if all contractions in the IFS are similarities. An IFS $\{\Omega, (w_l)_{l=1}^n\}$ is called *totally disconnected* (or disjoint) if the sets $w_1(Z^\star), w_2(Z^\star), \ldots, w_n(Z^\star)$ are pairwise disjoint. $\{\Omega, (w_l)_{l=1}^n\}$ will be called *just touching* if it is not totally disconnected but there exists a non-empty open set $U \subseteq \Omega$ such that $w_1(U), w_2(U), \ldots, w_n(U)$ are pairwise disjoint. Additionally to the operator $\mathcal{H}$ every IFSP also induces a (Markov) operator $\mathcal{V} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$, defined by

$$\mathcal{V}(\mu) := \sum_{i=1}^{n} p_i \, \mu^{w_i}. \tag{2}$$

The so-called *Hutchison metric* h (sometimes also called Kantorovich or Wasserstein metric) on $\mathcal{P}(\Omega)$ is defined by

$$h(\mu, \nu) := \sup \left\{ \int_\Omega f \, d\mu - \int_\Omega f \, d\nu : \; f \in Lip_1(\Omega, \mathbb{R}) \right\}. \tag{3}$$

Hereby $Lip_1(\Omega, \mathbb{R})$ is the class of all non-expanding functions $f : \Omega \to \mathbb{R}$, i.e. functions fulfilling $|f(x) - f(y)| \leq \rho(x, y)$ for all $x, y \in \Omega$. It is not difficult to show that $\mathcal{V}$ is a contraction on $(\mathcal{P}(\Omega), h)$, that h is a metrization of the topology of weak convergence on $\mathcal{P}(\Omega)$ and that $(\mathcal{P}(\Omega), h)$ is a compact metric space (see [6, 11]). Consequently, again by Banach's Fixed Point theorem, it follows that there is a unique, globally attractive fixed point $\mu^\star \in \mathcal{P}(\Omega)$ of $\mathcal{V}$, i.e. for every $\nu \in \mathcal{P}(\Omega)$ we have

$$\lim_{n \to \infty} h\big(\mathcal{V}^n(\nu), \mu^\star\big) = 0.$$

$\mu^\star$ will be called *invariant measure* - it is well known that the support of $\mu^\star$ is exactly the attractor $Z^\star$. The measure $\mu^\star$ will be called *self-similar* if $Z^\star$ is self-similar, i.e. if all contractions in the IFSP are similarities.

As mentioned already in the Introduction attractors of IFSs are strongly interrelated with symbolic dynamics via the so-called *address map* (see [6, 22]): For every $n \in \mathbb{N}$ the *code space of* n *symbols* will be denoted by Σ_n, i.e.

$$\Sigma_n := \{1, 2, \ldots, n\}^{\mathbb{N}} = \big\{(k_i)_{i \in \mathbb{N}} : 1 \leq k_i \leq n \; \forall i \in \mathbb{N}\big\}.$$

Bold symbols will denote elements of Σ_n. σ will denote the (left-) shift operator on Σ_n, i.e. $\sigma((k_1, k_2, \ldots)) = (k_2, k_3, \ldots)$. Define a metric ρ on Σ_n by setting

$$\rho(\mathbf{k}, \mathbf{l}) := \begin{cases} 0 & \text{if } \mathbf{k} = \mathbf{l} \\ 2^{1 - \min\{i : k_i \neq l_i\}} & \text{if } \mathbf{k} \neq \mathbf{l}, \end{cases}$$

then it is straightforward to verify that (Σ_n, ρ) is a compact ultrametric space and that ρ is a metrization of the product topology. Suppose now that $\{\Omega, (w_l)_{l=1}^n\}$ is an IFS with attractor $Z^\star$, fix an arbitrary $x \in \Omega$ and define the address map $G : \Sigma_n \to \Omega$ by

$$G(\mathbf{k}) := \lim_{m\to\infty} w_{k_1} \circ w_{k_2} \circ \cdots w_{k_m}(x), \tag{4}$$

then (see [22]) $G(\mathbf{k})$ is independent of x, $G : \Sigma_n \to \Omega$ is Lipschitz continuous and $G(\Sigma_n) = Z^\star$. Furthermore G is injective (and hence a homeomorphism) if and only if the IFS is totally disconnected. Given $z \in Z^\star$ every element of the preimage $G^{-1}(\{z\})$ will be called *address* of z. Considering a IFSP $\{\Omega, (w_l)_{l=1}^n, (p_l)_{l=1}^n\}$ with attractor $Z^\star$ and invariant measure $\mu^\star$ we can also define a probability measure P on $\mathcal{B}(\Sigma_n)$ by setting

$$P\Big(\big\{\mathbf{k} \in \Sigma_n : k_1 = i_1, k_2 = i_2, \ldots, k_m = i_m\big\}\Big) = \prod_{j=1}^m p_{i_j} \tag{5}$$

and extending in the standard way to full $\mathcal{B}(\Sigma_n)$. According to [22] $\mu^\star$ is the push-forward of P via the address map, i.e. $P^G(B) := P(G^{-1}(B)) = \mu^\star(B)$ holds for each $B \in \mathcal{B}(Z^\star)$.

Throughout the rest of the paper we will consider IFSP induced by so-called *transformation matrices*, for the original definition see [17], for the generalization to the multivariate setting we refer to [29].

Definition 1 ([17]). A $n\times m$-matrix $T = (t_{ij})_{i=1\ldots n, j=1\ldots m}$ is called *transformation matrix* if it fulfills the following four conditions: (i) $\max(n, m) \geq 2$, (ii) all entries are non-negative, (iii) $\sum_{i,j} t_{ij} = 1$, and (iv) no row or column has all entries 0.

Given T we define the vectors $(a_j)_{j=0}^m, (b_i)_{i=0}^n$ of cumulative column and row sums by $a_0 = b_0 = 0$ and

$$\begin{aligned} a_j &= \sum_{j_0\leq j} \sum_{i=1}^n t_{ij} \quad j \in \{1, \ldots, m\} \\ b_i &= \sum_{i_0\leq i} \sum_{j=1}^m t_{ij} \quad i \in \{1, \ldots, n\}. \end{aligned}$$

Since T is a transformation matrix both $(a_j)_{j=0}^m$ and $(b_i)_{i=0}^n$ are strictly increasing and $R_{ji} := [a_{j-1}, a_j] \times [b_{i-1}, b_i]$ are compact non-empty rectangles for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$. Set $\widetilde{I} := \{(i,j) : t_{ij} > 0\}$ and consider the IFSP $\{\mathbb{I}^2, (w_{ji})_{(i,j)\in\widetilde{I}}, (t_{ij})_{(i,j)\in\widetilde{I}}\}$, whereby the contraction $w_{ji} : \mathbb{I}^2 \to R_{ji}$ is defined by

$$w_{ji}(x,y) = \big(a_{j-1} + x(a_j - a_{j-1})\,,\, b_{i-1} + x(b_i - b_{i-1})\big).$$

The induced operator $\mathcal{V}_T$ on $\mathcal{P}(\mathbb{I}^2)$ is defined by

$$\mathcal{V}_T(\mu) := \sum_{j=1}^{m}\sum_{i=1}^{n} t_{ij}\,\mu^{w_{ji}} = \sum_{(i,j)\in\widetilde{I}} t_{ij}\,\mu^{w_{ji}} \tag{6}$$

and it is straightforward to see that $\mathcal{V}_T$ maps $\mathcal{P}_{\mathcal{C}}$ into itself so we can view $\mathcal{V}_T$ also as operator on $\mathcal{C}$ (see [17]). According to the before-mentioned facts there is exactly one copula $A_T^\star \in \mathcal{C}$, to which we will refer to as *invariant copula*, such that $\mathcal{V}_T(\mu_{A_T^\star}) = \mu_{A_T^\star}$ holds. Considering the conditions

(i) T contains at least one zero,

(ii) For each non-zero entry of T the row and column sums through for that entry are equal,

(iii) There is at least one row or column of T with two non-zero entries,

the following results hold (again see [17]): If T fulfills Condition (i) then $A_T^\star$ is singular with respect to the Lebesgue measure λ_2. $A_T^\star$ is self-similar if T satisfies Condition (ii). If T satisfies Conditions (i) and (iii) the support of $A_T^\star$ is a fractal with Hausdorff dimension between 1 and 2. As mentioned in the Introduction, for each $d \in (1,2)$ there exists a copula $A \in \mathcal{C}$ whose support is a fractal with Hausdorff dimension d. We use Mandelbrot's original definition of a *fractal set* as a set whose topological dimension is lower than its Hausdorff dimension (for basic properties concerning Hausdorff dimension and other notions that are useful to express fractal properties of sets, we refer to [14, 16]). For the analogous result on the subclass of idempotent copulas we refer to [29].

3. Support homeomorphisms

In this section we will mainly work with the following family $(T_r)_{r\in(0,1/2)}$ of transformation matrices already used in [4, 17]:

$$T_r = \begin{pmatrix} r/2 & 0 & r/2 \\ 0 & 1-2r & 0 \\ r/2 & 0 & r/2 \end{pmatrix} \tag{7}$$

Setting $A_r := A^\star_{T_r}$ as well as $\mu_r = \mu^\star_{T_r}$ for every $r \in (0, 1/2)$ and using the results mentioned in the previous section, it follows immediately that $\mu_r \in \mathcal{P}_\mathcal{C}$ is self-similar and that μ_r has fractal support. Furthermore (see [17]) for every $d \in (1, 2)$ there exists exactly one $r_d \in (0, 1/2)$ such that the Hausdorff dimension of the support S_{r_d} of A_{r_d} is d. We will rename the contractions induced by T_r as

$$w^r_1 := w^r_{11},\ w^r_2 := w^r_{13},\ w^r_3 := w^r_{31},\ w^r_4 := w^r_{33},\ w^r_5 := w^r_{22}$$

and set $Q^r_i = w^r_i(\mathbb{I}^2)$ as well as $S^i_r = Q^i_r \cap S_r$ for every $i \in \{1, \dots, 5\}$. In the sequel we will also write w_i instead of w^r_i etc., if no confusion can arise which r is meant. Figure 1 depicts the densities of $V^5_{T_r}(\Pi)$ for the cases $r = 1/4$ and $r = 1/3$, Figure 2 the copula $V^5_{T_r}(\Pi)$ and its density for $r = 1/3$. Due to the

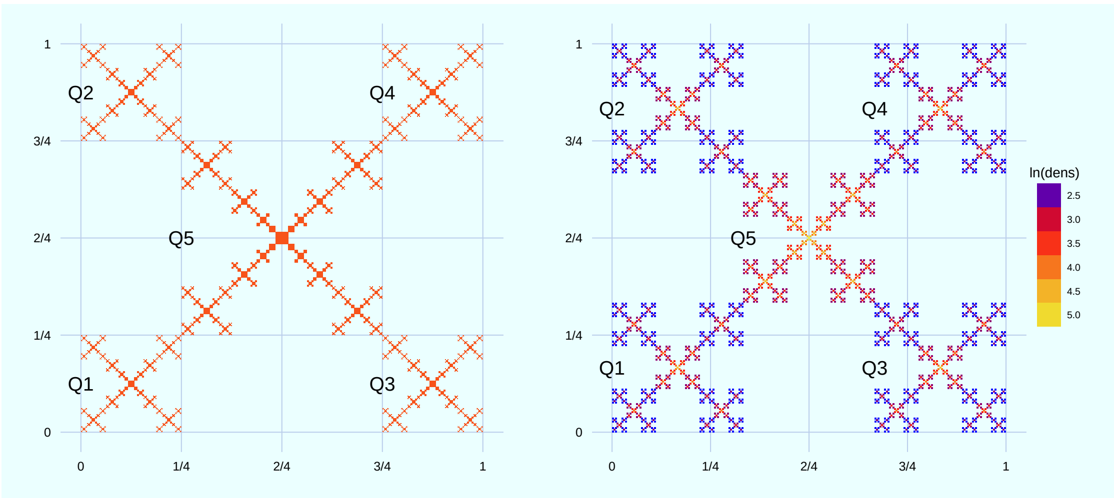


Figure 1: Image plot of the (natural) logarithm of the density of $\mathcal{V}^5_{T_r}(\Pi)$ for $r = 1/4$ (left) and $r = 1/3$ (right).

fact that the IFS induced by T_r is just-touching there can not be many points with more than one address - the following result holds (by a slight misuse of notation we will write $G_r^{-1}(x, y)$ instead of $G_r^{-1}(\{(x, y)\})$ in the sequel):

Lemma 2. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) and fix $r \in (0, 1/2)$. Then all but countable many points in S_r have a unique G_r-address. For every point (x, y) without unique G_r-address there exists a natural number n and $k_1, k_2, \dots, k_n \in \{1, 2, \dots, 5\}$ such that exactly one of the* *following four situations holds:*

$$\begin{array}{lcl}(S1)\ G_r^{-1}(x,y) &=& \big\{(k_1,\dots,k_n,5,1,1,1,\dots),(k_1,\dots,k_n,1,4,4,4,\dots)\big\}\\ (S2)\ G_r^{-1}(x,y) &=& \big\{(k_1,\dots,k_n,5,4,4,4,\dots),(k_1,\dots,k_n,4,1,1,1,\dots)\big\}\\ (S3)\ G_r^{-1}(x,y) &=& \big\{(k_1,\dots,k_n,5,2,2,2,\dots),(k_1,\dots,k_n,2,3,3,3,\dots)\big\}\\ (S4)\ G_r^{-1}(x,y) &=& \big\{(k_1,\dots,k_n,5,3,3,3,\dots),(k_1,\dots,k_n,3,2,2,2,\dots)\big\}\end{array} \quad (8)$$

Proof: Note that for every $\mathbf{k} \in \Sigma_5$ we have

$$G_r(\mathbf{k}) = w_{k_1}(G_r(\sigma\mathbf{k})). \quad (9)$$

Since $(0,0)$ is a fixed point of w_1 and $(0,0) \notin \cup_{i=2}^5 S_r^i$ we directly get that $(1,1,\dots)$ is the unique G_r-address of $(0,0)$. $G_r^{-1}(0,1) = \{(2,2,\dots)\}$ as well as $G_r^{-1}(1,0) = \{(3,3,\dots)\}$ and $G_r^{-1}(1,1) = \{(4,4,\dots)\}$ follows analogously. $(r,r) = w_1(1,1) = w_5(0,0)$ implies $G_r^{-1}(r,r) \subseteq \big\{(5,1,1,1,\dots),(1,4,4,4,\dots)\big\}$ from which, applying (9) together with the fact that $(0,0)$ and $(1,1)$ have unique addresses

$$G_r^{-1}(r,r) = \big\{(5,1,1,1,\dots),(1,4,4,4,\dots)\big\}$$

follows. Proceeding in the same manner we get

$$\begin{array}{rcl}G_r^{-1}(1-r,1-r) &=& \big\{(5,4,4,4,\dots),(4,1,1,1,\dots)\big\}\\ G_r^{-1}(r,1-r) &=& \big\{(5,2,2,2,\dots),(2,3,3,3,\dots)\big\}\\ G_r^{-1}(1-r,r) &=& \big\{(5,3,3,3,\dots),(3,2,2,2,\dots)\big\}.\end{array}$$

Having this, again using (9) and the fact that $(0,0)$ and $(1,1)$ have unique addresses yields, firstly,

$$\begin{array}{rcl}G_r\left((k_1,\dots,k_n,5,1,1,1,\dots)\right) &=& \left(w_{k_1}\circ\cdots\circ w_{k_n}\circ w_5\right)(0,0)\\ &=& \left(w_{k_1}\circ\cdots\circ w_{k_n}\circ w_1\right)(1,1)\\ &=& G_r\left((k_1,\dots,k_n,1,4,4,4,\dots)\right)\end{array}$$

implying that $(x,y) = G_r((k_1,\dots,k_n,5,1,1,1,\dots))$ has at least two addresses and, secondly, that there can not be more than two. The other three situations (S2-S4) in (8) follow in the same manner.
Finally suppose that a point $(x,y) \in S_r$ has two addresses $\mathbf{k},\mathbf{l} \in \Sigma_5$. Setting $j := \min\{i \in \mathbb{N} : k_i \neq l_i\}$ and once more using (9) it follows that

$$G_r\big((k_j,k_{j+1},\dots)\big) = G_r\big((l_j,l_{j+1},\dots)\big) \in \big\{(r,r),(1-r,1-r),(r,1-r),(1-r,r)\big\},$$

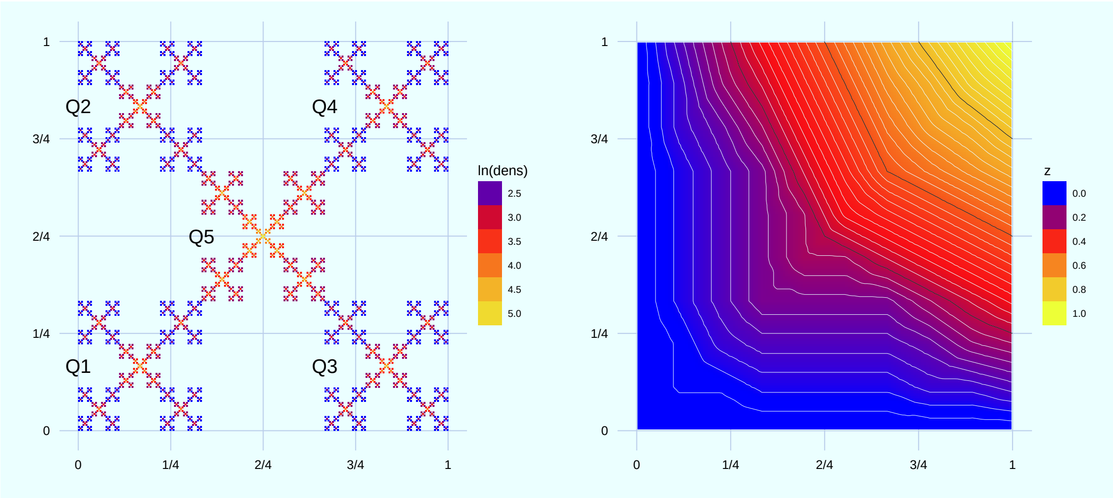


Figure 2: Image plot of the (natural) logarithm of the density of $\mathcal{V}^5_{T_r}(\Pi)$ (left) and image plot of the copula $\mathcal{V}^5_{T_r}(\Pi)$ (right) for $r = 1/3$ (white/gray lines depict contours).

which completes the proof. ■

Consider now $r, r' \in (0, 1/2)$ with $r \neq r'$. For every $(x, y) \in S_r$ the address map $G_{r'} : \Sigma_5 \to S_{r'}$ maps all possible G_r-addresses $G_r^{-1}(x, y)$ of (x, y) to the same point $S_{r'}$. Hence assigning

$$(x, y) \mapsto H_{rr'}(x, y) := G_{r'}(G_r^{-1}(x, y)) \tag{10}$$

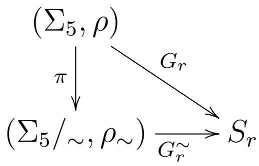
defines a mapping $H_{rr'} : S_r \to S_{r'}$ easily seen to be bijective. $H_{rr'}$ is also continuous - the following theorem holds:

Theorem 3. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7). Then for every pair $r, r' \in (0, 1/2)$ the mapping $H_{rr'}$ defined according to (10) is a homeomorphism.*

Proof: We will show that $H_{rr'}$ is continuous at every point (x, y) of S_r. Suppose that $(\mathbf{k}^n)_{n\in\mathbb{N}}$ is a sequence in Σ_5 such that $(x_n, y_n) \to (x, y)$ for $(x_n, y_n) = G_r(\mathbf{k}^n)$. Consider the following two cases: (a) If (x, y) has a unique G_r-address $\mathbf{k}$ then $(x, y) \in S_r^{k_1} \setminus \cup_{j\neq k_1} S_r^j$ and it follows immediately that there exists an index n_1 such that $k_1^n = k_1$ for all $n \geq n_1$. Obviously $G_r(\sigma^j \mathbf{k})$ has a unique address for every $j \in \mathbb{N}$ too, so, using $G_r(\mathbf{k}) = w_{k_1} \circ \cdots w_{k_i}(G_r(\sigma^i \mathbf{k}))$, we can find another index $n_2 > n_1$ such that $k_2^n = k_2$ for all $n \geq n_2$. Proceeding in the same manner shows $\rho(\mathbf{k}^n, \mathbf{k}) \to 0$ for

$n \to \infty$ which, using continuity of $G_{r'}$, in turn implies $\lim_{n\to\infty} H_{rr'}(x_n, y_n) = H_{rr'}(x, y)$. (b) Suppose that (x, y) has two addresses $(k_1, \ldots, k_l, 5, 1, 1, 1, \ldots)$ and $(k_1, \ldots, k_l, 1, 4, 4, 4, \ldots)$. Applying similar arguments we can show that there exists an index n_0 such that for each $n > n_0$ the address $\mathbf{k}^n$ is of the form $(k_1, \ldots, k_l, 5, *, *, * \ldots)$ or $(k_1, \ldots, k_l, 1, *, *, *, \ldots)$. Hence, using the fact that all corners of the unit square have unique addresses and proceed like in case (a), it follows that $H_{rr'}$ is continuous at (x, y). Completely the same line of argumentation shows that $H_{rr'}$ is also continuous in all points falling in categories (S2)-(S4) of Lemma 2. As continuous bijection on the compact metric space S_r $H_{rr'}$ is a homeomorphism, which completes the proof. ■

Remark 4. An alternative way for proving that $H_{rr'} : S_r \to S_{r'}$ is a homeomorphism without thinking much about possible double address would be the following: Define an equivalence relation $\sim$ on Σ_5^2 by setting $\sigma \sim \vartheta :\Leftrightarrow G_r(\mathbf{k}) = G_r(\mathbf{l})$ and consider the quotient space $\Sigma_5/_\sim$ with the quotient topology. π will denote the projection from Σ_5 to $\Sigma_5/_\sim$. According to [8] the quotient topology $\mathcal{O}_\sim$ is metrizable and the resulting quotient space $(\Sigma_5/_\sim, \rho_\sim)$ is compact again. Furthermore the new mapping $G_r^\sim : \Sigma_5/_\sim \to S_r$, defined via $G_r^\sim([\sigma]) := G_r(\pi^{-1}([\sigma]))$, is a bijection and continuous, hence a homeomorphism.



Since $\sim$ does not depend on the concrete choice of r we directly get that S_r and $S_{r'}$ are homeomorphic, which, considering $H_{rr'} = G_{r'}^\sim \circ (G_r^\sim)^{-1}$, completes the proof.

The homeomorphism $H_{rr'}$ can also be characterized through a system of functional equations - the following result holds:

Theorem 5. *Consider the family $(T_r)_{r\in(0,1/2)}$ in (7). Then, for every pair $r, r' \in (0, 1/2)$, $H_{rr'}$ defined according to (10) is the unique bounded function $h : S_r \to \mathbb{R}^2$ satisfying*

$$h \circ w_i^r(x, y) = w_i^{r'} \circ h(x, y) \tag{11}$$

for all $i \in \{1, \ldots, 5\}$.

Proof: Note that (11) is equivalent to

$$\begin{cases} h(rx, ry) = r'h(x,y) \\ h(rx, 1-r+ry) = (0, 1-r') + r'h(x,y) \\ h(1-r+rx, ry) = (1-r', 0) + r'h(x,y) \\ h(1-r+rx, 1-r+ry) = (1-r', 1-r') + r'h(x,y) \\ h(r+(1-2r)x, r+(1-2r)y) = (r,r) + (1-2r')h(x,y). \end{cases}$$

Direct calculations show that $H_{rr'}$ satisfies the above equalities. To prove that $H_{rr'}$ is the only solution we proceed as follows: Consider the Banach space $(B(S_r), \|\cdot\|_\infty)$ of all $\mathbb{R}^2$-valued bounded functions on S_r with $\|f\|_\infty = \sup_{z \in S_r} \|f(z)\|_2$ ($\|\cdot\|_2$ denoting the Euclidean norm) and apply the Contraction Mapping Theorem to $\Phi : B(S_r) \to B(S_r)$, defined by

$$\begin{array}{ll} \Phi(h)(x,y) = r'h\big(\frac{x}{r}, \frac{y}{r}\big) & \text{if } (x,y) \in S_r^1 \\ \Phi(h)(x,y) = (0, 1-r') + r'h\big(\frac{x}{r}, \frac{y+r-1}{r}\big) & \text{if } (x,y) \in S_r^2 \\ \Phi(h)(x,y) = (1-r', 0) + r'h\big(\frac{x+r-1}{r}, \frac{y}{r}\big) & \text{if } (x,y) \in S_r^3 \\ \Phi(h)(x,y) = (1-r', 1-r') + r'h\big(\frac{x+r-1}{r}, \frac{y+r-1}{r}\big) & \text{if } (x,y) \in S_r^4 \\ \Phi(h)(x,y) = (r,r) + (1-2r')h\big(\frac{x-r}{1-2r}, \frac{y-r}{1-2r}\big) & \text{if } (x,y) \in S_r^5. \ \blacksquare \end{array}$$

Although being a homeomorphism the push-forward $\mu_r^{H_{rr'}}$ of μ_r via $H_{rr'}$ is very different from $\mu_{r'}$:

Theorem 6. *Consider the family* $(T_r)_{r \in (0,1/2)}$ *defined according to (7) and fix* $r, r' \in (0, 1/2)$ *with* $r \neq r'$. *Then the measures* $\mu_r^{H_{rr'}}$ *and* $\mu_{r'}$ *on* $\mathcal{B}(S_{r'})$ *are singular with respect to each other.*

Proof: According to [2, Cor. 3.8] the set $M_{r'} \in \mathcal{B}(S_{r'})$ of points $(x,y) \in S_{r'}$ whose $G_{r'}$-address $\mathbf{k} \in \Sigma_5$ fulfills

$$\begin{cases} \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 1\}}{n} = r'/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 2\}}{n} = r'/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 3\}}{n} = r'/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 4\}}{n} = r'/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 5\}}{n} = 1 - 2r' \end{cases} \tag{12}$$

has full $\mu_{r'}$-measure. Considering both the fact that the set $N_r \in \mathcal{B}(S_r)$ of all $(x,y) \in S_r$ for which (12) holds has μ_r-measure zero and the fact that $H_{rr'}(N_r) = M_{r'}$ completes the proof. $\blacksquare$

Remark 7. It is straightforward to construct copulas $A, B \in \mathcal{C}$, $A \neq B$, with a support having λ_2-measure zero for which there exists a homeomorphism $H : S_A \to S_B$ between their supports which is at the same time an isomorphism of the corresponding doubly stochastic measure spaces $(S_A, \mathcal{B}(S_A), \mu_A)$ and $(S_B, \mathcal{B}(S_B), \mu_B)$. In fact, setting $A = M$ and $B = W$ yields a very simple example. For the copulas $(A_r)_{r\in(0,1/2)}$, however, Theorem 6 shows that the situation is completely different.

Remark 8. The function $H_{r,r'}$ could alternatively have been constructed on full $[0,1]^2$ as follows: Let v_1^r, v_4^r, v_5^r denote the first coordinates of the functions w_1^r, w_4^r, w_5^r for every $r \in (0, 1/2)$. Set $g_0(x) = x$ for every $x \in [0,1]$ and define a sequence $(g_n)_{n\in\mathbb{N}}$ of functions on $[0,1]$ recursively by

$$g_{n+1} \circ v_i^r(x) := v_i^{r'} \circ g_n(x).$$

for every $i \in \{1, 4, 5\}$. It is straightforward to verify that $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a homeomorphism $g : [0,1] \to [0,1]$, fulfilling $H_{rr'}(x, y) = (g(x), g(y))$ for all $x, y \in S_r$. Setting $G_{rr'}(x, y) := (g(x), g(y)$ therefore defines a homeomorphism $G_{rr'}$ on $[0,1]^2$ which is an extension of $H_{rr'}$. According to [1, 21] we can find λ-preserving transformations $f_1^r, f_2^r : [0,1] \to [0,1]$ such that $A_r(x, y) = \lambda\{z \in [0,1] : f_1^r(z) \leq x, f_2^r(z) \leq y)\}$ for all $x, y \in [0,1]$, so the push-forward of μ_M via (f_1^r, f_2^r) coincides with μ_r. The probability measure $\mu_M^{G\circ(f_1^r,f_2^r)}$ is an extension of $\mu_r^{H_{rr'}}$ to $\mathcal{B}(\mathbb{I}^2)$ assigning mass zero to all Borel sets $U \in \mathcal{B}(\mathbb{I}^2)$ with $U \cap S_{r'} = \emptyset$. Taking into account that g is not λ-preserving, $\mu_M^{G\circ(f_1^r,f_2^r)}(S_{r'})$ is not doubly stochastic.

As next step we will take a closer look to $H_{rr'}$ from the viewpoint of differentiable transformations of measure spaces. We start with the subsequent definitions containing the relevant ideas in the general setting, for more details see [24, 25].

Definition 9. A collection $\mathcal{U}$ of open sets in a metric space (Ω, ρ) is called a *substantial family* for a measure μ on $\mathcal{B}(\Omega)$ if the following conditions hold:

a) There exists a constant $\beta > 0$ such that for each $U \in \mathcal{U}$ there is an open ball B containing U and satisfying $0 < \mu(B) < \beta\mu(U)$.

b) For each $x \in \Omega$ and for each $\delta > 0$, there is a set $U = U(x, \delta) \in \mathcal{U}$ satisfying $\text{diam}(U) < \delta$ as well as $x \in U$.

Definition 10. Let (Ω, Λ, μ) and $(\Omega', \Lambda', \mu')$ be measure spaces, $f : \Omega \to \Omega'$ a function with $f(A) \in \Lambda'$ for all $A \in \Lambda$, and $\mathcal{U}$ a family of subsets in Λ. We say that f is $\mathcal{U}$-differentiable with respect to μ and μ' at $x \in \Omega$ if there exists a real number α satisfying

$$\begin{aligned}\alpha &= \lim_{\gamma \to 0}\left(\sup\left\{\frac{\mu'(f(U))}{\mu(U)} : x \in U \in \mathcal{U} \text{ and } \operatorname{diam}(U) < \gamma\right\}\right) \\ &= \lim_{\gamma \to 0}\left(\inf\left\{\frac{\mu'(f(U))}{\mu(U)} : x \in U \in \mathcal{U} \text{ and } \operatorname{diam}(U) < \gamma\right\}\right).\end{aligned}$$

If such an α exists it is called the *$\mathcal{U}$-derivative of f at x* (with respect to μ and μ').

For each $r \in (0, 1/2)$ let S_r^* denote the set of all points in S_r with unique G_r-address. The proof of the following lemma is straightforward:

Lemma 11. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7). For every $r \in (0, 1/2)$ the family $\mathcal{U}_r^*$ consisting of all sets of the form*

$$w^r_{k_1} \circ \cdots \circ w^r_{k_n}\left((0,1)^2 \cap S_r^*\right) : n \in \mathbb{N} \text{ and } k_i \in \{1,2,3,4,5\}$$

is substancial for μ_r on $\mathcal{B}(S_r^*)$.

Being doubly stochastic μ_r has no point masses, hence $\mu_r(S_r^*) = 1$ holds and we can also work with the class $\mathcal{U}_r$ consisting of all sets of the form

$$w^r_{k_1} \circ \cdots \circ w^r_{k_n}\left((0,1)^2 \cap S_r\right) : n \in \mathbb{N} \text{ and } k_i \in \{1,2,3,4,5\}.$$

Theorem 12. *Consider the family $(T_r)_{r\in(0,1/2)}$ according to (7) and fix $r, r' \in (0, 1/2)$ with $r \neq r'$. Then there exists a set $M_r \subseteq S_r$ with μ_r-measure one such that $H_{rr'} : S_r \longrightarrow S_{r'}$ is $\mathcal{U}_r$–differentiable with respect to μ_r and $\mu_{r'}$ at every $(x, y) \in M_r$. At every $(x, y) \in M_r$ the value of the $\mathcal{U}_r$ derivative is zero.*

Proof: Again applying Corollary 3.8 in [2] it follows that the set $M_r \in \mathcal{B}(S_r)$ of points $(x, y) \in S_r$ whose G_r-address $\mathbf{k} \in \Sigma_5$ fulfills

$$\begin{cases} \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 1\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 2\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 3\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 4\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : k_i = 5\}}{n} = 1 - 2r \end{cases} \tag{13}$$

has full μ_r-measure. Suppose now that $(x,y) \in M_r$, that $G_r(\mathbf{k}) = (x,y)$ and define $f^5_m(\mathbf{k}) = \mathrm{Card}\{i \le m : k_i = 5\}/m$ for every $m \in \mathbb{N}$. The function

$$g : z \mapsto \left(\frac{r'}{r}\right)^{1-z} \left(\frac{1-2r'}{1-2r}\right)^{z}$$

is continuous in $z_0 = (1-2r) \in (0,1)$ and fulfills $g(z_0) < 1$. Hence, for every $\varepsilon > 0$ we can find a constant $a < 1$ and an index $m_0 = m_0(\varepsilon)$ such that for all $m \ge m_0$ we have $g(f^5_m(\mathbf{k})) < a < 1$ as well as $a^m < \varepsilon$. Set $\gamma := g(f^5_{m_0}(\mathbf{k}))^{m_0}$, then every $U \in \mathcal{U}_r$ with $(x,y) \in U$ and $\mathrm{diam}(U) < \gamma$ is of the form

$$U_m := w^r_{k_1} \circ \cdots \circ w^r_{k_m}\left((0,1)^2 \cap S_r\right).$$

with $m \ge m_0$. For each such U_m we get

$$\begin{aligned} \frac{\mu_{r'}\big(H_{rr'}(w_{k_1} \circ \cdots \circ w_{k_m}(S_r))\big)}{\mu_r(w_{k_1} \circ \cdots \circ w_{k_m}(S_r))} &= \frac{\left(\frac{r'}{2}\right)^{m(1-f^5_m(\mathbf{k}))}(1-2r')^{mf^5_m(\mathbf{k})}}{\left(\frac{r}{2}\right)^{m(1-f^5_m(\mathbf{k}))}(1-2r)^{mf^5_m(\mathbf{k})}} \\ &= g(f^5_m(\mathbf{k}))^m < \varepsilon. \end{aligned}$$

This completes the proof since $(x,y) \in M_r$ was arbitrary. ■

So far in this paper we have only considered elements of the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) which all induce just-touching IFSP. To simplify matters we could also have started with transformation matrices that induce totally disconnected IFSP. The reasons for choosing $(T_r)_{r\in(0,1/2)}$ according to (7) were that (i) the family induces IFSPs that consist of only five transformations (which is impossible for the totally disconnected setting), (ii) the chosen approach shows that double addresses don't cause too much technical problems and, (iii) the family has already been discussed in various papers (see [4, 9, 17, 28, 29]). We will, however, close this section by taking a look to the totally disconnected setting and mention some alternative simple proofs valid in this situation. Note that the copulas we will consider are generalized shuffles of Min (see [13, 30]).

Consider the transformation matrices $(M_r)_{r\in(0,1/2)}$, defined by

$$M_r = \begin{pmatrix} \frac{r}{2} & 0 & 0 & 0 & 0 & \frac{r}{2} \\ 0 & 0 & \frac{1-2r}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2r}{4} & 0 \\ 0 & \frac{1-2r}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2r}{4} & 0 & 0 \\ \frac{r}{2} & 0 & 0 & 0 & 0 & \frac{r}{2} \end{pmatrix} \tag{14}$$

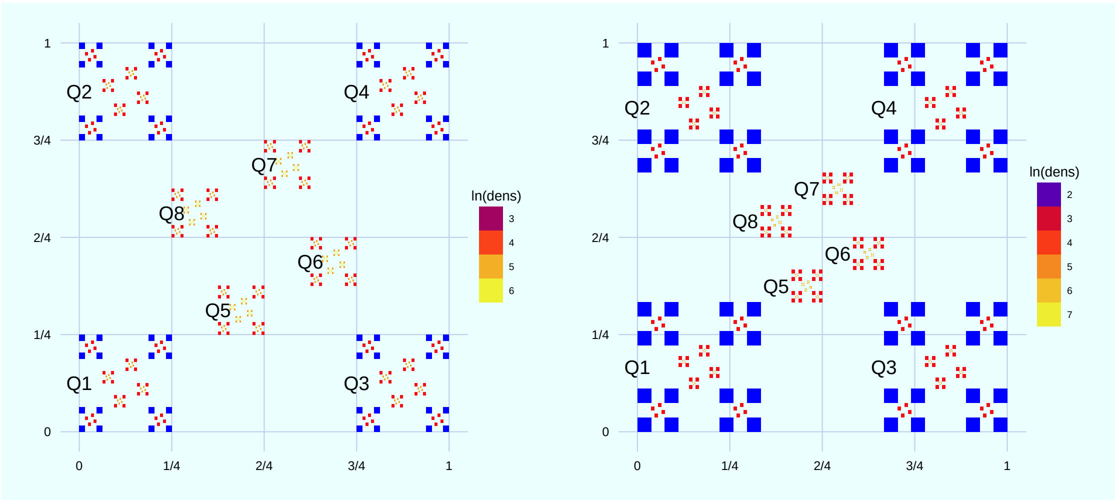


Figure 3: Image plot of the (natural) logarithm of the density of $\mathcal{V}^3_{M_r}(\Pi)$ for $r = 1/4$ (left) and $r = 1/3$ (right), M_r according to Eq.(14).

and, as before, let $w_1^r, \ldots, w_8^r : \mathbb{I}^2 \to \mathbb{I}^2$ denote the corresponding similarities of the IFSP, whereby the contraction factor of $w_1^r, w_2^r, w_3^r, w_4^r$ is r and that of $w_5^r, w_6^r, w_7^r, w_8^r$ is $(1-2r)/4$. Define the remaining quantities $\mu_r^\star, A_r^\star, S_i^r, Q_i^r$, etc. analogous to before. Figure 3 depicts the densities of $V^3_{M_r}(\Pi)$ for the cases $r = 1/4$ and $1/3$, Figure 4 the copula $V^3_{M_r}(\Pi)$ and its density for $r = 1/4$. The IFSP induced by M_r is totally disconnected, so the address map $G_r : \Sigma_8 \to S_r$, defined according to (4), is a homeomorphism for every $r \in (0, 1/2)$. Defining the function $F_r : S_r \to S_r$ (see [2] for the analogous construction in the just touching case) by

$$F_r(x,y) := \sum_{i=1}^{8} (w_i^r)^{-1}(x,y)\, \mathbf{1}_{w_i^r(S_r)}(x,y).$$

Then it follows immediately that the dynamical systems (Σ_8, σ) and (S_r, F_r) are topologically equivalent (see [31]), i.e. the following diagram is commutative:

$$\begin{array}{ccc} \Sigma_8 & \xrightarrow{G_r} & S_r \\ \sigma \downarrow & & \downarrow F_r \\ \Sigma_8 & \xrightarrow[G_r]{} & S_r \end{array}$$

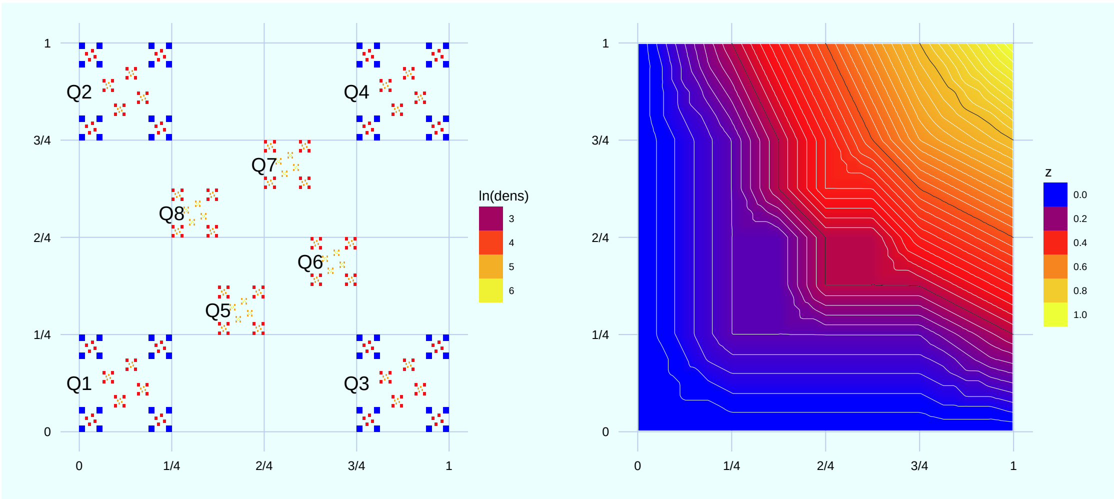


Figure 4: Image plot of the (natural) logarithm of the density of $\mathcal{V}^3_{M_r}(\Pi)$ (left) and image plot of the copula $\mathcal{V}^3_{M_r}(\Pi)$ (right) for $r = 1/4$ (white/gray lines depict contours).

As direct consequence we get that (S_r, F_r) is chaotic in the sense of Barnsley (see [6, pp. 168]), so F_r is topologically transitive and the set of period points in S_r with respect to F_r is dense. Additionally, for every pair $r, r' \in (0, 1/2)$ the dynamical systems (S_r, F_r) and $(S_{r'}, F_{r'})$ are topologically equivalent and

$$H_{rr'} := G_{r'} \circ G_r^{-1}$$

is a homeomorphism between S_r and $S_{r'}$. For every $r \in (0, 1/2)$ define the probability measure P_r on $\mathcal{B}(\Sigma_8)$ according to (5), whereby

$$p_j := \begin{cases} \frac{r}{2} & \text{if } j \in \{1, 2, 3, 4\} \\ \frac{1-2r}{4} & \text{if } j \in \{5, 6, 7, 8\}. \end{cases}$$

Then the dynamical systems (Σ_8, P_r, σ) and (S_r, μ_r, F_r) are isomorphic, i.e. the following diagram is commutative and the homeomorphism G_r is measure-preserving.

$$\begin{array}{ccc} (\Sigma_8, P_r) & \xrightarrow{G_r} & (S_r, \mu_r) \\ \sigma \downarrow & & \downarrow F_r \\ (\Sigma_8, P_r) & \xrightarrow[G_r]{} & (S_r, \mu_r) \end{array}$$

Since the shift operator σ on (Σ_8, P_r) is strongly mixing (see [31]) it follows that F_r is strongly mixing too. Moreover, considering $r, r' \in (0, 1/2)$ with

$r \neq r'$, Birkhoff's Ergodic theorem implies that P_r and $P_{r'}$ are singular with respect to each other, from which in turn it follows immediately that $\mu_r^{H_{rr'}}$ and $\mu_{r'}$ are singular with respect to each other.

4. Hausdorff dimensions of related sets

As mentioned before in this section we will consider some sets related to the function $H_{rr'}$ and calculate their Hausdorff dimensions. As straightforward consequence of the result [5] proved by Banach in 1925 characterizing monotone functions that are absolutely continuous, one has the following property (see [18, 24]): f transforms a set of measure zero onto a set of measure one if and only if f is a non-constant singular function. The results in Section 3 imply that we are in a similar situation here - the function $H_{rr'}$ maps a set of μ_r-measure zero onto a set of $\mu_{r'}$-measure one and, additionally, is $\mathcal{U}_r$-differentiable μ_r-almost everywhere (with derivative equal to zero).

We now return to the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) and calculate the Hausdorff dimension of the set $M_r \subseteq S_r$ fulfilling (13). Doing so we will apply the following Frostman-type lemma (for a proof see [16, pp. 60-61]) and consider (open) squares of the form $Q = w_{k_1} \circ \ldots, \circ w_{k_m}((0,1)^2)$ for m sufficiently big instead of open balls $B_\gamma(x)$ of radius γ around $x \in \mathbb{R}^d$ (the proof can easily be adjusted accordingly).

Lemma 13 ([16]). *Consider $M \in \mathcal{B}(\mathbb{R}^d)$ and a finite Borel measure μ on M. Then the following assertions hold for the Hausdorff dimension $\dim_{\mathrm{H}}(M)$ of M:*

1. *If $\limsup_{\gamma\to 0} \frac{\mu(B_\gamma(x))}{\gamma^s}$ is bounded on M then $\dim_{\mathrm{H}}(M) \leq s$.*
2. *If there exists a constant $a > 0$ such that $\liminf_{\gamma\to 0} \frac{\mu(B_\gamma(x))}{\gamma^s} > a > 0$ on M then $\dim_{\mathrm{H}}(M) \geq s$.*

Theorem 14. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) and fix $r, r' \in (0, 1/2)$ with $r \neq r'$. Then there exists a set $\Lambda_{r,r'} \subseteq S_r$ with $\mu_r(\Lambda_{r,r'}) = 0$, Hausdorff dimension*

$$\dim_{\mathrm{H}}(\Lambda_{r,r'}) = \frac{2r' \ln r' + (1 - 2r') \ln(1 - 2r') - 2r' \ln 2}{2r' \ln r + (1 - 2r') \ln(1 - 2r)}, \tag{15}$$

and $\mu_{r'}(H_{rr'}(\Lambda_{r,r'})) = 1$.

Proof: We consider the set $\Lambda_{r,r'} \subseteq S_r$ of all points (x,y) whose G_r-address fulfills (12). Obviously $\mu_r(\Lambda_{r,r'}) = 0$ and $\mu_{r'}(H_{rr'}(\Lambda_{r,r'})) = 1$ holds, so the theorem is proved if we can show that $\dim_{\mathrm{H}}(\Lambda_{r,r'})$ fulfills (15). Let s denote the right-hand-side of (15) and set $\mu(A) := \mu_{r'}(H_{rr'}(A))$ for every $A \in \mathcal{B}(S_r)$. Then we have to show that for each $(x,y) \in \Lambda_{r,r'}$ with G_r-address $\mathbf{k} \in \Sigma_5$

$$\lim_{n\to\infty} \frac{\mu\big(w_{k_1} \circ \cdots \circ w_{k_n}((0,1)^2)\big)}{\big|w_{k_1} \circ \cdots \circ w_{k_n}((0,1)^2)\big|^s} = 1$$

holds, whereby $|Q|$ denotes the side length of the square Q. Setting $f_n^5(\mathbf{k}) = \mathrm{Card}\{i \le n : k_i = 5\}/n$ for every $n \in \mathbb{N}$ it follows that

$$\frac{\mu\big(w_{k_1} \circ \cdots \circ w_{k_n}((0,1)^2)\big)}{\big|w_{k_1} \circ \cdots \circ w_{k_n}((0,1)^2)\big|^s} = \frac{\Big(\big(\frac{r'}{2}\big)^{1-f_n^5(\mathbf{k})}\,(1-2r')^{f_n^5(\mathbf{k})}\Big)^n}{\Big(r^{1-f_n^5(\mathbf{k})}\,(1-2r)^{f_n^5(\mathbf{k})}\Big)^{ns}}.$$

Using $\lim_{n\to\infty} f_n^5(\mathbf{k}) = (1-2r')$ it is straightforward to verify that the right-hand-side converges to 1 for $n \to \infty$. ■

Slightly modifying the proof of Theorem 14 and starting with the set $\Lambda_{r,r'} \subseteq S_r$ of all points $(x,y) \in S_r$ such that (13) instead of (12) holds, yields the following result:

Corollary 15. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) and fix $r, r' \in (0,1/2)$ with $r \neq r'$. Then there exists a set $\Lambda_{r,r'} \subseteq S_r$ with $\mu_r(\Lambda_{r,r'}) = 1$, Hausdorff dimension*

$$\dim_{\mathrm{H}}(\Lambda_{r,r'}) = \frac{2r\ln r + (1-2r)\ln(1-2r) - 2r\ln 2}{2r\ln r' + (1-2r)\ln(1-2r')}, \tag{16}$$

and $\mu_{r'}(H_{rr'}(\Lambda_{r,r'})) = 0$.

Obviously the strong interrelation between Σ_5 and S_r established by the address map G_r is closely related with the N-adic representation

$$x = \sum_{i=1}^{\infty} \frac{c_i(x)}{N^k}, \quad c_i(x) \in \{0,1,\ldots,N-1\} \;\; \forall i \in \mathbb{N}$$

of points x in the unit interval $\mathbb{I}$. Pursuing the work started by Besicovitch [7], Eggleston [15] proved that the set Γ of points $x \in \mathbb{I}$ satisfying

$$\lim_{n\to\infty} \frac{\mathrm{Card}\{i \le n : c_i(x) = j\}}{n} = d_j,$$

for every $j \in \{0, \cdots, N-1\}$ ($d_j \geq 0$ and $\sum_{j=0}^{N-1} d_j = 1$) has Hausdorff dimension

$$\dim_{\mathrm{H}}(\Gamma) = -\frac{\sum_{i=0}^{N-1} d_i \ln d_i}{\ln N}.$$

Taking this fact into account, we can prove the following Eggleston-Besicovitch-type result for subsets of S_r, that generalizes Theorem 14 and Corollary 15.

Theorem 16. *Consider the family $(T_r)_{r\in(0,1/2)}$ defined according to (7) and fix $r \in (0, 1/2)$ as well as five numbers $d_1, \ldots, d_5 > 0$ fulfilling $\sum_{j=1}^{5} d_j = 1$. Then the set $\Gamma \subseteq S_r$ consisting of all points $(x, y) \in S_r$ whose address $\mathbf{k} \in \Sigma_5$ fulfills*

$$\lim_{n\to\infty} \frac{\mathrm{Card}\{i \leq n : k_i = j\}}{n} = d_j$$

for every $j \in \{1, \ldots, 5\}$ has Hausdorff dimension

$$\dim_{\mathrm{H}}(\Gamma) = \frac{\sum_{i=1}^{5} d_i \ln d_i}{(d_1 + d_2 + d_3 + d_4) \ln r + d_5 \ln(1 - 2r)}.$$

Proof: The result can be proved in the same manner as Theorem 14 by defining the only self-similar measure μ satisfying $\mu(w_j(S_r)) = d_j$ for every $j \in \{1, \ldots, 5\}$ (also see [19]). ■

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