On the interrelation between Dempster-Shafer Belief Structures and their Belief Cumulative Distribution Functions

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Abstract

We consider Dempster-Shafer belief structures (DSBSs) $\mathfrak{m}$ having finitely many non-empty compact intervals as focal elements and prove several results describing the interrelation between DSBSs and their so-called Belief Cumulative Distribution Functions (BCDFs) $x \mapsto \mathbf{F}(x) = (\underline{F}(x), \overline{F}(x))$ induced by the corresponding belief and plausibility measures. In particular, we answer a question by Ronald R. Yager on the injectivity of the assignment $\mathfrak{m} \mapsto \mathbf{F}$.

Keywords: Dempster-Shafer belief structure, belief cumulative distribution function, random set

1. Introduction

Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X : \Omega \to \mathbb{R}$ is an unobservable random variable. Instead of $X(\omega)$, however, it is possible to observe a compact interval $\mathbf{X}(\omega) = [\underline{X}(\omega), \overline{X}(\omega)]$ containing the true value $X(\omega)$ for every $\omega \in \Omega$ (think, for instance, of a measurement device rounding

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to a certain digit or general interval-censored data, see Turnbull, 1976; Wang, 2008). Assuming that $\mathbf{X}$ is a random compact interval (i.e. that both $\underline{X}$ and $\overline{X}$ are random variables) according to Dempster (1967) $\mathbf{X}$ induces the so-called *lower and upper probability* $\underline{\pi}$ and $\overline{\pi}$ (also referred to as *belief and plausibility measure*) respectively via

$$\begin{aligned} \underline{\pi}(B) &= \mathcal{P}\big(\{\omega \in \Omega : \mathbf{X}(\omega) \subseteq B\}\big) \\ \overline{\pi}(B) &= \mathcal{P}\big(\{\omega \in \Omega : \mathbf{X}(\omega) \cap B \neq \emptyset\}\big) \end{aligned} \tag{1}$$

for every Borel set $B \in \mathcal{B}(\mathbb{R})$. Obviously $\underline{\pi}(B)$ and $\overline{\pi}(B)$ can be used as lower and upper bound for $\mathcal{P}(\{\omega \in \Omega : X(\omega) \in B\})$. For important properties of $\underline{\pi}$ and $\overline{\pi}$ as set functions (also in the general setting of random closed sets) see, for instance, Matheron (1975) and Molchanov (2005). In case the range of $\mathbf{X}$ only consists of (pairwise different) intervals $[a_1, b_1], \ldots, [a_n, b_n]$ the random interval $\mathbf{X}$ is fully characterized by the quantities $m_i = \mathcal{P}(\{\omega \in \Omega : \mathbf{X}(\omega) = [a_i, b_i]\})$ for $i \in \{1, \ldots, n\}$. Defining $\mathfrak{m} : 2^{\mathbb{R}} \to [0,1]$ by $\mathfrak{m}(A) = 0$ for $A \notin \{[a_1, b_1], \ldots, [a_n, b_n]\}$ and $\mathfrak{m}(A) = m_i$ for $A = [a_i, b_i]$ induces a Dempster-Shafer belief structure (DSBS, see Yager, 2004) which we will denote by $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$. Yager (2004) studied the interrelation between DSBSs and their so-called Belief Cumulative Distribution Function (BCDF) $\mathbf{F}_{\mathfrak{m}}$, defined by (notation as before)

$$\mathbf{F}_{\mathfrak{m}}(x) = \Big[\underline{\pi}\big((-\infty, x]\big), \overline{\pi}\big((-\infty, x]\big)\Big]$$

for every $x \in \mathbb{R}$, and asked the question under which (necessary and sufficient) conditions two DSBSs induce the same Belief Cumulative Distribution Function. In the current note we give an answer to this question and prove several related results. In fact we present two main theorems: Firstly, for each pair of non-decreasing right-continuous step functions F_1, F_2 with $F_1 \leq F_2$ we can find a (not necessarily unique) DSBS $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ such that $\mathbf{F}_{\mathfrak{m}} = [F_1, F_2]$. And secondly we show that $\mathfrak{m}$ is unique if and only if its focal elements $([a_i, b_i])_{i=1}^n$ fulfill a simple intersection-condition.

As main contribution the provides a concise description of the interrelation between DSBSs and BCDFs. Such a description is not only interesting from the theoretical mathematical point of view since it generalizes the one-to-one relationship between discrete probability measures and discrete distribution functions, which is, for instance, implicitly utilized in statistics whenever the empirical distribution function F_n of a sample $x_1, \ldots, x_n$ instead of the sample itself is considered. In fact it is also essential from the applied perspective since whenever working with DSBSs and BCDFs it has to be understood completely under which conditions, for instance, considering the BCDF instead of the DSBS does not imply a loss of (relevant) information.

2. Notation and preliminaries

For every set E the cardinality of E will be denoted by $\#E$. In the sequel $\mathcal{K}(\mathbb{R})$ will denote the family of all non-empty compact (i.e. closed and bounded) intervals in $\mathbb{R}$, $\mathcal{K}([0,1])$ the family of all non-empty compact subintervals of $[0,1]$. $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-field in $\mathbb{R}$, δ_x the Dirac measure in x, $\mathcal{P}$ the family of all probability measures on $\mathcal{B}(\mathbb{R})$ and $\mathcal{P}_d$ the family of all elements $\mu \in \mathcal{P}$ for which there exists a finite set $\Lambda \subseteq \mathbb{R}$ with $\mu(\Lambda) = 1$. $\mathcal{F}$ will denote the family of all distribution functions on $\mathbb{R}$. It is well known that there is a one-to-one correspondence between $\mathcal{F}$ and $\mathcal{P}$: In fact, given $F \in \mathcal{F}$ and defining $\mu_F : \mathcal{B}(\mathbb{R}) \to [0,1]$ by setting

$$\mu_F((-\infty, x]) = F(x)$$

for every $x \in \mathbb{R}$ and extending μ_F in the standard way (see for instance Elstrodt, 1999; Rudin, 1987) to full $\mathcal{B}(\mathbb{R})$ yields $\mu_F \in \mathcal{P}$. Moreover, given $\mu \in \mathcal{P}$ and defining $F_\mu(x) = \mu((-\infty, x])$ for every $x \in \mathbb{R}$, it is easily verified that $F_\mu \in \mathcal{F}$. Altogether the maps $\Phi : \mathcal{P} \to \mathcal{F}$ and $\Psi : \mathcal{F} \to \mathcal{P}$, given by

$$\Phi(\mu) = F_\mu \quad \text{and} \quad \Psi(F) = \mu_F, \tag{2}$$

fulfill $\Phi(\mu_F) = F$ for every $F \in \mathcal{F}$ as well as $\Psi(F_\mu) = F$ for every $\mu \in \mathcal{P}$ (again see Elstrodt, 1999; Rudin, 1987 for details), implying bijectivity of Φ, Ψ and $\Psi = \Phi^{-1}$. $\mathcal{F}_d$ will denote the family all elements in $\mathcal{F}$ corresponding to $\mu \in \mathcal{P}_d$, i.e. all functions of the form $F(x) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[x_i,\infty)}(x)$ with $n \in \mathbb{N}$, $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}$, $\{\alpha_1, \ldots, \alpha_n\} \subseteq (0,1]^d$ and $\sum_{i=1}^{n} \alpha_i = 1$. Obviously Φ maps $\mathcal{P}_d$ in a one-to-one manner to $\mathcal{F}_d$ and Ψ maps $\mathcal{F}_d$ in a one-to-one-manner to $\mathcal{P}_d$. Due to monotonicity of every $F \in \mathcal{F}$ the left- and right limit of F at $x \in \mathbb{R}$ exist and will be denoted by $F(x-)$ and $F(x+)$ respectively. Following Tavalera et al. (2013) and Yager (2004) we will only consider Dempster-Shafer belief structures (DSBS, for short) on $\mathbb{R}$ with pairwise different non-empty compact intervals $B_1 = [a_1, b_1], B_2 = [a_2, b_2], \cdots, B_n = [a_n, b_n]$ as focal elements, i.e. we consider mappings $\mathfrak{m} : 2^{\mathbb{R}} \to [0,1]$ fulfilling

i) $\mathfrak{m}(A) = 0$ if $A \neq B_i$ for all $i \in \{1, \ldots, n\}$

ii) $\sum_{i=1}^{n} \mathfrak{m}(B_i) = 1$

iii) $B_i \in \mathcal{K}(\mathbb{R})$ for every $i \in \{1, \ldots, n\}$ and $B_i \neq B_j$ for $i \neq j$.

Note that, contrary to half open intervals, considering non-empty compact intervals does not exclude the possibility of having focal elements containing only a single point. Without loss of generality we will also assume $\mathfrak{m}(B_i) > 0$ for every $i \in \{1, \ldots, n\}$. Each such DSBS will be denoted in the form $(\mathbb{R}, \mathfrak{m}, (B_i)_{i=1}^n)$, $\mathcal{D}$ will denote the family of all these DBSB. For every DSBS $(\mathbb{R}, \mathfrak{m}, (B_i)_{i=1}^n)$, the sets $L_\mathfrak{m}, R_\mathfrak{m}$ are defined by

$$L_\mathfrak{m} = \{a_1, \ldots, a_n\} \quad \text{and} \quad R_\mathfrak{m} = \{b_1, \ldots, b_n\}. \tag{3}$$

Obviously the cardinalities $\#L_\mathfrak{m}, \#R_\mathfrak{m}$ of $L_\mathfrak{m}, R_\mathfrak{m}$ fulfill $\#L_\mathfrak{m}, \#R_\mathfrak{m} \leq n$. Every DSBS $(\mathbb{R}, \mathfrak{m}, (B_i)_{i=1}^n)$ induces a *belief measure* $Bel_\mathfrak{m} : 2^\mathbb{R} \to [0, 1]$ and a *plausibility measure* $Pl_\mathfrak{m} : 2^\mathbb{R} \to [0, 1]$ by setting

$$Bel_\mathfrak{m}(A) = \sum_{i \colon B_i \subseteq A \neq \emptyset} \mathfrak{m}(B_i), \qquad Pl_\mathfrak{m}(A) = \sum_{i \colon B_i \cap A \neq \emptyset} \mathfrak{m}(B_i) \tag{4}$$

for every $A \subseteq \mathbb{R}$. Note that, using the interpretation of DSBS with compact intervals as focal elements given in the Introduction, the pair $(Bel_\mathfrak{m}, Pl_\mathfrak{m})$ coincides with the lower and upper probability $(\underline{\pi}, \overline{\pi})$ induced by the discrete random compact interval $\mathbf{X}$. Define functions $\underline{F}_\mathfrak{m}, \overline{F}_\mathfrak{m} : \mathbb{R} \to [0, 1]$ by $\underline{F}_\mathfrak{m}(x) = Bel((-\infty, x])$ and $\overline{F}_\mathfrak{m}(x) = Pl((-\infty, x])$ for every $x \in \mathbb{R}$. Then the function $\mathbf{F}_\mathfrak{m} : \mathbb{R} \to \mathcal{K}([0, 1])$, defined by $\mathbf{F}_\mathfrak{m}(x) = [\underline{F}_\mathfrak{m}(x), \overline{F}_\mathfrak{m}(x)]$ will be called *Belief Cumulative Distribution Function* (BCDF for short).

Figure 1 and Figure 2 depict two DSBSs and their corresponding BCDFs (colors according to masses of the focal elements). As stressed in Yager (2004), such graphs provide a very useful framework for representing information about an uncertain variable.

By using the obvious fact that $[a_i, b_i] \cap (-\infty, x] \neq \emptyset$ if and only if $a_i \leq x$, as well as $[a_i, b_i] \subseteq (-\infty, x]$ if and only if $b_i \leq x$, it follows immediately that

$$\underline{F}_\mathfrak{m}(x) = \sum_{i \colon b_i \leq x} \mathfrak{m}([a_i, b_i]) \quad \text{and} \quad \overline{F}_\mathfrak{m}(x) = \sum_{i \colon a_i \leq x} \mathfrak{m}([a_i, b_i]) \tag{5}$$

for every $x \in \mathbb{R}$. Obviously $\underline{F}_\mathfrak{m}$ and $\overline{F}_\mathfrak{m}$ are right-continuous non-decreasing step-functions. Hence, taking into account that there are only finitely many focal elements, $\underline{F}_\mathfrak{m}, \overline{F}_\mathfrak{m} \in \mathcal{F}_d$ follows. Letting $\underline{\mu}_\mathfrak{m} = \Psi(\underline{F}_\mathfrak{m}), \overline{\mu}_\mathfrak{m} = \Psi(\overline{F}_\mathfrak{m}) \in \mathcal{P}_d$

denote the corresponding elements in $\mathcal{P}_d$ we have $\underline{\mu}_{\mathfrak{m}}(L_{\mathfrak{m}}) = 1 = \overline{\mu}_{\mathfrak{m}}(R_{\mathfrak{m}})$ as well as $\underline{\mu}_{\mathfrak{m}}((-\infty, x]) = \underline{F}_{\mathfrak{m}}(x)$ and $\overline{\mu}_{\mathfrak{m}}((-\infty, x]) = \overline{F}_{\mathfrak{m}}(x)$ for every $x \in \mathbb{R}$. In the sequel we will simply write $L, R, Bel, Pl, \underline{F}, \overline{F}, \mathbf{F}, \underline{\mu}, \overline{\mu}$ (i.e. dropping $\mathfrak{m}$ in the notation) if it is clear which DSBS we are referring to.

Remark 1. Note that we would have gotten the same BCDF $\mathbf{F}_{\mathfrak{m}}$ if, as in Yager (2004), we would have considered non-empty half open intervals of the form $[a_i, b_i)$ as focal elements. We chose to work with compact intervals since this makes it possible to have degenerated focal elements containing only single points and therefore view DSBSs as generalization of discrete probability distributions assigning full mass to a finite set.

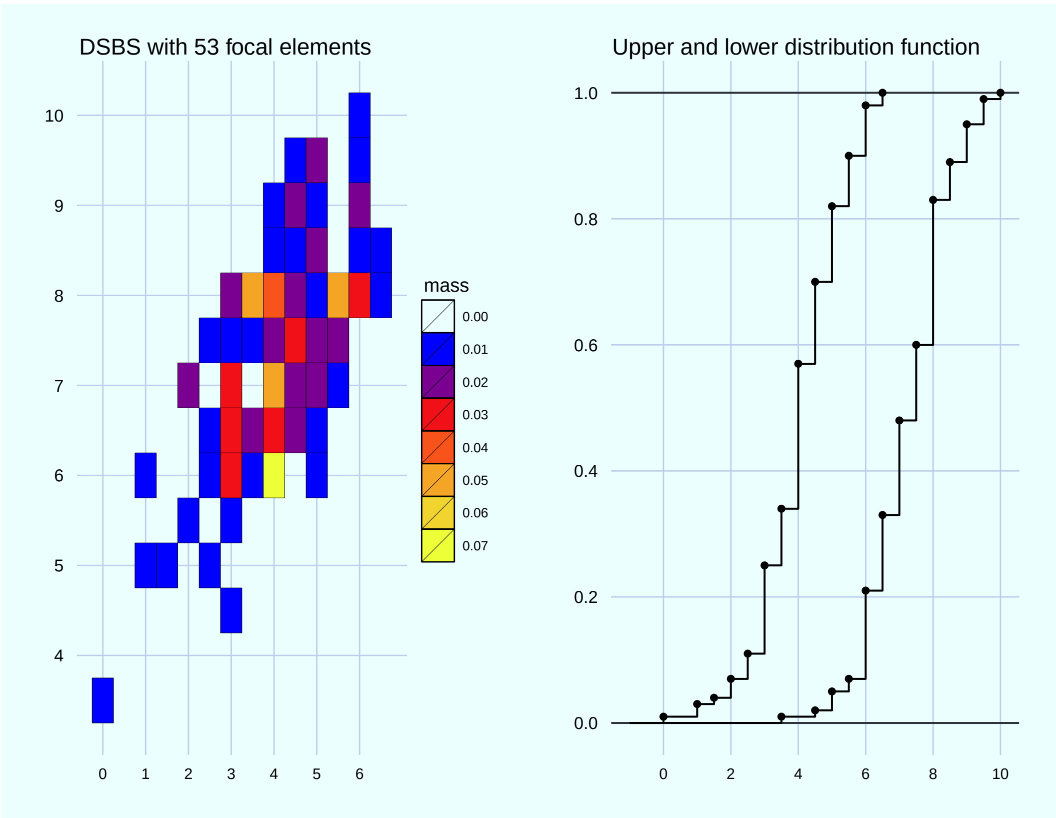

Figure 1: DSBS with 53 intervals $[a_i, b_i]$ as focal elements (left) and the corresponding BCDF (right); all a_i, b_i are elements of $\mathbb{N}/2 = \{0, 1/2, 1, 3/2, 2, \ldots\}$

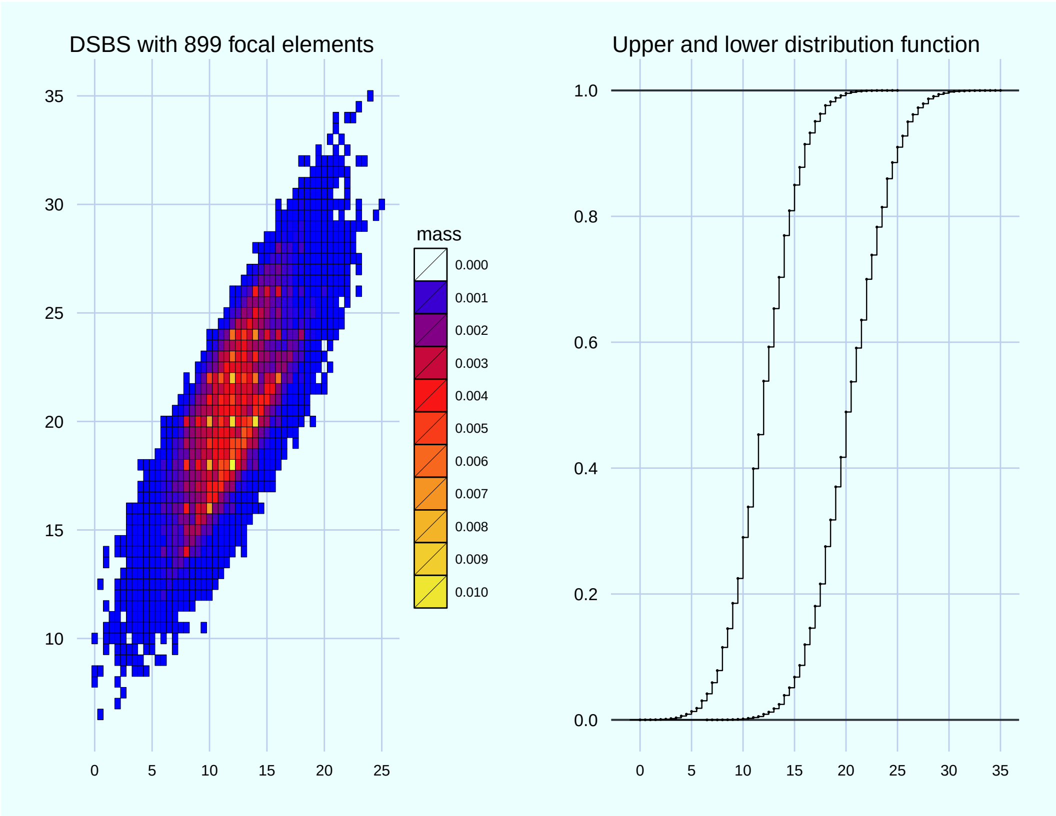

Figure 2: DSBS with 899 intervals $[a_i, b_i]$ as focal elements (left) and the corresponding BCDF (right); all a_i, b_i are elements of $\mathbb{N}/2 = \{0, 1/2, 1, 3/2, 2, \ldots\}$

3. Results

In Yager (2004, page 2083) Ronald R. Yager wrote 'One interesting question, which at least to the author does not have an obvious answer, is under what general conditions do two belief structures generate the same B-CD.' Note that this question actually concerns injectivity of the natural generalization of the mapping $\Phi : \mathcal{P} \to \mathcal{F}$ assigning every probability distribution the corresponding distribution function to the DSBS setting. In fact, if $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ is a DSBS having exclusively single points as focal elements then it may be regarded as probability distribution and the induced BCDF is crisp, i.e. $\mathbf{F}_\mathfrak{m}(x)$ only contains one point for each $x \in \mathbb{R}$. We start this section with an answer to Yager's question and then prove several related results.

Theorem 2. *Consider DSBSs* $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ *and* $(\mathbb{R}, \hat{\mathfrak{m}}, ([\hat{a}_i, \hat{b}_i]_{i=1}^{\hat{n}})$*. Then we have* $\boldsymbol{F}_{\mathfrak{m}} = \boldsymbol{F}_{\hat{\mathfrak{m}}}$ *if and only if the following two conditions hold:*

1. $L_{\mathfrak{m}} = L_{\hat{\mathfrak{m}}} = L$, $R_{\mathfrak{m}} = R_{\hat{\mathfrak{m}}} = R$
2. *For every* $l \in L$ *and every* $r \in R$ *we have*

$$\sum_{i \in \{1,\dots,n\}\colon a_i = l} \mathfrak{m}([a_i, b_i]) = \sum_{j \in \{1,\dots,\hat{n}\}\colon \hat{a}_j = l} \hat{\mathfrak{m}}([\hat{a}_j, \hat{b}_j])) \tag{6}$$

$$\sum_{i \in \{1,\dots,n\}\colon b_i = r} \mathfrak{m}([a_i, b_i]) = \sum_{j \in \{1,\dots,\hat{n}\}\colon \hat{b}_j = r} \hat{\mathfrak{m}}([\hat{a}_j, \hat{b}_j]) \tag{7}$$

Proof: First of all note that the left and right hand-side of equation (6) coincide with the point mass (jump) of $\overline{F}_{\mathfrak{m}}$ and $\overline{F}_{\hat{\mathfrak{m}}}$ at $l \in L$ respectively. Analogously, the left and right hand-side of equation (7) coincide with the point mass (jump) of $\underline{F}_{\mathfrak{m}}$ and $\underline{F}_{\hat{\mathfrak{m}}}$ at $r \in R$. **(i)** If we have $\mathbf{F}_{\mathfrak{m}} = \mathbf{F}_{\hat{\mathfrak{m}}}$ then $\overline{F}_{\mathfrak{m}} = \overline{F}_{\hat{\mathfrak{m}}}$ as well $\underline{F}_{\mathfrak{m}} = \underline{F}_{\hat{\mathfrak{m}}}$ follows. Since $\underline{F}_{\mathfrak{m}}, \underline{F}_{\hat{\mathfrak{m}}}, \overline{F}_{\mathfrak{m}}, \overline{F}_{\hat{\mathfrak{m}}} \in \mathcal{F}_d$, using the one-to-one correspondence between $\mathcal{F}_d$ and $\mathcal{P}_d$ both $\underline{\mu}_{\mathfrak{m}} = \underline{\mu}_{\hat{\mathfrak{m}}}$ and $\overline{\mu}_{\mathfrak{m}} = \overline{\mu}_{\hat{\mathfrak{m}}}$ follow. Since two elements in $\mathcal{P}_d$ coincide if and only if their corresponding point masses coincide (see Elstrodt, 1999; Rudin, 1987) this implies that $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ and $(\mathbb{R}, \hat{\mathfrak{m}}, ([\hat{a}_i, \hat{b}_i]_{i=1}^{\hat{n}})$ fulfill both assertions of Theorem 2, completing the proof of one direction. **(ii)** On the other hand, if both assertions hold, then $\underline{\mu}_{\mathfrak{m}} = \underline{\mu}_{\hat{\mathfrak{m}}}$ and $\overline{\mu}_{\mathfrak{m}} = \overline{\mu}_{\hat{\mathfrak{m}}}$ follows since the corresponding point masses coincide. Having this, again using the one-to-one correspondence between $\mathcal{F}_d$ and $\mathcal{P}_d$, shows both $\overline{F}_{\mathfrak{m}} = \overline{F}_{\hat{\mathfrak{m}}}$ as well $\underline{F}_{\mathfrak{m}} = \underline{F}_{\hat{\mathfrak{m}}}$, implying $\mathbf{F}_{\mathfrak{m}} = \mathbf{F}_{\hat{\mathfrak{m}}}$ ■

Corollary 3. *If* $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ *and* $(\mathbb{R}, \hat{\mathfrak{m}}, ([\hat{a}_i, \hat{b}_i]_{i=1}^{\hat{n}})$ *are DSBSs fulfilling* $\boldsymbol{F}_{\mathfrak{m}} = \boldsymbol{F}_{\hat{\mathfrak{m}}}$*, then* $\bigcup_{i=1}^n [a_i, b_i] = \bigcup_{i=1}^{\hat{n}} [\hat{a}_i, \hat{b}_i]$ *holds.*

Proof: Suppose that $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ is an arbitrary DSBS and suppose that x is not a boundary point of any focal element, i.e. $x \notin (L_{\mathfrak{m}} \cup R_{\mathfrak{m}})$. If we have $\underline{F}_{\mathfrak{m}}(x) = \overline{F}_{\mathfrak{m}}(x)$ then

$$\sum_{i\colon b_i \leq x} \mathfrak{m}([a_i, b_i]) = \underline{F}_{\mathfrak{m}}(x) = \overline{F}_{\mathfrak{m}}(x) = \sum_{i\colon a_i \leq x} \mathfrak{m}([a_i, b_i]),$$

implying that x can not be an inner point of any focal element. Considering $x \notin (L_{\mathfrak{m}} \cup R_{\mathfrak{m}})$ this shows $x \notin \bigcup_{i=1}^n [a_i, b_i]$. On the other hand, if $x \notin \bigcup_{i=1}^n [a_i, b_i]$, then, using once more $x \notin (L_{\mathfrak{m}} \cup R_{\mathfrak{m}})$, x can not be an inner point of any focal element, so $\underline{F}_{\mathfrak{m}}(x) = \overline{F}_{\mathfrak{m}}(x)$ follows. Altogether this proves that for $x \notin (L_{\mathfrak{m}} \cup R_{\mathfrak{m}})$ we have $\underline{F}_{\mathfrak{m}}(x) = \overline{F}_{\mathfrak{m}}(x)$ if and only if $x \notin \bigcup_{i=1}^{n}[a_i, b_i]$. Suppose now that $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^{n})$ and $(\mathbb{R}, \hat{\mathfrak{m}}, ([\hat{a}_i, \hat{b}_i]_{i=1}^{\hat{n}})$ are DSBSs with $\mathbf{F}_{\mathfrak{m}} = \mathbf{F}_{\hat{\mathfrak{m}}}$. Then Theorem 2 implies $L_{\mathfrak{m}} \cup R_{\mathfrak{m}} = L_{\hat{\mathfrak{m}}} \cup R_{\hat{\mathfrak{m}}}$. If $x \notin \bigcup_{i=1}^{n}[a_i, b_i]$ and $x \notin L_{\mathfrak{m}} \cup R_{\mathfrak{m}}$ then $x \notin L_{\hat{\mathfrak{m}}} \cup R_{\hat{\mathfrak{m}}}$ and $\underline{F}_{\mathfrak{m}}(x) = \overline{F}_{\mathfrak{m}}(x)$, so $\underline{F}_{\hat{\mathfrak{m}}}(x) = \overline{F}_{\hat{\mathfrak{m}}}(x)$ and $x \notin \bigcup_{i=1}^{\hat{n}}[\hat{a}_i, \hat{b}_i]$ follows. Starting from $x \notin \bigcup_{i=1}^{\hat{n}}[\hat{a}_i, \hat{b}_i]$ with $x \notin L_{\hat{\mathfrak{m}}} \cup R_{\hat{\mathfrak{m}}}$ and applying the same argument yields $x \notin \bigcup_{i=1}^{n}[a_i, b_i]$. Hence we have shown $\bigcup_{i=1}^{n}[a_i, b_i] = \bigcup_{i=1}^{\hat{n}}[\hat{a}_i, \hat{b}_i]$, which completes the proof. ■

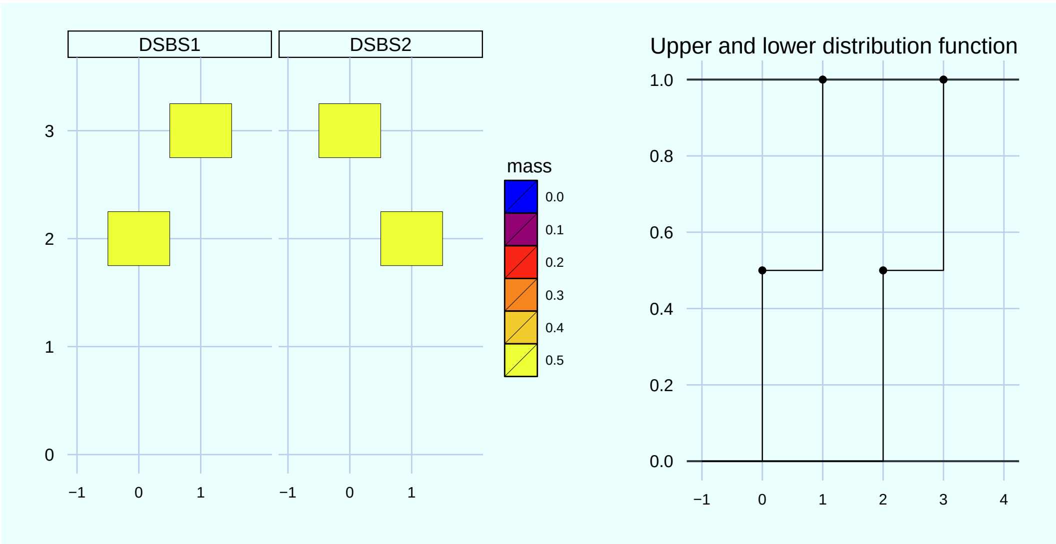

Figure 3: The two DSBS from Example 4 and their BCDF

Example 4. Consider the DSBSs $(\mathbb{R}, \mathfrak{m}, ([0,2], [1,3]))$ and $(\mathbb{R}, \hat{\mathfrak{m}}, ([0,3], [1,2]))$, defined by

$$\mathfrak{m}([0,2]) = \mathfrak{m}([1,3]) = \frac{1}{2} = \hat{\mathfrak{m}}([0,3]) = \hat{\mathfrak{m}}([1,2]).$$

Then we have

$$\overline{F}_{\mathfrak{m}}(x) = \overline{F}_{\hat{\mathfrak{m}}}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{1}{2} & \text{if } x \in [0,1) \\ 1 & \text{if } x \geq 1, \end{cases} \qquad \underline{F}_{\mathfrak{m}}(x) = \underline{F}_{\hat{\mathfrak{m}}}(x) = \begin{cases} 0 & \text{if } x < 2, \\ \frac{1}{2} & \text{if } x \in [2,3) \\ 1 & \text{if } x \geq 3, \end{cases}$$

so, in particular, $\mathbf{F}_{\mathfrak{m}} = \mathbf{F}_{\hat{\mathfrak{m}}}$. Obviously $\mathfrak{m}$ and $\hat{\mathfrak{m}}$ fulfill the two conditions of Theorem 2.

There are two main questions that naturally arise from Theorem 2: The first one being if each pair (F_1, F_2) with $F_1, F_2 \in \mathcal{F}_d$ and $F_1 \leq F_2$ is the BCDF of a DSBS. And the second one asking for which BCDF $\mathbf{F}$ there is only one DSBS $\mathfrak{m}$ with $\mathbf{F}_{\mathfrak{m}} = \mathbf{F}$. We start with a positive answer to the first question - note that this generalizes surjectivity of the mapping $\Phi : \mathcal{P} \to \mathcal{F}$ from equation (2).

Theorem 5. *Suppose that $F_1, F_2 \in \mathcal{F}_d$ fulfill $F_1(x) \leq F_2(x)$ for all $x \in \mathbb{R}$. Then there exists a DSBS $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ such that $\underline{F}_{\mathfrak{m}} = F_1$ and $\overline{F}_{\mathfrak{m}} = F_2$.*

Proof: Although the subsequent proof is inevitably a bit technical it is based on one single idea: We consider the DSBS whose focal elements are all intervals of the form $[a_i, b_j]$ with a_i being a discontinuity point of F_2 and b_j a discontinuity point of F_1, and then prove the existence of numbers $m_{i,j}$ such that $\mathfrak{m}([a_i, b_j]) = m_{i,j}$ is a DSBS with BCDF $\mathbf{F}_{\mathfrak{m}}(x) = [F_1(x), F_2(x)]$ by induction. Let $\Lambda_{F_2} = \{a_1, \ldots, a_{n_a}\}$ and $\Lambda_{F_1} = \{b_1, \ldots, b_{n_b}\}$ denote the discontinuity points of F_2 and F_1 respectively. Without loss of generality we assume that $a_1 < a_2 < \cdots < a_{n_a}$ as well as $b_1 < b_2 \cdots < b_{n_b}$. Setting

$$\mathcal{I} = \Big\{(i, j) \in \{1, \ldots, n_a\} \times \{1, \ldots, n_b\} : a_i \leq b_j\Big\}$$

yields $\mathcal{I} \neq \emptyset$ since, by assumption, $F_1 \leq F_2$. It suffices to prove the existence of $(m_{i,j})_{(i,j)\in\mathcal{I}}$ with $\sum_{(i,j)\in\mathcal{I}} m_{i,j} = 1$ and $m_{i,j} > 0$ for all $(i, j) \in \mathcal{I}$ fulfilling

$$\begin{aligned} 0 < \beta_{j_0} = F_1(b_{j_0}) - F_1(b_{j_0}-) &= \sum_{i:\,(i,j_0)\in\mathcal{I}} m_{i,j_0} \quad \text{for every } j_0 \in \Lambda_{F_1} \quad \text{and} \\ 0 < \alpha_{i_0} = F_2(a_{i_0}) - F_2(a_{i_0}-) &= \sum_{j:\,(i_0,j)\in\mathcal{I}} m_{i_0,j} \quad \text{for every } i_0 \in \Lambda_{F_2}. \end{aligned}$$

We proceed by induction on the cardinality of Λ_{F_2}:
(i) If $n_a = 1$ then $\mathcal{I} = \{(1, 1), (1, 2), \ldots (1, n_b)\}$ and setting $m_{1,j} = \beta_j$ for every $j \in \{1, \ldots, n_b\}$ produces the desired result.
(ii) Suppose now that the statement holds for F_2 having $n_a - 1 \geq 2$ discontinuity points. $F_2 \in \mathcal{F}_d$ with discontinuity points $\Lambda_{F_2} = \{a_1, \ldots, a_{n_a}\}$ corresponds to the measure $\mu_2 = \sum_{i=1}^{n_a} \alpha_i \delta_{a_i} \in \mathcal{P}_d$. Consider $\mu_2' = \sum_{i=1}^{n_a-1} \frac{\alpha_i}{1-\alpha_{n_a}} \delta_{a_i} \in \mathcal{P}_d$ and let $F_2' \in \mathcal{F}_d$ denote the corresponding distribution function. Since, by assumption $-\infty = b_0 < b_1 < b_2 \cdots < b_{n_b}$, there exists a unique s with $a_{n_a} \in (b_s, b_{s+1}]$. Define

$$\beta_j' = \begin{cases} \frac{\beta_j}{1-\alpha_{n_a}} & \text{for } j \leq s, \\ \frac{\beta_j}{1-\alpha_{n_a}} \frac{1-\alpha_{n_a}-\beta_1-\beta_2-\ldots-\beta_s}{1-\beta_1-\beta_2-\ldots-\beta_s} & \text{for } j > s, \end{cases}$$

then $\sum_{j=1}^{n_b} \beta'_j = 1$ and $\mu'_1 = \sum_{j=1}^{n_b} \beta'_j \delta_{b_j} \in \mathcal{P}_d$ follows. Let $F'_1 \in \mathcal{F}_d$ denote the distribution function corresponding to μ'_1. By the induction hypothesis we can find numbers $(m'_{i,j})_{(i,j)\in\mathcal{I}'}$ with $\mathcal{I}' = \mathcal{I} \setminus \{(n_a, s+1), \ldots, (n_a, n_b)\}$, $\sum_{(i,j)\in\mathcal{I}'} m'_{i,j} = 1$ and $m'_{i,j} > 0$ for all $(i,j) \in \mathcal{I}'$ fulfilling

$$\begin{aligned} 0 < \beta'_{j_0} = F'_1(b_{j_0}) - F'_1(b_{j_0}-) &= \sum_{i\colon (i,j_0)\in\mathcal{I}'} m'_{i,j_0} \quad \text{for every } j_0 \in \Lambda_{F_1} \quad \text{and} \\ 0 < \alpha'_{i_0} = F'_2(a_{i_0}) - F'_2(a_{i_0}-) &= \sum_{j\colon (i_0,j)\in\mathcal{I}'} m_{i_0,j} \quad \text{for every } i_0 \in \Lambda_{F_2} \setminus \{n_a\}. \end{aligned}$$

For every pair $(i,j) \in \mathcal{I}$ define $m_{i,j} \in (0,1]$ by

$$m_{i,j} = \begin{cases} (1-\alpha_{n_a}) m'_{i,j} & \text{for } (i,j) \in \mathcal{I}', \\ \frac{\alpha_{n_a} \beta_j}{1-\beta_1-\beta_2-\ldots-\beta_s} & \text{for } i = a_{n_a}, j > s, \end{cases}$$

then $\sum_{(i,j)\in\mathcal{I}} m_{i,j} = 1$ follows immediately. Furthermore, for $j_0 \le s$ we have

$$\begin{aligned} F_1(b_{j_0}) - F_1(b_{j_0}-) &= \beta_{j_0} = (1-\alpha_{n_a})\beta'_{j_0} = (1-\alpha_{n_a}) \sum_{i:(i,j_0)\in\mathcal{I}'} m'_{i,j_0} \\ &= \sum_{i\colon (i,j_0)\in\mathcal{I}} m_{i,j_0} \end{aligned}$$

and for $j_0 > s$ it follows that

$$\begin{aligned} \sum_{i\colon (i,j_0)\in\mathcal{I}} m_{i,j_0} &= \sum_{i\colon (i,j_0)\in\mathcal{I}'} (1-\alpha_{n_a}) m'_{i,j_0} + m_{n_a,j_0} = (1-\alpha_{n_a})\beta'_{j_0} + m_{n_a,j_0} \\ &= (1-\alpha_{n_a}) \frac{\beta_{j_0}}{1-\alpha_{n_a}} \frac{1-\alpha_{n_a}-\beta_1-\beta_2-\ldots-\beta_s}{1-\beta_1-\beta_2-\ldots-\beta_s} + m_{n_a,j_0} \\ &= \beta_{j_0} = F_1(b_{j_0}) - F_1(b_{j_0}-). \end{aligned}$$

The fact that $F_2(a_{i_0}) - F_2(a_{i_0}) = \sum_{j:(i_0,j)\in\mathcal{I}} m_{i_0,j}$ for every $i_0 \in \{1,\ldots,n_a\}$ can be verified analogously. Setting $\mathfrak{m}([a_i,b_j]) = m_{i,j}$ for each pair $(i,j) \in \mathcal{I}$ obviously yields $\underline{F}_{\mathfrak{m}} = F_1$ as well as $\overline{F}_{\mathfrak{m}} = F_2$, which completes the proof. ■

Remark 6. Theorem 5 does not hold for the case of DSBSs with half-open non-empty intervals of the form $[a_i, b_i)$ as focal elements. In fact, considering the case $F_1(x) = F_2(x) = \mathbf{1}_{[a,\infty)}(x)$ for all $x \in \mathbb{R}$ and some fixed $a \in \mathbb{R}$ yields a simple counterexample.

We now tackle the second afore-mentioned question. Given an arbitrary DSBS $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ and its corresponding BCDF $\mathbf{F}_\mathfrak{m}$ let $\mathcal{M}_\mathfrak{m}$ denote the family of all DSBSs having $\mathbf{F}_\mathfrak{m}$ as BCDF. In the sequel we will say that $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ fulfills the *intersection-condition* if and only if for each pair $[a_i, b_i], [a_j, b_j]$ of focal elements with $i \neq j$ we have either (i) $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ or (ii) $a_i = a_j$ or $b_i = b_j$. In other words, a DSBS $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ fulfills the intersection condition if each pair of focal elements either has empty intersection or a common left or right boundary point. The two DSBSs considered in Example 4 do not fulfill the intersection condition - in fact, for $\mathfrak{m}$ we have $[0,2] \cap [1,3] = [2,3] \neq \emptyset$ and for $\hat{\mathfrak{m}}$ we have $[0,3] \cap [1,2] = [1,2] \neq \emptyset$ although neither the corresponding right- nor the left boundary points coincide. Moreover, neither the DSBS depicted in Figure 1 nor the DSBS depicted in Figure 2 fulfills the intersection conditions. In fact, in the first case, for instance, the focal elements $[1,5]$ and $[3,4.5]$ have non-empty intersection and in the second case the same is true for the focal elements $[5,15]$ and $[10,12]$.

Example 7. Consider the DSBSs $(\mathbb{R}, \mathfrak{m}, ([0,2],[1,2]))$, $(\mathbb{R}, \hat{\mathfrak{m}}, ([0,2],[0,1]))$ and $(\mathbb{R}, \mathfrak{n}, ([0,3],[0,1],[2,3]))$ defined by

$$\begin{aligned} \mathfrak{m}([0,2]) &= \mathfrak{m}([1,2]) = \frac{1}{2} = \hat{\mathfrak{m}}([0,2]) = \hat{\mathfrak{m}}([0,1]), \\ \mathfrak{n}([0,3]) &= \mathfrak{n}([0,1]) = \mathfrak{n}([2,3]) = \frac{1}{3}. \end{aligned}$$

Then all three DSBSs fulfill the intersection condition.

Remark 8. Every DSBS $(\mathbb{R}, \mathfrak{m}, ([a_i, b_i])_{i=1}^n)$ having pairwise disjoint focal elements certainly fulfills the intersection condition. Working, for instance, with discretizations of real scales pairwise disjoint focal elements seem not at all unnatural.

The subsequent theorem will show that the intersection-condition decides if $\mathcal{M}_\mathfrak{m} = \{\mathfrak{m}\}$ or not - note that the different cases considered in the 'Sufficieny Step' of the proof generalize the situations sketched in Example 7.

Theorem 9. *Suppose that* $(\mathbb{R}, \mathfrak{m}, ([l_i, r_i])_{i=1}^n)$ *is a DSBS and let* $\boldsymbol{F}_\mathfrak{m}$ *denote the corresponding BCDF. Then* $\mathcal{M}_\mathfrak{m} = \{\mathfrak{m}\}$ *if and only if* $(\mathbb{R}, \mathfrak{m}, ([l_i, r_i])_{i=1}^n)$ *fulfills the intersection-condition.*

Proof: Step I - Sufficiency: We first show that the intersection-condition implies $\mathcal{M}_{\mathfrak{m}} = \{\mathfrak{m}\}$ and start with some notations that will facilitate the proof. The main idea of this step is to decompose the union $\bigcup_{v=1}^{n} [l_v, r_v]$ of all focal elements in sets Z_1, Z_2, Z_3 fulfilling some disjointness properties that allow for a proof (considering four different cases that are handled easily) of the fact that the intersection condition is sufficient for $\mathcal{M}_{\mathfrak{m}} = \{\mathfrak{m}\}$. Suppose that $(\mathbb{R}, \mathfrak{m}, ([l_i, r_i])_{i=1}^{n})$ fulfills the intersection-condition, set $L_{\mathfrak{m}} = \{l_1, \ldots, l_n\}, R_{\mathfrak{m}} = \{r_1, \ldots, r_n\}$ as well as $n_a = \#L_{\mathfrak{m}}, n_b = \#R_{\mathfrak{m}}$. Consider $a_1 < a_2 < \cdots < a_{n_a}$ and $b_1 < b_2 < \cdots < b_{n_b}$ with $L_{\mathfrak{m}} = \{a_1, \ldots, a_{n_a}\}, R_{\mathfrak{m}} = \{b_1, \ldots, b_{n_b}\}$ and set

$$\mathcal{I} = \Big\{(i,j) \in \{1, \ldots, n_a\} \times \{1, \ldots, n_b\} : \exists v \in \{1, \ldots, n\} \text{ with } [a_i, b_j] = [l_v, r_v]\Big\}.$$

Define functions $\underline{j}, \overline{j}, \{1, \ldots, n_a\} \rightarrow \{1, \ldots, n_b\}$ and $\underline{i}, \overline{i} : \{1, \ldots, n_b\} \rightarrow \{1, \ldots, n_a\}$ as follows:

$$\begin{aligned}
\underline{j}(i) &= \min\{v \in \{1, \ldots, n_b\} : (i, v) \in \mathcal{I}\},\\
\overline{j}(i) &= \max\{v \in \{1, \ldots, n_b\} : (i, v) \in \mathcal{I}\},\\
\underline{i}(j) &= \min\{v \in \{1, \ldots, n_a\} : (v, j) \in \mathcal{I}\},\\
\overline{i}(j) &= \max\{v \in \{1, \ldots, n_a\} : (v, j) \in \mathcal{I}\}
\end{aligned}$$

Furthermore let Ω denote the set of all $i_0 \in \{1, \ldots, n_a\}$ for which $\underline{j}(i_0) < \overline{j}(i_0)$ and Λ the set of all $j_0 \in \{1, \ldots, n_b\}$ for which $\underline{i}(j_0) < \overline{i}(j_0)$. Then it follows immediately that

$$\bigcup_{v=1}^{n} [l_v, r_v] = \bigcup_{(i,j) \in \mathcal{I}} [a_i, b_j] = \underbrace{\bigcup_{i \in \Omega} [a_i, b_{\overline{j}(i)}]}_{=Z_1} \cup \underbrace{\bigcup_{j \in \Lambda} [a_{\underline{i}(j)}, b_j]}_{=Z_2} \cup \underbrace{\bigcup_{(i,j) \in \Omega^c \times \Lambda^c} [a_i, b_j]}_{=Z_3} \tag{8}$$

holds. We will now show the afore-mentioned *disjointness properties* of the sets Z_1, Z_2, Z_3 that will prove useful:
If $Z_1 \cap Z_3 \neq \emptyset$ then we could find $i_0 \in \Omega$, $(l, s) \in \Omega^c \times \Lambda^c$ with $[a_{i_0}, b_{\overline{j}(i_0)}] \cap [a_l, b_s] \neq \emptyset$, from which either $a_{i_0} = a_l$ (hence $l \in \Omega$) or $a_{i_0} \neq a_l$ and $b_{\overline{j}(i_0)} = b_s$ (hence $s \in \Lambda$) would follow. Therefore $Z_1 \cap Z_3 = \emptyset$.
$Z_2 \cap Z_3 = \emptyset$ follows analogously.
As next step we show that, given $i_0 \in \Omega$ and $j_0 \in \Lambda$, we have either $[a_{i_0}, b_{\overline{j}(i_0)}] \cap [a_{\underline{i}(j_0)}, b_{j_0}] = \emptyset$ or $[a_{i_0}, b_{\overline{j}(i_0)}] = [a_{\underline{i}(j_0)}, b_{j_0}]$. In fact, if the intersection is non-empty then there are two possibilities: (i) If $a_{i_0} = a_{\underline{i}(j_0)}$ then

$b_{j_0} \leq b_{\overline{j}(i_0)}$ follows. Since $j_0 \in \Lambda$ we can find $a_s > a_{\underline{i}(j_0)}$ such that $[a_s, b_{j_0}]$ is focal, implying $b_{j_0} = b_{\overline{j}(i_0)}$. (ii) If $b_{j_0} = b_{\overline{j}(i_0)}$ then we proceed analogously to conclude equality of the two intervals. Furthermore, if $i_1, i_2 \in \Omega$ with $i_1 \neq i_2$, then $[a_{i_1}, b_{\overline{j}(i_1)}] \cap [a_{i_2}, b_{\overline{j}(i_2)}] \neq \emptyset$ and $a_{i_1} < a_{i_2}$ would imply $b_{\overline{j}(i_1)} = b_{\overline{j}(i_2)}$ as well as $\overline{j}(i_1) = \overline{j}(i_2) = j_0 \in \Lambda$, so, according to the previous step, $[a_{i_1}, b_{\overline{j}(i_1)}] = [a_{\underline{i}(j_0)}, b_{j_0}] = [a_{i_2}, b_{\overline{j}(2_1)}]$ would follow. This shows that $[a_{i_1}, b_{\overline{j}(i_1)}]$ and $[a_{i_2}, b_{\overline{j}(i_2)}]$ are disjoint. The fact that $[a_{\underline{i}(j_1)}, b_{j_1}]$ and $[a_{\underline{i}(j_2)}, b_{j_2}]$ are disjoint for $j_1 \neq j_2$ and $j_1, j_2 \in \Lambda$ can be verified analogously.
Finally, $[a_{i_1}, b_{j_1}] \cap [a_{i_2}, b_{j_2}] \neq \emptyset$ for $(i_1, j_2), (i_2, j_2) \in \Omega^c \times \Lambda^c$ and $(i_1, j_2) \neq (i_2, j_2)$ would imply (i) $a_{i_1} = a_{i_2}$ (hence $b_{j_1} = b_{j_2}$ since $i_1 \in \Omega^c$) or $b_{j_1} = b_{j_2}$ (hence $a_{i_1} = b_{i_2}$ since $j_1 \in \Lambda^c$). Therefore $[a_{i_1}, b_{j_1}]$ and $[a_{i_2}, b_{j_2}]$ must be disjoint.
The just-mentioned disjointness properties will now be used to show that $\mathfrak{m}$ is the only DSBS with BCDF $\mathbf{F}_{\mathfrak{m}}$: Suppose that $(\mathbb{R}, \hat{\mathfrak{m}}, ([\hat{l}_i, \hat{r}_i])_{i=1}^{\hat{n}})$ is a DSBS with BCDF $\mathbf{F}_{\mathfrak{m}}$. According to Theorem 2 and Corollary 3 we have $\{\hat{l}_1, \ldots \hat{l}_{\hat{n}}\} = \{a_1, \ldots, a_{n_a}\}$, $\{\hat{r}_1, \ldots \hat{r}_{\hat{n}}\} = \{b_1, \ldots, b_{n_b}\}$ as well as $\bigcup_{i=1}^{\hat{n}} [\hat{l}_i, \hat{r}_i] = \bigcup_{(i,j) \in \mathcal{I}} [a_i, b_j]$. Hence, for every focal element $[\hat{l}_v, \hat{r}_v]$ we can find indices $(k, w) \in \{1, \ldots, n_a\} \times \{1, \ldots, n_b\}$ with $[\hat{l}_v, \hat{r}_v] = [a_k, b_w]$. In order to show that $[a_k, b_w]$ is a focal element of $\mathfrak{m}$, the disjointness properties imply that we only have to consider the following four cases:
(i) Suppose that $[a_k, b_w] \subseteq [a_{i_0}, b_{\overline{j}(i_0)}] \subseteq Z_1$ with $i_0 \in \Omega$ and $\overline{j}(i_0) \in \Lambda^c$. If $a_k > a_{i_0}$ then $[a_k, b_{\overline{j}(k)}] \cap [a_{i_0}, b_{\overline{j}(i_0)}] \neq \emptyset$ would imply $b_{\overline{j}(i_0)} = b_{\overline{j}(k)}$ and therefore $\overline{j}(i_0) \in \Lambda$, which contradicts the assumption. Hence $a_k = a_{i_0}$ follows. If $b_w = b_{\overline{j}(i_0)}$ then obviously $[a_k, b_w]$ is focal for $\mathfrak{m}$. If $b_w < b_{\overline{j}(i_0)}$ then, considering $[a_{\underline{i}(w)}, b_w] \cap [a_{i_0}, b_{\overline{j}(i_0)}] \neq \emptyset$, $a_{\underline{i}(w)} = a_{i_0} = a_k$ follows, implying that $[a_k, b_w] = [a_{\underline{i}(w)}, b_w]$ is focal for $\mathfrak{m}$.
(ii) The case $[a_k, b_w] \subseteq [a_{\underline{i}(j_0)}, b_{j_0}] \subseteq Z_2$ with $j_0 \in \Lambda$ and $\underline{i}(j_0) \in \Omega^c$ can be handled analogously.
(iii) If $[a_k, b_w] \subseteq [a_{i_0}, b_{j_0}] \subseteq Z_3$ for some focal element $[a_{i_0}, b_{j_0}]$ of $\mathfrak{m}$ with $(i_0, j_0) \in \Omega^c \times \Lambda^c$ then, considering $[a_k, b_{\overline{j}(k)}] \cap [a_{i_0}, b_{j_0}] \neq \emptyset$, $[a_{\underline{i}(w)}, b_w] \cap [a_{i_0}, b_{j_0}] \neq \emptyset$ together with the afore-mentioned disjointness properties of Z_1, Z_2, Z_3, directly implies $a_k = a_{i_0}$ and $b_w = b_{j_0}$. Hence $[a_k, b_w]$ is also focal for $\mathfrak{m}$ in this case.
(iv) Finally, suppose that $[a_k, b_w] \subseteq [a_{i_0}, b_{j_0}] \subseteq Z_1 \cap Z_2$ with $(i_0, j_0) \in \Omega \times \Lambda$. Let $\widetilde{j} < j_0 = \overline{j}(i_0)$ denote the maximal $v < j_0$ such that $(i_0, v) \in \mathcal{I}$ and $\widetilde{i} > i_0 = \underline{i}(j_0)$ denote the minimal $v > i_0$ such that $(v, j_0) \in \mathcal{I}$. Then the intersection-condition implies $[a_{i_0}, b_{\widetilde{j}}] \cap [a_{\widetilde{i}}, b_{j_0}] = \emptyset$, so $b_{\widetilde{j}} < a_{\widetilde{i}}$. If $a_k > a_{i_0}$

then, considering $[a_k, b_{\overline{j}(k)}] \cap [a_{i_0}, b_{j_0}] \neq \emptyset$ implies $b_{\overline{j}(k)} = b_{j_0}$, hence $a_k \geq a_{\widetilde{i}}$. The fact that $b_w \leq b_{\widetilde{j}}$ holds in case of $b_w < b_{j_0}$ follows analogously. Therefore, for $a_k > a_{i_0}$ we get $b_w = b_{j_0} = b_{\overline{j}(k)}$, and for $b_w < b_{j_0}$ we get $a_k = a_{i_0} = a_{\underline{i}(b)}$, implying that $[a_k, b_w]$ is focal for $\mathfrak{m}$.
Since $[a_k, b_w]$ was an arbitrary focal element of $\hat{\mathfrak{m}}$ the afore-mentioned arguments prove that every focal element of $\hat{\mathfrak{m}}$ is also a focal element of $\mathfrak{m}$. As direct consequence $\hat{\mathfrak{m}}$ also fulfills the intersection-condition and, repeating the previous steps for $\hat{\mathfrak{m}}$ replaced by $\mathfrak{m}$ and vice versa, we finally conclude that the focal elements of $\mathfrak{m}$ and $\hat{\mathfrak{m}}$ coincide. Since the remaining step of showing $\mathfrak{m}([a_i, b_j]) = \hat{\mathfrak{m}}([a_i, b_j])$ for all $(i, j) \in \mathcal{I}$ now is a straightforward consequence of the considered four cases (i)-(iv) in combination with (5) this completes the proof of Step I.
Step II - Necessity: Suppose that the intersection-condition does not hold, i.e. we can find focal elements $[l_s, r_s], [l_j, r_j]$ of $\mathfrak{m}$ with $l_s \neq l_j$ and $r_s \neq r_j$ but $[l_s, r_s] \cap [l_j, r_j] \neq \emptyset$. Without loss of generality we may assume $l_s < l_j$ (implying $r_s \geq l_j$). Choose $\alpha < \min\{\mathfrak{m}([l_s, r_s]), \mathfrak{m}([l_j, r_j])\}$. Setting

$$[l_{n+1}, r_{n+1}] = [l_s, r_j] \quad \text{and} \quad [l_{n+2}, r_{n+2}] = [l_j, r_s]$$

and considering the DSBS $(\mathbb{R}, \hat{\mathfrak{m}}, ([l_i, r_i])_{i=1}^{n+2})$ with $\hat{\mathfrak{m}}([l_i, r_i]) = \mathfrak{m}([l_i, r_i])$ for $i \in \{1, \ldots, n\} \setminus \{s, j\}$ and

$$\begin{aligned} \hat{\mathfrak{m}}([l_s, r_s]) &= \mathfrak{m}([l_s, r_s]) - \alpha, \ \hat{\mathfrak{m}}([l_j, r_j]) = \mathfrak{m}([l_j, r_j]) - \alpha \\ \hat{\mathfrak{m}}[l_{n+1}, r_{n+1}] &= \hat{\mathfrak{m}}[l_{n+2}, r_{n+2}] = \alpha \end{aligned}$$

obviously yields $\mathbf{F}_{\hat{\mathfrak{m}}} = \mathbf{F}_{\mathfrak{m}}$ and $\mathcal{M}_{\mathfrak{m}} \neq \{\mathfrak{m}\}$. ■

We conclude the paper with the following corollary that directly follows from the decomposition (8) and the disjointness properties of the sets Z_1, Z_2, Z_3. For a given DSBS $(\mathbb{R}, \mathfrak{m}, ([l_i, r_i])_{i=1}^n)$ let the points $a_1 < a_2 < \cdots < a_{n_a}$ and $b_1 < b_2 < \cdots < b_{n_b}$ be defined as in the sufficiency-step in the proof of Theorem 9.

Corollary 10. *Suppose that $(\mathbb{R}, \mathfrak{m}, ([l_i, r_i])_{i=1}^n)$ is a DSBS. Then we have $\mathcal{M}_{\mathfrak{m}} = \{\mathfrak{m}\}$ if and only if we can find points $x_1 < x_2 < \cdots < x_h < x_{h+1}$ with $x_1 < a_1$, $x_{h+1} \geq b_{n_b}$ and $\underline{F}_{\mathfrak{m}}(x_l) = \overline{F}_{\mathfrak{m}}(x_l)$ for every $l \in \{1, \ldots, h+1\}$, such that for every $l \in \{1, \ldots, h\}$ one of the following two conditions holds:*

(A) $\#\big(\{a_1, \ldots, a_{n_a}\} \cap [x_l, x_{l+1}]\big) = 1$ *or* $\#\big(\{b_1, \ldots, b_{n_b}\} \cap [x_l, x_{l+1}]\big) = 1$

(B) $\#\big(\{a_1,\dots,a_{n_a}\}\cap[x_l,x_{l+1}]\big)>1$, $\#\big(\{b_1,\dots,b_{n_b}\}\cap[x_l,x_{l+1}]\big)>1$, *and, letting* $\tilde{i}$ *denote the minimal* $i\in\{1,\dots,n_a\}$ *with* $a_i\in[x_l,x_{l+1}]$ *as well as* $\tilde{j}$ *the maximal* $j\in\{1,\dots,n_b\}$ *with* $b_j\in[x_l,x_{l+1}]$*, we have* $a_{\tilde{i}+1}>b_{\tilde{j}-1}$.

Remark 11. Apart from being a mathematical result, Theorem 9 is of vital importance in representing information via BCDFs. In fact, while the BCDF is generally less informative than the DSBS, the two ways of providing information coincide under some quite weak assumptions. The conditions in Theorem 9 are some basic conditions that we can argue a "typical" agent will supply (see Figure 6 in Yager, 2004). In other words, the agent only needs to provide a BCDF satisfying the (simple) intersection condition, and the system may provide a unique DSBS. In our opinion, this opens the door for a more extensive use of Yager's BCDF representation in practice.

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