Some members of the class of (quasi-)copulas with given diagonal from the Markov kernel perspective

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Abstract

Calculating Markov kernels of copulas allows not only for a precise description of the way Bertino- and diagonal copulas distribute mass, but also enables a simply proof of the fact that, for certain diagonals, both may degenerate to proper generalized shuffles of the minimum copula. After extending the kernel approach to the case of the maximum quasi-copula $A_\delta$ with given diagonal $\delta$, a conjecture on singularity of $A_\delta$ by Nelsen et al. (2008) is established and an alternative simple and short proof of the result by Úbeda-Flores (2008) characterizing diagonals for which $A_\delta$ is a copula is given.

Keywords: Copula, Quasi-copula, Markov Kernel, Doubly Stochastic Measure, Shuffle

1 Introduction

The importance of copulas for fields like probability theory and statistics is underlined by Sklar’s famous theorem (see Sklar, 1959), saying that the joint distribution function $H$ of a pair $(X,Y)$ of real-valued random variables and the (marginal) distribution functions $F$ and $G$ of $X$ and $Y$ respectively are linked by a copula $C$ via $H(x,y) = C(F(x),G(y))$ for all $x,y \in \mathbb{R}$. If $F$ and $G$ are continuous, then the copula is unique; otherwise, the copula is only uniquely determined on $\text{Range}(F) \times \text{Range}(G)$ (see, for instance, de Amo et al. 2012). As pointed out by Jaworski (2009) there are various reasons for the interest in copulas with given diagonal $\delta$ - the facts that (i) tail dependence of a copula $A$ only uses the copula’s diagonal and (ii) that $(X,Y) \sim A$ implies $\max\{X,Y\} \sim \delta_A$ being two of the most important ones. It is well known that the class $\mathcal{C}_\delta$ of all copulas with given diagonal is non-empty for
every diagonal $\delta$, that the Bertino copula $B_\delta$ is the lower bound in $C_\delta$, and that the diagonal copula $E_\delta$ is the upper bound of the class of all symmetric elements in $C_\delta$ (see Fredricks and Nelsen, 2002, Nelsen and Fredricks, 1997, Nelsen et al., 2008). Durante and Jaworski (2008) and Jaworski (2009) have given necessary and sufficient conditions for a diagonal $\delta$ to be the diagonal of an absolutely continuous copula. Durante et al. (2007) constructed asymmetric elements in $C_\delta$ via patchworks, de Amo et al. (2013) analyzed the generalization of $C_\delta$ to so-called sub- and super-diagonals.

In the current paper we will first take a closer look to Bertino- and diagonal copulas from the perspective of regular conditional distributions. We will calculate the Markov kernels both of Bertino- and of diagonal copulas. It will be shown that the kernel approach, firstly, provides a concise description of the way $B_\delta$ ($E_\delta$) concentrates all mass on the union of the graphs of three (two) measurable functions implying singularity both of $B_\delta$ and of $E_\delta$ (loosely speaking we will also say the copulas 'live' on the graph of functions). And secondly, that it also serves as a handy tool for proving the existence of diagonals $\delta$ for which $B_\delta$ and $E_\delta$ degenerate to completely dependent copulas concentrating all mass on the graph of a Lebesgue-measure-preserving bijection $S: [0, 1] \to [0, 1]$ fulfilling $S \circ S = id_{[0, 1]}$, which, however, is not monotonic on any interval. After that we focus on the maximum quasi-copula $A_\delta$ with diagonal $\delta$ introduced and studied by ´Ubeda-Flores (2008) and Nelsen et al. (2010), such a signed measure does not exist for every quasi-copula. We will construct a signed Markov kernel $K: [0, 1] \times B(\{0, 1\}) \to [-1, 2]$ (see Section 2 for the definition of kernels) fulfilling

$$A_\delta(x, y) = \int_{[0, x]} K(t, [0, y])d\lambda(t)$$

for all $x, y \in [0, 1]$, and, based on that, prove the existence of a doubly stochastic signed measure $\mu$ on the Borel $\sigma$-field $B([0, 1]^2)$ of $[0, 1]^2$ fulfilling

$$A_\delta(x_2, y_2) - A_\delta(x_1, y_2) - A_\delta(x_2, y_1) + A_\delta(x_1, y_1) = \mu([x_1, x_2] \times [y_1, y_2])$$

for all intervals $[x_1, x_2], [y_1, y_2] \subseteq [0, 1]$. Note that, according to Fernández Sánchez et al. (2010) and Nelsen et al. (2010), such a signed measure does not exist for every quasi-copula. Equation (1), together with the simple form of the signed Markov kernel $K(\cdot, \cdot)$, will also be useful for proving the fact that the (positive) measures $\mu^+, \mu^-$ in the Hahn decomposition $\mu = \mu^+ - \mu^-$ of $\mu$ live on the graph of at most five functions, confirming singularity of $\mu$ as conjectured by Nelsen et al. (2008). Finally, usefulness of the kernel approach will be underlined once more by giving an alternative short and simple proof of the main result in ´Ubeda-Flores (2008) on the characterization of all diagonals for which the quasi-copula $A_\delta$ actually is a copula.

The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations. In Section 3 we construct a diagonal $\delta_0$ for which the function $\delta_0: t \mapsto t - \delta_0(t)$ is not monotonic on any interval. Section 4 contains the calculation of Markov kernels of diagonal copulas $E_\delta$ and shows that $E_{\delta_0}$ is a proper generalized shuffle of $M$ living on the graph of a Lebesgue-measure-preserving bijection $S: [0, 1] \to [0, 1]$ which is not monotonic on any interval. Section 5 contains the analogous results for Bertino copulas. Finally, the afore-mentioned quasi-copula $A_\delta$ is studied in Section 6.
2 Notation and preliminaries

Throughout the rest of the paper \( \mathcal{B}([0,1]) \) and \( \mathcal{B}([0,1]^2) \) will denote the Borel \( \sigma \)-fields in \([0,1]\) and \([0,1]^2\) respectively, \( \lambda \) denotes the Lebesgue measure on \([0,1]\), and \( \epsilon_x \) the Dirac measure at \( x \). \( \mathcal{T} \) will denote the class of all \( \lambda \)-preserving transformations \( T : [0,1] \to [0,1] \). \( \mathcal{T}_p \) the subset of all bijective \( T \in \mathcal{T} \) fulfilling \( T^{-1} \in \mathcal{T} \). \( \mathcal{C} \) will denote the family of all copulas, i.e. the restrictions to \([0,1]^2\) of distribution functions with uniform \( U_{0,1} \)-marginals. \( Q \) will denote the family of all quasi-copulas, i.e. the family of all functions \( Q : [0,1]^2 \to [0,1] \) fulfilling (i) \( Q(x,0) = Q(0,x) = 0 \) and \( Q(x,1) = Q(1,x) = x \) for all \( x \in [0,1] \), (ii) \( Q \) is non-decreasing in each variable, and (iii) \( |Q(x_1,y_2) - Q(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2| \) for all \( x_1, x_2, y_1, y_2 \in [0,1] \).

\( M \) will denote the minimum copula, \( W \) the copula defined by \( W(x,y) = \max(x + y - 1,0) \). For properties of copulas and quasi-copulas we refer to Genest et al (1999), Nelsen (2006), Durante and Sempi (2010), and Sempi (2011).

We will call a real-valued set function \( \mu \) on a measurable space \((\Omega, \mathcal{A})\) \textit{signed measure} if we can find finite positive measures \( \mu^+, \mu^- \) on \((\Omega, \mathcal{A})\) such that \( \mu(F) = \mu^+(F) - \mu^-(F) \) holds for all \( F \in \mathcal{A} \) (this is slightly more restrictive than the standard definition given, for instance, in Rudin, 1987, allowing at most one of \( \mu^+, \mu^- \) to be infinite, but sufficient for the rest of the paper). It is well known that for every signed measure \( \mu \) we can choose \( \mu^+, \mu^- \) to be mutually singular w.r.t. each other - in this case we will refer to \( \mu^+, \mu^- \) as \textit{Hahn decomposition} of \( \mu \) (again see Rudin, 1987). In the sequel ‘measure’ always means non-negative measure - in case \( \mu \) also assumes negative values we will explicitly mention the word ‘signed’. A signed measure \( \mu \) on \( \mathcal{B}([0,1]^2) \) will be called \textit{singular} if and only if both measures \( \mu^+, \mu^- \) in the Hahn decomposition are singular w.r.t. the Lebesgue measure \( \lambda_2 \). A (signed) measure \( \mu \) on \( \mathcal{B}([0,1]^2) \) will be called \textit{doubly stochastic} if we have \( \mu(E \times [0,1]) = \mu([0,1] \times E) = \lambda(E) \) for every \( E \in \mathcal{B}([0,1]) \). For every copula \( A \in \mathcal{C} \), setting \( \mu_A([0,x] \times [0,y]) := A(x,y) \) for all \( x,y \in [0,1] \) and extending \( \mu_A \) to full \( \mathcal{B}([0,1]^2) \) in the usual way yields a doubly stochastic measure \( \mu_A \). \( \mathcal{P}_\mathcal{C} \) will denote the family of all doubly stochastic measures on \( \mathcal{B}([0,1]^2) \).

Throughout this paper a \textit{signed kernel} is a mapping \( K : [0,1] \times \mathcal{B}([0,1]) \to \mathbb{R} \) such that \( x \mapsto K(x,B) \) is measurable for every fixed \( B \in \mathcal{B}([0,1]) \) and \( B \mapsto K(x,B) \) is a signed measure for every fixed \( x \in [0,1] \). \( K(\cdot,\cdot) \) will simply be called \textit{kernel} if it is a signed kernel assuming only non-negative values. A (signed) kernel \( K(\cdot,\cdot) \) will be called \textit{(signed) Markov kernel} if we have \( K(x,[0,1]) = 1 \) for every \( x \in [0,1] \). Suppose that \( X,Y \) are \([0,1] \)-valued random variables on a probability space \((\Omega, \mathcal{A}, \mathcal{P})\), then a Markov kernel \( K : [0,1] \times \mathcal{B}([0,1]) \to [0,1] \) is called (a version of the) \textit{regular conditional distribution of} \( Y \) \textit{given} \( X \) if for every \( B \in \mathcal{B}([0,1]) \)

\[
K(X(\omega), B) = \mathbb{E}(1_B \circ Y | X)(\omega)
\]

holds \( \mathcal{P} \)-a.e. It is well known that for each pair \((X,Y)\) of \([0,1] \)-valued random variables a regular conditional distribution \( K(\cdot,\cdot) \) of \( Y \) given \( X \) exists, that \( K(\cdot,\cdot) \) is unique \( \mathcal{P}^X \)-almost everywhere (i.e. unique for \( \mathcal{P}^X \)-almost all \( x \in [0,1] \)) and that \( K(\cdot,\cdot) \) only depends on \( \mathcal{P}^{X \otimes Y} \).

**Definition 2.1** Suppose that \( A \in \mathcal{C} \) and that the vector \((X,Y)\) has joint distribution function \( A \). Then we will denote (a version of) the regular conditional distribution of \( Y \) given \( X \) by \( K_A(\cdot,\cdot) \) and refer to \( K_A(\cdot,\cdot) \) simply as regular conditional distribution of \( A \) or as Markov kernel of \( A \).
Note that for every $A \in \mathcal{C}$, its conditional regular distribution $K_A(\cdot, \cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have $(G_x := \{y \in [0,1] : (x,y) \in G\}$ denoting the $x$-section of $G$ for every $x \in [0,1]$)

$$\int_{[0,1]} K_A(x, G_x) \, d\lambda(x) = \mu_A(G),$$

so in particular

$$\int_{[0,1]} K_A(x, F) \, d\lambda(x) = \lambda(F)$$

for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (4) obviously induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (3). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see Kallenberg (1997) and Klenke (2007).

A copula $A \in \mathcal{C}$ will be called completely dependent if and only if there exists $T \in \mathcal{T}$ such that $K(x, E) := 1_E(Tx)$ is a regular conditional distribution of $A$ (see Lancaster, 1963, and Trutschnig, 2011, for equivalent definitions and main properties). For every $T \in \mathcal{T}$ the induced completely dependent copula will be denoted by $C_T$ throughout the rest of the paper. It is straightforward to verify (see Trutschnig, 2013) that the star product $C_T \ast C_S$ of $C_T$ and $C_S$ coincides with $C_{S \circ T}$. A copula $A$ is called shuffle of the minimum copula $M$ (see Durante et al., 2009, Nelsen, 2006) if $A = C_T$ for some interval-exchange transformation $T \in \mathcal{T}$. We will call $A \in \mathcal{C}$ generalized shuffle of $M$ if $A = C_T$ for some $T \in \mathcal{T}_p$ (for a further generalization see Trutschnig and Fernández Sánchez, 2013).

**Definition 2.2** A function $\delta : [0,1] \to [0,1]$ will be called diagonal if and only if the following four conditions are fulfilled (i) $\delta(0) = 0$, $\delta(1) = 1$, (ii) $\delta$ is monotonically non-decreasing, (iii) $\delta$ is Lipschitz continuous with Lipschitz constant $L = 2$, and (iv) $\delta(t) \leq t$ for all $t \in [0,1]$. $\mathcal{D}$ will denote the family of all diagonals.

$\mathcal{C}_\delta$ ($\mathcal{Q}_\delta$) will be the family of all copulas (quasi-copulas) $A$ fulfilling $A(x, x) = \delta(x)$ for all $x \in [0,1]$ and a given $\delta \in \mathcal{D}$. It is well known (see Durante et al., 2005, Durante and Jaworski, 2008, Nelsen, 2006, Úbeda-Flores, 2009) that $\mathcal{C}_\delta$ (and hence $\mathcal{Q}_\delta$) is non-empty for every $\delta \in \mathcal{D}$. For every diagonal $\delta$ the function $\hat{\delta} : [0,1] \to [0,1/2]$ is defined by $\hat{\delta}(t) = t - \delta(t)$. It is straightforward to verify that $\hat{\delta}(0) = \hat{\delta}(1) = 0$, that $\hat{\delta}$ is Lipschitz continuous with Lipschitz constant $L = 1$, and that $0 \leq \hat{\delta}(t) \leq \min\{t, 1-t\}$ for all $t \in [0,1]$. Figure 1 depicts two diagonals $\delta_1$ and $\delta_2$ which will serve as ongoing example in Sections 4 - 6. Due to Lipschitz continuity both $\delta$ and $\hat{\delta}$ are differentiable $\lambda$-almost everywhere and we can find Borel measurable functions $w_\delta : [0,1] \to [0,2]$ and $\tilde{w}_\delta : [0,1] \to [-1,1]$ fulfilling $\delta'(x) = w_\delta(x)$ and $\hat{\delta}'(x) = \tilde{w}_\delta(x)$ respectively for $\lambda$-almost every $x \in [0,1]$ (see Rudin, 1987). In the sequel we will refer to $w_\delta$ and $\tilde{w}_\delta$ as (a versions of) the derivative of $\delta$ and $\hat{\delta}$ respectively. Finally, when working with *Dini derivatives* in Section 5 - 6 we will write $D^+ f(x), D^+_+ f(x), D^- f(x), D^-_- f(x)$ for the upper right-, the lower right-, the upper left- and the lower left Dini derivative of a real-valued function $f$ at $x$ respectively, see Hewitt and Stromberg (1965).
Markov kernels for copulas with given diagonal

Figure 1: Piecewise linear diagonal $\delta_1$ with $\delta_1' \in \{0, 2\}$ $\lambda$-almost everywhere; $\delta_2$ defined by $\delta_2(t) = \frac{1}{2}(\delta_1(t) + t^2)$; the blue lines depict the corresponding functions $\hat{\delta}_1, \hat{\delta}_2$.

3 A diagonal $\delta$ for which $\hat{\delta}$ is not monotonic on any interval

Although $\hat{\delta}$ is Lipschitz continuous it does not need to be monotonic on any interval - the following simple lemma provides the basis for one possible construction of such a $\hat{\delta}$.

Lemma 3.1 There exists a Borel set $\Omega \in \mathcal{B}([0, 1])$ with $\lambda(\Omega) = 1/2$ fulfilling the following conditions:

(i) $\lambda((a, b) \cap \Omega) > 0$ and $\lambda((a, b) \cap \Omega^c) > 0$ for every open non-empty interval $(a, b) \subset [0, 1]$.

(ii) $\lambda([0, x] \cap \Omega) > \frac{x}{2}$ for every $x \in (0, 1)$.

Proof: Let $C_0 \subseteq [0, 1]$ denote the classical Smith-Volterra-Cantor set constructable as follows: Start with the unit interval $[0, 1]$, remove an open interval of length $\frac{1}{4}$ around the mid point $\frac{1}{2}$ and let $C_1$ denote the remaining compact set. Remove an interval of length $\frac{1}{4}$ centered at the mid points of the 2 intervals constituting $C_1$, denote the remaining set by $C_2$ and proceed analogously, i.e. remove intervals of length $\frac{1}{4^{n+1}}$ centered at the mid points of the $2^n$ intervals constituting $C_n$. It is easily verified that $C_\infty := \bigcap_{n=1}^{\infty} C_n$ is a totally disconnected compact set fulfilling $\lambda(C_\infty) = \frac{1}{2}$.

In the next step we will paste affine copies of $C_\infty$ into the closures $(J_{1,n})_{n \in \mathbb{N}}$ of the countably many pairwise disjoint open intervals $(U_{1,n})_{n \in \mathbb{N}}$ constituting $C_\infty^c$. Doing so we use the
following notation: For each compact interval \( J = [a, b] \subseteq [0, 1] \) let \( S_J : [0, 1] \to J \) denote the function \( S_J(x) = a + (b - a)x \). Set \( H_1 = C_\infty \) and \( L_\infty := C_\infty \cap [0, \frac{1}{2}] \). Then the set \( H_2 := H_1 \cup \bigcup_{n=1}^{\infty} S_{J_{1,n}}(C_\infty) \) fulfills \( \lambda(H_2) = \frac{3}{2} \) and we have \( \lambda \left( \bigcup_{n=1}^{\infty} S_{J_{1,n}}(L_\infty) \right) = \frac{1}{2} \). Let \( (J_{2,n})_{n \in \mathbb{N}} \) denote the closures of the countably many pairwise disjoint open intervals \( (U_{2,n})_{n \in \mathbb{N}} \) constituting \( H_2 \). The set \( H_3 := H_2 \cup \bigcup_{n=1}^{\infty} S_{J_{2,n}}(C_\infty) \) fulfills \( \lambda(H_3) = \frac{2^k+1}{2^k} \) and we have \( \lambda \left( \bigcup_{n=1}^{\infty} S_{J_{2,n}}(L_\infty) \right) = \frac{1}{2^k} \). We proceed inductively, i.e. for given \( H_k \) with \( \lambda(H_k) = \frac{2^k-1}{2^k} \) let \( (J_{k,n})_{n \in \mathbb{N}} \) denote the closures of the countably many pairwise disjoint open intervals \( (U_{k,n})_{n \in \mathbb{N}} \) constituting \( H_k \) and set \( H_{k+1} = H_k \cup \bigcup_{n=1}^{\infty} S_{J_{k,n}}(C_\infty) \). Then \( \lambda(H_{k+1}) = \frac{2^{k+1}-1}{2^{k+1}} \) as well as \( \lambda \left( \bigcup_{n=1}^{\infty} S_{J_{k,n}}(L_\infty) \right) = \frac{1}{2^{k+1}} \) follows immediately. We will show now that \( \Omega \), defined by

\[
\Omega := L_\infty \cup \bigcup_{n=1}^{\infty} S_{J_{k,n}}(L_\infty),
\]

fulfills the properties stated in the lemma. Since both \( \lambda(\Omega) = \frac{1}{2} \) and condition (i) follow immediately from the construction it suffices to prove condition (ii): Set \( R_\infty := C_\infty \cap [\frac{1}{2}, 1] \) as well as \( \Omega' := R_\infty \cup \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} S_{J_{k,n}}(R_\infty) \). Considering that for each pair \( (k, n) \in \mathbb{N}^2 \) both sets \( R_\infty \cap S_{J_{k,n}}(L_\infty) \) and \( L_\infty \cap S_{J_{k,n}}(R_\infty) \) contain at most one point \( \lambda(\Omega' \cap \Omega) = 0 \) follows, which in turn implies that \( \lambda(\Omega' \Delta \Omega) = 0 \). Since for every \( x \in (0, 1) \) we obviously have \( \lambda([0, x] \cap L_\infty) > \lambda([0, x] \cap R_\infty) \) we get

\[
\lambda([0, x] \cap \Omega) = \lambda([0, x] \cap L_\infty) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambda([0, x] \cap S_{J_{k,n}}(L_\infty)) \\
\geq \lambda([0, x] \cap L_\infty) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambda([0, x] \cap S_{J_{k,n}}(R_\infty)) \\
> \lambda([0, x] \cap R_\infty) + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \lambda([0, x] \cap S_{J_{k,n}}(R_\infty)) = \lambda([0, x] \cap \Omega'),
\]

which completes the proof. ■

**Theorem 3.2** There exists a diagonal \( \delta_0 \in \mathcal{D} \) such that \( \hat{\delta}_0 \) is not monotonic on any non-empty open interval.

**Proof:** Consider a Borel set \( \Omega \) fulfilling the properties of Lemma 3.1 and set

\[
\delta_0(x) := 2 \int_{[0,x]} 1_{\Omega'}(t) d\lambda(t) = 2\lambda([0, x] \cap \Omega').
\]

(5)

for every \( x \in [0, 1] \). Then obviously \( \delta_0 \) is monotonically non-decreasing, Lipschitz continuous with Lipschitz constant \( 2 \) and fulfills \( \delta_0(0) = 0, \delta_0(1) = 1 \). Furthermore, using condition (ii) in Lemma 3.1, it follows that \( \delta_0(x) < x \) for every \( x \in (0, 1) \), so \( \delta_0 \in \mathcal{D} \). According to Rudin (1987) \( \delta_0 \) is as absolutely continuous function differentiate \( \lambda \)-a.e. and we have \( \delta_0'(x) \in \{0, 2\} \) for \( \lambda \)-almost every \( x \in [0, 1] \). In terms of \( \delta_0 \) this means \( \delta_0'(x) \in \{-1, 1\} \) for \( \lambda \)-almost every \( x \in [0, 1] \), from which, using condition (i) in Lemma 3.1, it follows immediately that \( \hat{\delta} \) is not monotonic on any interval. ■
4 Markov kernels of diagonal copulas

For every diagonal $\delta$ define the so-called diagonal copula (see Nelsen et al., 2004, 2008, and Úbeda-Flores, 2008) $E_{\delta}$ by

$$E_{\delta}(x, y) = \min\left\{ x, y, \frac{\delta(x) + \delta(y)}{2} \right\}.$$  

(6)

(We will use the symbol $E_{\delta}$ instead of $K_{\delta}$ since the letter $K$ will denote kernels throughout the whole paper.) It is well known that $E_{\delta}$ is singular and that for every symmetric copula $A \in \mathcal{C}_\delta$ we have $A \leq E_{\delta}$ (see again Nelsen et al., 2004, 2008, and Úbeda-Flores, 2008). The following lemma will be useful for the calculation of the Markov kernel $K_{E_{\delta}}$ of $E_{\delta}$:

**Lemma 4.1** Suppose that $\delta \in \mathcal{D}$ and set $g(x) = 2x - \delta(x)$ for $x \in [0, 1]$. Furthermore define two functions $L, U : [0, 1] \to [0, 1]$ by

$$L(x) := \min\{ z \in [0, 1] : g(z) \geq \delta(x) \} , \quad U(x) := \min\{ z \in [0, 1] : \delta(z) \geq g(x) \}.$$  

Then the following assertions hold:

1. $L(x) \leq x$ for all $x \in [0, 1]$. Furthermore $L$ is non-decreasing and lower semicontinuous (hence left-continuous).

2. $U(x) \geq x$ for all $x \in [0, 1]$. Furthermore $U$ is non-decreasing and upper semicontinuous (hence right-continuous).
3. \( \delta = \delta \circ U \circ L \) and \( g = g \circ L \circ U \).

4. \( \delta(x) < x \) implies \( L(x) < x \) and \( U(x) > x \).

**Proof:** Obviously the function \( g \) is Lipschitz-continuous with Lipschitz constant \( L = 2 \), non-decreasing, and fulfills \( g(0) = 0, g(1) = 1 \) as well as \( g \geq \delta \). Hence, in particular, \( L(x) \leq x \) and \( U(x) \geq x \) for every \( x \in [0,1] \). The fact that \( L, U \) are non-decreasing is a direct consequence of the fact that \( \delta, g \) are non-decreasing. If \( L(x) > \alpha \) then \( g(\alpha) < \delta(x) \) and continuity of \( \delta \) implies the existence of \( r > 0 \) such that \( g(\alpha) = \delta(z) \) for all \( z \in B(x, r) \), which shows that the set \( \{ x \in [0,1] : L(x) > \alpha \} \) is open. Since \( \alpha \) was arbitrary lower semicontinuity of \( L \) follows. Using the fact that \( L(x) \leq y \) if and only if \( U(y) \geq x \) this shows upper semicontinuity of \( U \), which completes the proof of the first two assertions. The third assertion is a direct consequence of continuity of \( \delta \) and \( g \). If \( L(x) = x \) then, using \( g(L(x)) = \delta(x) \) we have \( 2x - \delta(x) = \delta(x) \), so \( \delta(x) = x \) follows. The fact that \( U(x) = x \) implies \( \delta(x) = x \) follows analogously. \( \blacksquare \)

\( E_\delta \) lives on the graph of two functions - the following result holds:

**Theorem 4.2** Suppose that \( \delta \in \mathcal{D} \) and let \( w_\delta : [0,1] \rightarrow [0,2] \) be measurable with \( w_\delta(x) = \delta'(x) \) for \( \lambda \)-almost every \( x \in [0,1] \). Then the Markov kernel \( K_{E_\delta}(\cdot,\cdot) \) of \( E_\delta \) is given by

\[
K_{E_\delta}(x,F) = \frac{w_\delta(x)}{2} \epsilon_{L(x)}(F) + \left( 1 - \frac{w_\delta(x)}{2} \right) \epsilon_{U(x)}(F)
\]

for \( \lambda \)-almost every \( x \in [0,1] \).

**Proof:** Obviously \( K_{E_\delta} : [0,1] \times \mathcal{B}([0,1]) \rightarrow [0,1] \) according to (7) is a Markov kernel. Using the fact that \( L(x) \leq y \) if and only if \( U(y) \geq x \) it follows that for every \( y \in [0,1] \) we have \( L^{-1}([0,y]) = [0,U(y)] \) and \( U^{-1}([0,y]) = [0,L(y)] \).

Fix \( y \in [0,1] \). If \( U^{-1}([y]) \) contains a compact interval of the form \([a,b]\) with \( a < b \) then we have \( U(a) = U(b) \) from which \( \delta(b) - \delta(a) = 2(b-a) \), \( \delta'(x) = 2 \) for each \( x \in (a,b) \), and \( w_\delta(x) = 2 \) for \( \lambda \)-almost every \( x \in [a,b] \) follows. Having this it is straightforward to verify that \( K_{E_\delta} \) corresponds to \( E_\delta \). In fact, taking into account

\[
\int_{[0,y]} K_{E_\delta}(t,[0,x]) \ d\lambda(t) = \int_{[0,y]} \frac{w_\delta}{2} 1_{L^{-1}([0,x])} d\lambda + \int_{[0,y]} \left( 1 - \frac{w_\delta}{2} \right) 1_{U^{-1}([0,x])} d\lambda
\]

\[
= \int_{[0,y]} \frac{w_\delta}{2} 1_{[0,U(x)]} d\lambda + \int_{[0,y]} \left( 1 - \frac{w_\delta}{2} \right) 1_{[0,L(x)]} d\lambda
\]

\[
= \frac{1}{2} \delta(\min\{y,U(x)\}) + \frac{1}{2} g(\min\{y,L(x)\}),
\]

using symmetry of \( E_\delta \), and considering the three cases (i) \( y \leq L(x) \), (ii) \( y \in [L(x),U(x)] \), (iii) \( y \geq U(x) \) immediately yields the desired result. \( \blacksquare \)

The following result gives necessary and sufficient conditions for \( E_\delta \) to be completely dependent.
Figure 3: Image plots of the functions \((x, y) \mapsto K_{E_1}(x, [0, y])\) and \((x, y) \mapsto K_{E_2}(x, [0, y])\), whereby \(\delta_1, \delta_2\) are as in Figure 1.

**Theorem 4.3** Suppose that \(\delta\) is a diagonal. Then \(E_\delta\) is a generalized shuffle of \(M\) living on the graph of a \(\lambda\)-preserving bijection \(S : [0, 1] \to [0, 1]\) fulfilling \(S \circ S = \text{id}_{[0,1]}\) if and only if for \(\lambda\)-almost every \(x \in [0, 1]\) either \(\delta'(x) \in \{0, 2\}\) or \(\delta(x) = x\) holds.

**Proof:** Suppose that \(E_\delta\) lives on the graph of a \(\lambda\)-preserving transformation \(S : [0, 1] \to [0, 1]\). Let \(\Lambda \in \mathcal{B}([0, 1])\) denote a Borel set such that \(\delta'(x) = w_\delta(x)\) and \(K_A(x, \cdot) = \epsilon_{Sx}(\cdot)\) for every \(x \in \Lambda\). Fix \(x \in \Lambda\). If \(\delta(x) < x\) then, applying Lemma 4.1, yields \(L(x) < x\) and \(U(x) > x\), so, using (7) we have \(w_\delta(x) \in \{0, 2\}\), which proves one implication. On the other hand, if for \(\lambda\)-almost every \(x \in [0, 1]\) either \(\delta'(x) \in \{0, 2\}\) or \(\delta(x) = x\) holds, then Theorem 4.2 and Lemma 4.1 imply complete dependence of \(E_\delta\), i.e. \(E_\delta\) lives on the graph of a \(\lambda\)-preserving function \(\tilde{S} : [0, 1] \to [0, 1]\). Symmetry implies

\[
\lambda(F \times \tilde{S}^{-1}(G)) = \lambda(G \times \tilde{S}^{-1}(F))
\]

for all \(F, G \in \mathcal{B}([0, 1])\), from which, setting \(F = \tilde{S}^{-1}(G)\), we immediately get that every Borel set \(G\) is \(\tilde{S}^2\)-invariant (i.e. \(\lambda(\tilde{S}^{-2}(G)\Delta G) = 0\)). Using the fact that \(C_{\tilde{S}} * C_{\tilde{S}} = C_{\tilde{S}^2}\) it follows that \(C_{\tilde{S}^2} = M\), so \(\tilde{S}^2 = \text{id}_{[0,1]}\) \(\lambda\)-almost everywhere. Set \(\Psi := \{x \in [0, 1] : \tilde{S}^2(x) = x\}\), then the function \(S : [0, 1] \to [0, 1]\), defined by \(S(x) = \tilde{S}(x)1_\Psi(x) + x 1_{\Psi^c}(x)\) has the desired properties, \(E_\delta\) lives on the graph of \(S\) and \(E_\delta\) is a generalized shuffle of \(M\). 

The following direct consequence of Theorem 4.3 has already been proved in Nelsen and Fredricks (1997).
Proposition 4.4 \( E_\delta \) is a (straight) shuffle of \( M \), if and only if, \( \delta \) is piecewise linear and for \( \lambda \)-almost every \( x \in [0, 1] \) either \( \delta'(x) \in \{0, 2\} \) or \( \delta(x) = x \) holds.

Example 4.5 Consider \( E_{\delta_0} \) for the diagonal \( \delta_0 \) from Example 3.1. According to Proposition 4.3 \( E_{\delta_0} \) is mutually completely dependent and lives on the graph of a \( \lambda \)-preserving bijection \( S \). Furthermore, for \( \lambda \)-almost every \( x \) with \( \delta_0'(x) = 2 \) we have \( S(x) = L(x) \) and for \( \lambda \)-almost every \( x \) with \( \delta_0'(x) = 0 \) we have \( S(x) = U(x) \). It follows directly from the construction of \( \delta_0 \) and Lemma 4.1 that both \( \delta_0 \) and \( g \) are bijections and that \( L(x) < x \) as well as \( U(x) > x \) for every \( x \in (0, 1) \). Having this it is straightforward to verify that \( S \) is neither monotonic nor continuous on any open non-empty interval \( (a, b) \subseteq [0, 1] \): Choose \( x_0 \in (a, b) \) such that \( \delta_0'(x) = 0 \) and \( S(x_0) = U(x_0) \). Then there exists \( R > 0 \) such that \( S(x_0) = U(x_0) = x_0 + R \) and for every integer \( m \geq 2 \) we can find points \( z_1, z_2 \in (x_0 - \frac{R}{m}, x_0 + \frac{R}{m}) \) with \( z_1 < x_0 < z_2 \), \( \delta'(z_1) = \delta'(z_2) = 2 \), and \( S(z_1), S(z_2) < x_0 + \frac{R}{m} = S(x_0) - R \frac{m-1}{m} \). Obviously \( E_{\delta_0} \) is a proper generalized shuffle of \( M \).

5 Markov kernels of Bertino copulas

Given a diagonal \( \delta \) the Bertino copula \( B_\delta \) is defined by (see Fredricks and Nelsen, 2002)

\[
B_\delta(x, y) = M(x, y) - \min \left\{ \hat{\delta}(t) : t \in [\min\{x, y\}, \max\{x, y\}] \right\}. 
\]  

(8)

It is well known that \( B_\delta \) is the minimal element in \( C_\delta \) (see Fredricks and Nelsen, 2002, Nelsen et al., 2008). Analogous to the previous section we will now calculate the Markov kernel for \( B_\delta \) and, as direct consequence of that, extend some results from Fredricks and Nelsen (2002).

We start with the following two functions \( l, u : [0, 1] \rightarrow [0, 1] \):

\[
\begin{align*}
    u(x) & := \max\{y \geq x : \hat{\delta}(t) \geq \hat{\delta}(x) \text{ for all } t \in [x, y]\} \\
    l(x) & := \min\{y \leq x : \hat{\delta}(t) \geq \hat{\delta}(x) \text{ for all } t \in [y, x]\}
\end{align*}
\]  

(9)

The following lemma gathers some properties of \( l \) and \( u \):

Lemma 5.1 Suppose that \( \delta \) is a diagonal and let \( u, l \) be defined according to (9). Then the following assertions hold:

1. \( u \) is upper semicontinuous, \( l \) lower semicontinuous.
2. \( u(0) = u(1) = 1 \), \( u(x) \geq x \) and \( \hat{\delta}(u(x)) = \hat{\delta}(x) \) for every \( x \in [0, 1] \).
3. \( l(0) = l(1) = 0 \), \( l(x) \leq x \) and \( \hat{\delta}(l(x)) = \hat{\delta}(x) \) for every \( x \in [0, 1] \).
4. \( \hat{\delta}'(x) > 0 \) implies \( u(x) > x \) and \( l(x) = x \), \( \hat{\delta}'(x) < 0 \) implies \( l(x) < x \) and \( u(x) = x \).
5. If \( u(x) > x \) and \( \hat{\delta} \) is differentiable at \( x \) then \( \hat{\delta}'(x) \geq 0 \) follows. If \( l(x) < x \) and \( \hat{\delta} \) is differentiable at \( x \) then \( \hat{\delta}'(x) \leq 0 \) follows.
6. Suppose that \( x < y \); then we have \( u(x) < y \) if and only if \( \hat{\delta}(x) > \min\{\hat{\delta}(t) : t \in [x, y]\} \).
7. Suppose that \( y < x \); then we \( l(x) > y \) if and only if
\[
\hat{\delta}(x) = \min \{ \hat{\delta}(t) : t \in [y, x] \}
\]

**Proof:** We start with showing upper semicontinuity of \( u \). Let \( \alpha \in (0, 1] \) and suppose that \( u(x) < \alpha \). Then, by definition, we can find \( t_m \in (u(x), \alpha] \) such that
\[
\hat{\delta}(t_m) = \min \{ \hat{\delta}(t) : t \in [u(x), \alpha] \} < \hat{\delta}(x).
\]
Continuity of \( \hat{\delta} \) implies the existence of an interval \( (x - r, x + r) \) with \( r > 0 \) such that \( \hat{\delta}(x) > \hat{\delta}(t_m) \) and therefore \( u(z) < t_m \leq \alpha \) for each \( z \in B(x, r) \). This shows that the set \( \{ y \in [0, 1] : u(y) < \alpha \} \) is open proving upper semicontinuity of \( u \) since \( \alpha \) was arbitrary. Lower semicontinuity of \( l \) can be proved in the same manner. Assertions two and three are direct consequences of continuity of \( \hat{\delta} \), assertions four and five follow directly from the definition of the derivative. Assume that \( x < y \). If \( \hat{\delta}(x) > \min \{ \hat{\delta}(t) : t \in [x, y] \} \), then there exists \( t_0 \in (x, y] \) such that \( \hat{\delta}(x) > \hat{\delta}(t_0) \), implying \( u(x) < t_0 \leq y \). On the other hand, if \( u(x) < y \) holds, then there exists \( t_0 \in (x, y] \) with \( \hat{\delta}(t_0) < \hat{\delta}(x) \). Assertion seven follows analogously.

**Remark 5.2** Proposition 2.1 in Fredricks and Nelsen (2002) does not cover all possible cases. In fact, for the diagonal \( \delta_0 \) from Theorem 3.2 \( \hat{\delta}_0 \) is not monotonic on any interval and, using Lemma 5.1 it is straightforward to see that the same is true for \( u \) and \( l \). Furthermore, neither \( u \) nor \( l \) needs to be right- or leftcontinuous. Counterexamples are easily constructed: for the piecewise linear diagonal \( \delta_1 \) fulfilling \( \delta_1(1/4) = 0, \delta_1(1/2) = 3/8 \) and \( \delta_1(7/8) = 3/4 \) obviously \( u \) is not right-continuous at \( 1/8 \) and not left-continuous at \( 1/2 \).

The following two lemmata help to calculate the Markov kernel \( K_{B_\delta}(\cdot, \cdot) \) of the Bertino copula \( B_\delta \) for every \( \delta \in \mathcal{D} \).

**Lemma 5.3** Suppose that \( \delta \in \mathcal{D} \), that the corresponding \( \hat{\delta} \) is differentiable at \( x_0 \in (0, 1) \) and that \( x_0 < y_0 \). Define a non-decreasing function \( g \) on \( [0, y_0] \) by \( g(z) = \min \{ \hat{\delta}(t) : t \in [z, y_0] \} \). Then the following assertions hold:

(a) If \( y_0 > u(x_0) \) then \( g \) is differentiable at \( x_0 \) and we have \( g'(x_0) = 0 \).

(b) If \( y_0 < u(x_0) \) then we have \( D^- g(x_0) = D^- g(x_0) = \hat{\delta}'(x_0) \geq 0 \), i.e. \( g \) is left-differentiable at \( x_0 \) with derivative \( \hat{\delta}'(x_0) \geq 0 \).

**Proof:** If (a) holds, then assertion five in Lemma 5.1 implies \( \hat{\delta}(x_0) > \min \{ \hat{\delta}(t) : t \in [x_0, y_0] \} := M \). Using continuity of \( \hat{\delta} \) there exists an open ball \( B(x_0, r) \) with \( r > 0 \) such that \( \hat{\delta} > M \) on \( B(x_0, r) \). Hence \( g \) is constant on \( B(x_0, r) \) and \( g'(x_0) = 0 \) follows immediately.

To prove assertion (b) we show that both the lower and upper left Dini derivative (see Hewitt and Stromberg, 1965) of \( g \) at \( x_0 \) coincide with \( \hat{\delta}(x_0) \). Since, by assumption, \( u(x_0) > y_0 > x_0 \), Lemma 5.1 implies \( \hat{\delta}'(x_0) \geq 0 \) as well as \( \hat{\delta}(x_0) = \min \{ \hat{\delta}(t) : t \in [x_0, y_0] \} \). By definition of \( g \) we have
\[
\frac{g(x_0) - g(x_0 - t)}{t} \geq \frac{g(x_0) - \hat{\delta}(x_0 - t)}{t} = \frac{\hat{\delta}(x_0) - \hat{\delta}(x_0 - t)}{t}
\]
from which it follows immediately that the lower left Dini derivative \( D_- g(x_0) \) of \( g \) at \( x_0 \) fulfills \( D_- g(x_0) \geq \hat{\delta}'(x_0) \). Furthermore, by definition of the upper left Dini derivative, for each \( \varepsilon > 0 \) there exists \( \Delta_0 = \Delta_0(\varepsilon) > 0 \) such that

\[
\sup_{t \in (0, \Delta)} \frac{\hat{\delta}(x_0) - \hat{\delta}(x_0 - t)}{t} \leq \hat{\delta}'(x_0) + \varepsilon
\]

for each \( \Delta \in (0, \Delta_0] \). Hence, for each \( t \in (0, \Delta) \) we have \( \hat{\delta}(x_0 - t) \geq \hat{\delta}(x_0) - t(\hat{\delta}'(x_0) + \varepsilon) \), which, considering \( \hat{\delta}(x_0) + \varepsilon > 0 \), implies \( g(x_0 - t) \geq \hat{\delta}(x_0) - t(\hat{\delta}'(x_0) + \varepsilon) \) for every \( t \in (0, \Delta) \). Consequently the upper left Dini derivative \( D^- g(x_0) \) of \( g \) at \( x_0 \) fulfills \( D^- g(x_0) \leq \hat{\delta}'(x_0) + \varepsilon \), from which \( D^- g(x_0) \leq \hat{\delta}'(x_0) \) follows since \( \varepsilon > 0 \) was arbitrary.

**Lemma 5.4** Suppose that \( \delta \in \mathcal{D} \), that the corresponding \( \hat{\delta} \) is differentiable at \( x_0 \in (0, 1) \) and that \( y_0 < x_0 \). Define a non-increasing function \( g \) on \([y_0, 1]\) by \( g(z) = \min \{ \hat{\delta}(t) : t \in [y_0, z] \} \). Then the following two assertions hold:

- If \( y_0 < l(x_0) \) then \( g \) is differentiable at \( x_0 \) and we have \( g'(x_0) = 0 \).
- If \( y_0 > l(x_0) \) then we have \( D_+ g(x_0) = D^+ g(x_0) = \hat{\delta}'(x_0) \leq 0 \), i.e. \( g \) is right-differentiable at \( x_0 \) with derivative \( \hat{\delta}'(x_0) \leq 0 \).

**Proof:** Analogous to the proof of Lemma 5.3.
Theorem 5.5 Suppose that \( \delta \in \mathcal{D} \) and let \( \tilde{w}_\delta : [0, 1] \to [-1, 1] \) be measurable with \( \tilde{w}_\delta(x) = \tilde{\delta}'(x) \) for \( \lambda \)-almost every \( x \in [0, 1] \). Then the Markov kernel \( K_{B_\delta}(\cdot, \cdot) \) of \( B_\delta \) is given by

\[
K_{B_\delta}(x, E) = \begin{cases} 
(1 - \tilde{w}_\delta(x))c_1(E) + \tilde{w}_\delta(x)c_0(x)(E) & \text{if } \tilde{w}_\delta(x) > 0 \\
(1 + \tilde{w}_\delta(x))c_2(E) - \tilde{w}_\delta(x)c_1(x)(E) & \text{if } \tilde{w}_\delta(x) \leq 0,
\end{cases}
\]

for \( \lambda \)-almost every \( x \in [0, 1] \).

**Proof:** Fix \( A \in \mathcal{C} \), (a version of) the corresponding Markov kernel \( K_A \in \mathcal{K} \), \( \delta \in \mathcal{D} \) and (a version of) the derivative \( \tilde{w}_\delta \) of \( \tilde{\delta} \). Then for all \( x, y \in [0, 1] \) we have \( A(x, y) = \int_{[0, x]} K_A(t, [0, y])d\lambda(t) \). Hence (see Rudin, 1987) for every fixed \( y \in [0, 1] \) there exists a Borel set \( \Lambda_y \) with \( \lambda(\Lambda_y) = 1 \) such for every \( x \in \Lambda_y \) the function \( f_y : x \mapsto A(x, y) \) is differentiable at \( x_0 \) and fulfills \( f_y'(x_0) = K_A(x_0, [0, y]) \). Use Lipschitz continuity of \( \tilde{\delta} \) to find a set \( \Lambda' \in \mathcal{B}([0, 1]) \) with \( \lambda(\Lambda') = 1 \) and \( \tilde{\delta}'(x) = \tilde{w}_\delta(x) \) for every \( x \in \Lambda' \) and set \( \Lambda_A := \Lambda' \cap \bigcap_{y \in \mathbb{Q} \cap [0, 1]} \Lambda_y \). Then obviously \( \Lambda_A \in \mathcal{B}([0, 1]) \) and \( \lambda(\Lambda_A) = 1 \) follows. Now consider the case \( A = B_\delta \), set \( \Lambda := \Lambda_A \cup \bigcup_{y \in \mathbb{Q} \cap [0, 1]} \Lambda_y \). In case \( \mathbb{Q} \cap [0, 1] \neq \emptyset \) the result follows immediately from right-continuity of \( y \mapsto K(x, y) \) and the fact that \( \mathbb{Q} \) is dense in \( [0, 1] \).

**Proposition 5.6** The support of the Bertino copula \( B_\delta \) is contained in the union of the diagonal and the closure of the graph of the measurable function \( S : [0, 1] \to [0, 1] \), defined by

\[
S(x) = \begin{cases} 
u(x) & \text{if } w_\delta(x) > 0 \\
l(x) & \text{if } w_\delta(x) \leq 0,
\end{cases}
\]

A result similar to Theorem 4.3 also holds for Bertino copulas:

**Theorem 5.7** Suppose that \( \delta \) is a diagonal. If \( \delta'(x) \in \{0, 2\} \) holds for \( \lambda \)-almost every \( x \in [0, 1] \) then the Bertino copula \( B_\delta \) is a generalized shuffle of \( M \) and lives on the graph of a \( \lambda \)-preserving bijection \( S : [0, 1] \to [0, 1] \) fulfilling \( S \circ S = \text{id}_{[0, 1]} \). In case \( \delta \), in addition, is piecewise linear then \( B_\delta \) is a (straight) shuffle of \( W \).

**Proof:** Analogous to the proof of Theorem 4.3.

**Example 5.8** For \( \delta_0 \) from Example 3.1 Proposition 5.7 implies that \( B_{\delta_0} \) is a generalized shuffle of \( M \) living on the graph of a \( \lambda \)-preserving bijection \( S : [0, 1] \to [0, 1] \) fulfilling \( S \circ S = \text{id}_{[0, 1]} \). Since for every \( x \in (0, 1) \) with \( \delta_0'(x) > 0 \) we have \( u(x) > x \) and for every \( x \in (0, 1) \) with \( \delta_0'(x) < 0 \) we have \( l(x) < x \) we can proceed analogously to Example 4.5 to show that \( S \) is neither monotonic nor continuous on any non-empty open interval \( (a, b) \subset [0, 1] \). As a consequence Theorem 2.2, in Fredricks and Nelsen (2002) does not cover all possible supports of Bertino copulas.
Markov kernels for copulas with given diagonal

Figure 5: Image plots of the functions \((x, y) \mapsto K_{B_1}(x, [0, y])\) and \((x, y) \mapsto K_{B_2}(x, [0, y])\), whereby \(\delta_1, \delta_2\) are as in Figure 1.

6 Signed Markov kernels for the maximum quasi-copulas with given diagonal

Given \(\delta \in \mathcal{D}\) in the following \(A_\delta\) will denote the quasi-copula introduced and studied in Nelsen et al. (2008) and Úbeda-Flores (2008), i.e.

\[
A_\delta(x, y) := \min \left\{ x, y, \max\{x, y\} - \max \left\{ \delta(t) : t \in [\min\{x, y\}, \max\{x, y\}] \right\} \right\}
\]

(12)

for all \(x, y \in [0, 1]\). It is well known (see Nelsen et al., 2008, Úbeda-Flores, 2008) that \(A_\delta\) is the maximal quasi-copula with given diagonal \(\delta\) - in the sequel we will therefore refer to \(A_\delta\) as the MQC with diagonal \(\delta\). Following a similar approach as in the last sections we will prove the conjecture stated in Nelsen et al. (2008), saying that \(A_\delta\) is singular. Working with Markov kernels will also allow for a very simple and short alternative proof of the characterization of diagonals for which \(A_\delta\) is a copula given in Úbeda-Flores (2008). As in the previous two sections we start with the construction of some functions that will be useful in the sequel:

For every \(x \in [0, 1]\) define two functions \(g_x : [0, x] \to [0, 1]\) and \(\overline{g}_x : [x, 1] \to [0, 1]\) by

\[
g_x(z) = z + \max \left\{ \delta(t) : t \in [z, x] \right\}, \quad \overline{g}_x(z) = z - \max \left\{ \delta(t) : t \in [x, z] \right\}.
\]

It is straightforward to verify that both \(g_x\) and \(\overline{g}_x\) are non-decreasing and Lipschitz continuous with Lipschitz constant \(L = 1\). Furthermore we have \(g_x(0) \leq x, g_x(x) \geq x\) as well as \(\overline{g}_x(x) \leq x\).
and \( g_x(1) \geq x \). Given the functions \( g_x, \bar{g}_x \) for every \( x \in [0,1] \) define \( \overline{f}, \underline{f}, u, l : [0,1] \to [0,1] \) by

\[
\begin{align*}
\overline{f}(x) &= \min \{ z \in [x,1] : \overline{g}_x(z) \geq x \} \\
\underline{f}(x) &= \max \{ z \in [0,x] : \underline{g}_x(z) \leq x \} \\
u(x) &= \max \{ y \in [x,1] : \delta(t) \leq \delta(x) \text{ for all } t \in [x,y] \} \\
l(x) &= \min \{ y \in [0,x] : \delta(t) \leq \delta(x) \text{ for all } t \in [y,x] \}
\end{align*}
\]

The following lemma gathers some properties of the latter four functions:

**Lemma 6.1** Suppose that \( \delta \) is a diagonal and let \( \overline{f}, \underline{f}, u, l \) be defined according to (13). Then the following assertions hold:

1. \( f(x) \leq x \) for all \( x \in [0,1] \). Furthermore \( \underline{f} \) is non-decreasing and upper semicontinuous (hence right-continuous).

2. \( \overline{f}(x) \geq x \) for all \( x \in [0,1] \). Furthermore \( \overline{f} \) is non-decreasing and lower semicontinuous (hence left-continuous).

3. \( \overline{f}(x) = \max \{ z \in [0,x] : A_\delta(x,z) \geq z \} \), \( \overline{f}(x) = \min \{ z \in [x,1] : A_\delta(x,z) \geq x \} \).

4. For every \( x \in [0,1] \) we have \( \underline{f}(x) < x \) if and only if \( \delta(x) < x \) if and only if \( \overline{f}(x) > x \).

5. \( u \) is upper semicontinuous, \( l \) is lower semicontinuous.
6. $\hat{\delta}'(x) < 0$ implies $u(x) > x$ and $l(x) = x$, $\hat{\delta}'(x) > 0$ implies $l(x) < x$ and $u(x) = x$.

7. If $u(x) > x$ and $\hat{\delta}$ is differentiable at $x$ then $\hat{\delta}'(x) \leq 0$ follows. If $l(x) < x$ and $\hat{\delta}$ is differentiable at $x$ then $\hat{\delta}'(x) \geq 0$ follows.

8. Suppose that $x < y$; then we have $u(x) < y$ if and only if
\[ \hat{\delta}(x) < \max \{ \hat{\delta}(t) : t \in [x, y] \} \]

9. Suppose that $y < x$; then we have $l(x) > y$ if and only if
\[ \hat{\delta}(x) < \max \{ \hat{\delta}(t) : t \in [y, x] \} \]

**Proof:** First notice that the third assertion is a direct consequence of the definition of $A_\delta$ since, in case of $z \leq x$ we have $A_\delta(x, z) \geq z$ if and only if $g_x(z) \leq x$ and in case of $z \geq x$ we have $A_\delta(x, z) \geq x$ if and only if $g_x(z) \geq x$. In particular we get $A_\delta(x, f(x)) = f(x)$ and $A_\delta(x, \overline{f}(x)) = x$ for every $x \in [0, 1]$. Having this showing monotonicity and upper semicontinuity of $f$ is straightforward. In fact, $x_1 < x_2$ implies $A_\delta(x_2, f(x_1)) \geq f(x_1)$, from which $f(x_1) \leq f(x_2)$ directly follows. Furthermore, considering a sequence $(x_n)_{n \in \mathbb{N}}$ in $[0, 1]$ converging to $x$ and fulfilling $f(x_n) \geq \alpha$ for every $n \in \mathbb{N}$ it follows that $A_\delta(x_n, \alpha) = \alpha$, so, by continuity of $A_\delta$, $A_\delta(x, \alpha) = \alpha$ and $f(x) \geq \alpha$ follows. Hence the set $\{ z \in [0, 1] : f(z) \geq \alpha \}$ is closed and upper semicontinuity of $\overline{f}$ follows, which completes the proof of the first assertion.

The second assertion can be proved analogously. If $\hat{\delta}(x) = x$ then, using again assertion three, $f(x) = \overline{f}(x) = x$ follows. On the other hand $f(x) = x$ implies $\delta(x) = x$, which completes the proof of assertion four. Suppose now that $l(x) > \alpha$. Then there exists $t_0 \in [\alpha, l(x))$ such that $\hat{\delta}(t_0) = \max \{ \hat{\delta}(t) : t \in [\alpha, x] \} > \hat{\delta}(x)$. Lipschitz continuity of $\hat{\delta}$ implies the existence of an interval $(x-r, x+r)$ with $r > 0$ such that $\delta(z) < \delta(t_0)$ holds for every $z \in B(x, r)$. This shows $l(z) > t_0 \geq \alpha$ for every $z \in B(x, r)$, so $\{ y \in [0, 1] : l(y) > \alpha \}$ is open and $l$ is lower semicontinuous since $\alpha$ was arbitrary. Upper semicontinuity of $u$ can be proved in the same manner. Assertions six and seven follow directly from the definition of the derivative and assertions eight and nine are straightforward to verify.

**Lemma 6.2** Suppose that $\delta$ is a diagonal and let $\hat{w}_\delta : [0, 1] \to [-1, 1]$ denote the derivative of the corresponding $\hat{\delta}$. Then there exists a Borel set $\Lambda \in \mathcal{B}([0, 1])$ fulfilling $\lambda(\Lambda) = 1$ as well as $\hat{\delta}'(x) = \hat{w}_\delta(x)$ for every $x \in \Lambda$, such that for every $y \in \mathbb{Q}$ the derivative $s'_y$ of the function $s_y : x \mapsto A_\delta(x, y)$ exists for every $x \in \Lambda$ and fulfills

\[
s'_y(x) = \begin{cases} 
1 & \text{if } y \geq \overline{f}(x) \\
0 & \text{if } y \in (u(x), \overline{f}(x)) \\
-\hat{w}_\delta(x) & \text{if } y \in (x, u(x)) \cap (x, \overline{f}(x)) \\
1 - \hat{w}_\delta(x) & \text{if } y \in [l(x), x) \cap [f(x), x) \\
1 & \text{if } y \in [f(x), l(x)) \\
0 & \text{if } y < \overline{f}(x). 
\end{cases}
\]  

**Proof:** For every $y \in [0, 1]$ the function $s_y : x \mapsto A_\delta(x, y)$ is Lipschitz continuous with Lipschitz constant $L = 1$ and non-decreasing, so (see Rudin, 1987) there exists a Borel set
\( \Lambda_y \subseteq (0, 1) \) such that \( s_y \) is differentiable at every \( x \in \Lambda_y \) and fulfills \( s'_y(x) \in [0, 1] \). Moreover, Lipschitz continuity of \( \hat{\delta} \) implies the existence of another Borel set \( \Gamma \in \mathcal{B}(0, 1) \) with \( \Lambda(\Gamma) = 1 \) such that \( \hat{\delta} \) is differentiable at every \( x \in \Gamma \) and fulfills \( \hat{\delta}'(x) = \hat{\omega}_\delta(x) \). Finally, let \( \mathcal{J} \) denote the (countable) set of all \( x \in [0, 1] \) such \( x \) is a discontinuity point of \( f \) or \( \overline{f} \), and define \( \Lambda = \Gamma \cap \mathcal{J}^c \cap \bigcap_{y \in \mathbb{Q} \cap [0,1]} \Lambda_y \). Then obviously \( \Lambda \in \mathcal{B}([0, 1]) \) and \( \Lambda(\Lambda) = 1 \). Suppose now that \( x \in \Lambda \), \( y \in \mathbb{Q} \cap [0, 1] \) and distinguish the following two cases (Lemma 6.1 will be applied multiple times without reference):

**Case I:** \( y < x \) (i) If \( y < f(x) \) then there exists \( r > 0 \) such that for all \( z \in (x - r, x + r) \) we have \( A_3(z, y) = y \), from which \( s'_y(x) = 0 \) immediately follows. (ii) \( y \in [f(x), l(x)] \) implies \( \hat{\delta}(x) < \max \{ \hat{\delta}(t) : t \in [y, x] \} \). Hence, taking into account that \( f \) is non-decreasing and \( \hat{\delta} \) is Lipschitz continuous, we can find \( r > 0 \) such that \( A_3(z, y) = z - \max_{t \in [y, z]} \hat{\delta}(z) \) for every \( z \in (x - r, x) \) and the function \( g : z \mapsto \max_{t \in [y, z]} \hat{\delta}(t) \) is constant on \( (x - r, x) \). Since \( s_y(x) \) exists \( s'_y(x) = 1 \) follows. (iii) If \( y \geq l(x) \) and \( y > f(x) \), then \( \hat{\delta}(x) \geq 0 \) as well as \( \hat{\delta}(x) = \max_{t \in [y, x]} \hat{\delta}(t) \) follows. Furthermore we can find \( r > 0 \) such that \( A_3(z, y) = z - \max_{t \in [y, z]} \hat{\delta}(t) \) for every \( z \in (x - r, x + r) \). Setting \( g(z) := \max_{t \in [y, z]} \hat{\delta}(t) \) for \( z \in (x - r, x + r) \) and considering that

\[
\frac{g(x + t) - g(x)}{t} \quad \frac{\hat{\delta}(x + t) - \hat{\delta}(x)}{t} \quad \frac{g(x) - g(x - t)}{t} \quad \frac{\hat{\delta}(x) - \hat{\delta}(x - t)}{t}
\]

holds for every \( t \in (0, r) \), \( s'_y(x) = 1 - \hat{\delta}'(x) \) follows immediately. (iv) If \( y \geq l(x) \) and \( y = f(x) \), then, as in (iii), \( \hat{\delta}'(x) \geq 0 \) as well as \( \hat{\delta}(x) = \max_{t \in [y, x]} \hat{\delta}(t) \) follows. It suffices to consider the case that \( f(z) > y \) for every \( z > x \) (otherwise the arguments in (iii) may be applied). In this case \( A_3(z, y) = y \) for all \( z > x \) follows, which implies \( s'_y(x) = 0 \). Furthermore, we can find \( r > 0 \) such that \( A_3(z, y) = z - g(z) \) for every \( z \in (x - r, x) \), whereby \( g \) is defined as in (iii). Applying (15) it follows that \( g'(x) \leq \hat{\delta}'(x) \), from which, using \( s'_y(x) = 0 \), \( \hat{\delta}'(x) = 1 \) follows, i.e. \( s'_y(x) = 1 - \hat{\delta}'(x) \) as in (iii). This completes the proof of the case \( y < x \).

**Case II:** \( y > x \) (i) If \( y \geq f(x) \) then we have \( A_3(z, y) = z \) for all \( z < x \) from which \( s'_y(x) = 1 \) directly follows. (ii) If \( y > l(x) \) and \( y < f(x) \) then we have \( \hat{\delta}(x) < \max_{t \in [x, y]} \hat{\delta}(t) \). Hence the function \( g : z \mapsto \max_{t \in [x, y]} \hat{\delta}(t) \) is constant on an interval \( (x - r, x + r) \) with \( r > 0 \), implying \( s'_y(x) = -g'(z) = 0 \). (iii) If \( y \leq u(x) \) and \( y < f(x) \) then \( \hat{\delta}'(x) \leq 0 \) as well as \( \hat{\delta}(x) = \max_{t \in [x, y]} \hat{\delta}(t) \) follows. Furthermore there exists \( r > 0 \) such that for every \( z \in (x - r, x + r) \) we have \( A_3(z, y) = y - g(z) \) whereby \( g(z) = \max_{t \in [x, y]} \hat{\delta}(t) \). Since \( g \) obviously fulfills (15) it follows that \( g'(x) = \hat{\delta}'(x) \) and \( s'_y(x) = -\hat{\delta}'(x) \).

It is well known (see Fernández Sánchez, 2010, Nelsen et al., 2010) that, given a quasi-copula \( Q \) there need not exist a doubly stochastic signed measure \( \mu_Q : \mathcal{B}([0, 1]^2) \to \mathbb{R} \) fulfilling

\[
\mu([x_1, x_2] \times [y_1, y_2]) = Q(x_2, y_2) - Q(x_1, y_2) - Q(x_2, y_1) + Q(x_1, y_1) =: V_Q([x_1, x_2] \times [y_1, y_2])
\]

for all intervals \([x_1, x_2], [y_1, y_2] \subseteq [0, 1]\). Nevertheless, we will show now that in case of the MQC \( A_3 \) a signed measure \( \mu \) with the afore-mentioned properties can be constructed.
For every $x \in \Lambda$ the function $y \mapsto s'_y(x) \in [0,1]$ is a step-function that is right-continuous at all $y \in \Q \setminus \{x, u(x)\}$. Additionally, for given $y \in \Q$ and arbitrary $x \in [0,1]$ we obviously have

$$A_\delta(x,y) = \int_{[0,x]} s'_y(t)d\lambda(t).$$

(17)

For every $x \in \Lambda$ let $y \mapsto K(x,[0,y])$ denote the right-continuous extension of $y \mapsto s'_y(x)$ to full $[0,1]$, for every $x \in \Lambda^c$ set $K(x,[0,y]) = 1$, i.e.

$$K(x,[0,y]) = \begin{cases} 1 & \text{if } x \in \Lambda \text{ and } y \geq \overline{f}(x) \\ 0 & \text{if } x \in \Lambda \text{ and } y \in [u(x), \overline{f}(x)) \\ -\hat{w}_\delta(x) & \text{if } x \in \Lambda \text{ and } y \in [x, u(x)) \cap [x, \overline{f}(x)) \\ 1 - \hat{w}_\delta(x) & \text{if } x \in \Lambda \text{ and } y \in [\overline{f}(x), \overline{f}(x) \cap [x, \overline{f}(x)) \\ 1 & \text{if } x \in \Lambda \text{ and } y \in [l(x), x) \\ 0 & \text{if } x \in \Lambda \text{ and } y < \overline{f}(x) \\ 1 & \text{if } x \in \Lambda^c. \end{cases}$$

(18)

Then $y \mapsto K(x,[0,y])$ is a step-function too and for every $x \in \Lambda$ we have $s_y(x) = K_A(x,[0,y])$ for all $y \in \Q \setminus \{x, u(x)\}$. Fix $y \in \Q$ and consider $\{z \in \Lambda : u(z) = y < \overline{f}(z)\}$. If the latter

contains two points $x_1 < x_2$ then we have $\delta(x_1) = \delta(x_2)$ as well as $A_\delta(x_1,y) = A_\delta(x_2,y)$, from which $s'_y(x_1) = s'_y(x_2) = 0$ immediately follows. Since $s'_y(x) = 0$ is exactly the case where

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{image.png}
\caption{Image plots of the functions $(x,y) \mapsto K_{A_\delta}(x,[0,y])$ and $(x,y) \mapsto K_{A_{\delta_2}}(x,[0,y])$, whereby $\delta_1, \delta_2$ are as in Figure 1.}
\end{figure}
For $x$, for every $\mu$ K\{ the set $\vartheta$ whose Hahn decomposition $[0,1]$ $\rightarrow \mu$ markov kernels for copulas with given diagonal $E$ for every $\mu$ K\ are finite discrete kernels fulfilling $\mu$. Moreover, using disintegration (see Kallenberg, 1997), we get that $K$ will therefore write $K(x, [0,y])$ for every $\mu$ K\ $\rightarrow \mu$ $\rightarrow \mu$ $\rightarrow \mu$ $\rightarrow \mu$.

Suppose now that $x_1 < x_2$, that $y \in [0,1]$, and that $(y_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence in $[0,y)$ with limit $y$, then we have

$$\int_{[x_1,x_2]} K(t, [0,y])d\lambda(t) = \lim_{n \rightarrow \infty} \int_{(x_1,x_2)} K(t, [0,y_n])d\lambda(t) = \lim_{n \rightarrow \infty} (A_\delta(x_2, y_n) - A_\delta(x_1, y_n))$$

$$= A_\delta(x_2, y) - A_\delta(x_1, y) = \int_{[x_1,x_2]} K(t, [0,y])d\lambda(t),$$

where $A_\delta(x, y) = \lim_{n \rightarrow \infty} A_\delta(x, y_n)$.
so, in particular \( \int_{[x_1, x_2]} K(t, \{y\})d\lambda(t) = 0 \). Having this, equation (16) follows from

\[
V_{A_0}([x_1, x_2] \times [y_1, y_2]) = \int_{[x_1, x_2]} K(t, [0, y_2])d\lambda(t) - \int_{[x_1, x_2]} K(t, [0, y_1])d\lambda(t) = \int_{[x_1, x_2]} K(t, (y_1, y_2))d\lambda(t) = \mu([x_1, x_2] \times [y_1, y_2]).
\]

Altogether we have the following result confirming singularity of \( A_\lambda \) as conjectured in Nelsen et al. (2008) (note that we used the standard definition of singularity of a signed measure and not \( \frac{\partial^2 A_\lambda}{\partial x \partial y} = 0 \) \( \lambda \)-almost everywhere):

**Theorem 6.3** Suppose that \( \delta \) is a diagonal. Then there exists a doubly stochastic (finite) signed measure \( \mu : \mathcal{B}([0, 1]^2) \to [-1, 2] \) such that

\[
A_\delta(x_2, y_2) - A_\delta(x_1, y_2) - A_\delta(x_2, y_1) + A_\delta(x_1, y_1) = \mu([x_1, x_2] \times [y_1, y_2])
\]

holds for all intervals \( [x_1, x_2], [y_1, y_2] \subseteq [0, 1] \). Additionally, both measures \( \mu^+, \mu^- \) of the Hahn decomposition \( \mu = \mu^+ - \mu^- \) of \( \mu \) live on the graph of at most three measurable functions, i.e. \( \mu \) is singular.

We conclude the paper by showing that the chosen approach with kernels also allows for a very simply and short proof of the main result in Ubeda-Flores (2008) characterizing all diagonals for which \( A_\delta \) is a copula.

**Lemma 6.4** Suppose that \( \delta \) is a diagonal for which \( A_\delta \) is a copula. Then for almost every \( x \in [0, 1] \) we have either \( \delta(x) \in \{-1, 1\} \) or \( \delta(x) = x \) and \( A_\delta \) is completely dependent.

**Proof:** Consider \( \Lambda \) according to Lemma 6.2 and \( K(\cdot, \cdot) \) as in equation (18). Since, by assumption, \( A_\delta \) is a copula, there exists a subset \( \Lambda' \subseteq \Lambda \) with \( \lambda(\Lambda') = 1 \) such that \( K(x, \cdot) \) is a probability measure for every \( x \in \Lambda' \). (i) If \( x \in \Lambda' \) and \( \delta'(x) > 0 \) then \( l(x) < x = u(x) \) follows. In this case \( f(x) < l(x) \) can not hold since for every \( y \in [f(x), l(x)] \) we have \( K(x, [0, y]) = 1 \), for every \( y \in [l(x), x] \) we have \( K(x, [0, y]) = 1 - \delta'(x) \), so monotonicity would imply \( \delta'(x) = 0 \). Hence \( f(x) \geq l(x) \) follows. Since, additionally, \( f(x) = x \) would imply \( f(x) = \overline{f(x)} = \delta(x) = x \), \( \delta(x) = 0 \) and therefore \( \delta'(x) = 0 \), it suffices to consider the case \( l(x) \leq f(x) < x \). Since \( K(x, [0, y]) = 1 - \delta'(x) \) for \( y \in [f(x), x] \) and \( K(x, [0, y]) = 0 \) for \( y \in [u(x), \overline{f(x)}] = [x, \overline{f(x)}] \) it follows immediately that \( \delta'(x) = 1 \) and that \( K(x, E) = \epsilon_{\overline{f(x)}}(E) \) for every \( E \in \mathcal{B}([0, 1]) \). (ii) If \( x \in \Lambda' \) and \( \delta'(x) < 0 \) then \( \delta'(x) = -1 \) and \( K(x, E) = \epsilon_{f(x)}(E) \) can be shown analogously.

(iii) Finally, suppose that \( x \in \Lambda' \) and \( \delta'(x) = 0 \). Since \( \delta(x) < x \) would imply \( f(x) < x < \overline{f(x)}, K(x, [0, y]) = 1 \) for \( y \in [f(x), 1] \) and \( K(x, [0, y]) = 0 \) for \( y \in [x, \overline{f(x)}] \), both \( \delta(x) = x \) and \( K(x, E) = \epsilon_x(E) \) follow.

**Theorem 6.5** (Ubeda-Flores, 2008) Suppose that \( \delta \) is a diagonal. Then \( A_\delta \) is a copula if and only if for \( \lambda \)-almost every \( x \in [0, 1] \) one of the following three conditions holds:
(a) $\delta(x) = x$

(b) $\delta(x) < x$, $\delta'(x) = 1$ and $\delta$ is constant on the interval $[f(x), x]$

(c) $\delta(x) < x$, $\delta'(x) = -1$ and $\delta$ has slope two on the interval $[x, f(x)]$

Proof: Suppose that $A_\delta$ is a copula. Then $A_\delta = E_\delta$ and, using Lemma 4.1 and Lemma 6.1, $L = f$ as well as $U = \overline{f}$ follows. Consider $\Lambda'$ from the proof of Lemma 6.4 and suppose that $x \in \Lambda'$. (i) If $\delta(x) < x$ and $\delta'(x) = 1$, then we have $l(x) \leq f(x) < x$ and $\hat{\delta}(t) = \max_{t \in [y,x]} \hat{\delta}(t)$ for every $y \in [f(x), x]$. Hence $x - \hat{\delta}(x) = A_\delta(x, y) = E_\delta(x, y) = \frac{\delta(x) + \delta(y)}{2}$ and $\delta(y) = \delta(x)$ for every $y \in [f(x), x]$. (ii) If $\delta(x) < x$ and $\delta'(x) = -1$ then we have $u(x) \geq \overline{f}(x) > x$ and $\hat{\delta}(t) = \max_{t \in [x,y]} \hat{\delta}(t)$ for every $y \in [x, \overline{f}(x)]$. Hence $y - \hat{\delta}(x) = A_\delta(x, y) = E_\delta(x, y) = \frac{\delta(x) + \delta(y)}{2}$ and $\delta(y) = \delta(x) + 2(y - x)$ for every $y \in [x, \overline{f}(x)]$. This completes the proof of one implication. Suppose now that $\Gamma \in B([0, 1])$ fulfills $\lambda(\Gamma) = 1$ and for every $x \in \Gamma$ (a), (b) or (c) holds. Let $\Lambda$ as in Lemma 6.2 and consider $x \in \Gamma \cap \Lambda$. (i) If $\delta(x) = x$ then $\delta'(x) = 0$ follows and equation (18) implies $K(x, E) = \epsilon_x(E)$. (ii) If (b) holds then $l(x) < x = u(x) < \overline{f}(x)$ and $\hat{\delta}$ has slope 1 on $[f(x), x]$. Hence $l(x) \leq f(x)$ and it follows immediately that $K(x, E) = \epsilon_{\overline{f}(x)}(E)$ for every $E \in B([0, 1])$. (iii) If (c) holds then $f(x) < l(x) = x = u(x)$ and $\hat{\delta}$ has slope $-1$ on $[x, \overline{f}(x)]$. Hence $u(x) \geq \overline{f}(x)$ and $K(x, E) = \epsilon_{f(x)}(E)$ for every $E \in B([0, 1])$ follows. Altogether we have shown that $K(x, \cdot)$ is a probability measure for $\lambda$-almost every $x \in [0, 1]$, which implies that $\mu$ is a doubly stochastic measure. Applying Theorem 6.3 completes the proof. \qed

Theorem 6.5 may be reformulated as follows (for the definition of ordinal sums see Nelsen, 2006):

**Proposition 6.6** $A_\delta$ is a copula if and only if it is an ordinal sum of $W$.

**Remark 6.7** Considering, for instance, the set $C_\infty$ and the family $(J_{1,n})_{n \in \mathbb{N}}$ used in the proof of Lemma 3.1 we can easily construct a diagonal $\delta_3$ for which $A_{\delta_3}$ is an ordinal sum of $W$ although $\delta_3$ is not piecewise linear (compare with Corollary 10 in Úbeda-Flores, 2008). In fact, setting $\delta_3(t) := t$ for every $t \in C_\infty$ and filling the holes $(J_{1,n})_{n \in \mathbb{N}}$ with affine copies of the diagonal $\delta_W$ of $W$ yields a diagonal $\delta_3$ with the desired property.

**References**


Markov kernels for copulas with given diagonal


