Multivariate copulas with hairpin support

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Abstract

The notion of a two-dimensional hairpin allows for two different extensions to the general multivariate setting - that of a sub-hairpin and that of a super-hairpin. We study existence and uniqueness of ρ-dimensional copulas whose support is contained in a sub- (or super-) hairpin and extend various results about doubly stochastic measures to the general multivariate setting. In particular, we show that each copula with hairpin support is necessarily an extreme point of the convex set of all ρ-dimensional copulas. Additionally, we calculate the corresponding Markov kernels and, using a simple analytic expression for sub- (or super-) hairpin copulas, analyze the strong interrelation with copulas having a fixed diagonal section. Several examples and graphics illustrate both the chosen approach and the main results.

Key words: Copula, Stochastic measure, Hairpin Support, Extreme points

1. Introduction

The study of doubly stochastic measures originated with Birkhoff's 1948 problem (Birkhoff, 1948) related to the extension of the notion of doubly stochastic matrices. A doubly stochastic measure on the unit square is a probability measure on $[0,1]^2$ whose (horizontal and vertical) marginals both coincide with the Lebesgue measure λ on $[0,1]$. Since the set of doubly stochastic measures is convex and, in addition, compact under the sup–norm $\|\cdot\|$ of continuous functions on $[0,1]^2$ (see, e.g., Durante et al. (2012)), there has been a growing interest in the determination of extremal points, i.e. doubly stochastic measures that cannot be expressed as (non-trivial) convex combination of another two doubly stochastic measures. The famous Krein–Milman theorem (Rudin, 1991) asserts that convex combinations of extreme points are dense in the set of doubly stochastic measures.

Contributions about extremal doubly stochastic measures started with seminal papers from the end of the 1950's (see, e.g., Peck (1959); Brown (1965)) and have been continuously present in the literature since then (see, among others, Losert (1982); Hestir and Williams (1995)). Recently, Ahmad et al. (2011) stressed the importance of such investigations in connection with the Monge–Kantorovich transportation problem.

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From the viewpoint of applied probability and multivariate statistics, doubly stochastic measures are a class of probability measures that is in one-to-one correspondence with the class of copulas (see, e.g., Nelsen (2006)), which represent the pillars that build many multivariate stochastic models (see, e.g., Jaworski et al. (2010); Mai and Scherer (2012); Jaworski et al. (2013)). Special subclasses of extremal copulas (e.g. shuffles) often appear in several studies about the approximation of dependence structures: see, for instance, Vitale (1991); Durante et al. (2009); Mikusiński and Taylor (2010); Durante and Fernández-Sánchez (2012); Trutschnig and Fernández Sánchez (2013). In this work, we focus on the class of extremal copulas whose support is contained in a hairpin set, which is, in dimension two, the union of the graphs of two increasing homeomorphisms h, g with $g = h^{-1}$ and $h(x) < x$ for all $x \in (0, 1)$. Such bivariate copulas have been considered by Seethoff and Shiflett (1977/78), and further analyzed by Sherwood and Taylor (1988), where, among other things, an existence theorem for such copulas is proved. Further constructions were also provided by Kamiński et al. (1987/88, 1990); Quesada Molina and Rodríguez Lallena (1994). Here, we first revisit some results concerning bivariate copulas with hairpin support by proving them in a different (and simplified) way. Afterwards we state and prove conditions for the existence and uniqueness of copulas with (sub- or super-) hairpin support in the general multivariate setting. More precisely, we consider two possible extensions of a two-dimensional hairpin to the general ρ-dimensional setting - that of a sub-hairpin and that of a super-hairpin - separately and, additionally, study a simple transformation mapping super-hairpin copulas to sub-hairpin ones and vice versa. To the best of the authors' knowledge, with the exception of multivariate shuffles of M, sub- (and super-) hairpin copulas are the first example of extreme points of $\mathcal{C}_\rho$ appearing in the literature for $\rho > 2$.
It is worth mentioning that these constructions are connected with the problem of determining the dependence structure of a random vector $\mathbf{X} = (X_1, \dots, X_\rho)$ in presence of some information (e.g., the probability distribution function) about the order statistics of $\mathbf{X}$. In fact, hairpin constructions are related to some early works about distribution functions by Rychlik (1993, 1994) and were recently reconsidered in a copula framework by Jaworski and Rychlik (2008); Jaworski (2009). For the bivariate case, it is already known that the the construction coincides with copulas with given diagonal sections as considered in Fredricks and Nelsen (1997); Nelsen and Fredricks (1997).
Finally, we would like to stress that copulas with singular support (or with singular component) gained some popularity in recent years due to their ability to conveniently describe the occurrence of joint defaults in a vector of credit risks (or lifetimes): see, for instance, Mai and Scherer (2012). In fact, under inhomogeneous marginal distribution functions the only dependence structures that are capable of including the occurrence of joint defaults (with non-zero probability) are exactly those structures that spread probability mass along graphs of functions (see, e.g., Mai and Scherer (2009, 2013a,b)). To this end, the multivariate copulas presented here (often in combination with other copulas) may provide more realistic tools for credit risk simulations.

2. Notation and preliminaries

Throughout the paper $\rho \geq 1$ will denote the dimension, $\mathcal{B}([0,1]^\rho)$ the Borel-σ-field on $[0,1]^\rho$, λ the Lebesgue measure on $\mathcal{B}([0,1])$ and $\pi_i : [0,1]^\rho \to [0,1]$ the projection onto coordinate i, i.e. $\pi_i(x_1, \dots, x_\rho) = x_i$. Instead of λ-a.e. we will simply write a.e. in the sequel. $\mathcal{C}_\rho$ will denote the class of ρ–dimensional copulas ($\rho \geq 2$), $\mathcal{P}_\mathcal{C}$ the class of all ρ-stochastic measures, i.e. probability measures μ on $[0,1]^\rho$ for which all one-dimensional marginals μ^{π_i} coincide with λ whereby μ^{π_i} denotes the push-forward of μ under π_i. By definition, the support $Supp(A)$ of a copula A is the support of the corresponding ρ-stochastic measure μ_A. We will call $\delta\colon [0,1] \to [0,1]$ a *ρ-dimensional diagonal* if it has the following three properties: (a) $\delta(t) \leq t$ for all $t \in [0,1]$; (b) $\delta(1) = 1$; (c) $0 \leq \delta(t) - \delta(s) \leq \rho(t-s)$ for all $s,t \in [0,1]$. The set of all ρ-dimensional diagonals will be denoted by $\mathcal{D}_\rho$. According to (Cuculescu and Theodorescu, 2001, Proposition 5.1) for every $A \in \mathcal{C}_\rho$, the diagonal section δ_A of A, given by $\delta_A(t) = A(t,\ldots,t)$, fulfills $\delta_A \in \mathcal{D}_\rho$ and, conversely, for every $\delta \in \mathcal{D}_\rho$ there exists a copula $A \in \mathcal{C}_\rho$ such that $\delta_A = \delta$. Since diagonals are Lipschitz-continuous functions they are absolutely continuous and differentiable a.e. (see Rudin (1987)). For every $\delta \in \mathcal{D}_2$ the *diagonal copula* E_δ (see Nelsen and Fredricks (1997)) is defined by

$$E_\delta(x,y) = \min\Big\{x, y, \frac{\delta(x)+\delta(y)}{2}\Big\}.$$

As shown in Fernández Sánchez and Trutschnig (2014a) (also see Durante et al. (2014)) the mass of E_δ is concentrated on the union of the graphs of two functions $L, U : [0,1] \to [0,1]$, defined as

$$L(x) := \min\big\{z \in [0,1] : g(z) \geq \delta(x)\big\}\ ,\ \ U(x) := \max\big\{z \in [0,1] : \delta(z) \geq g(x)\big\},$$

whereby $g(x) := 2x - \delta(x)$ for every $x \in [0,1]$.
A *Markov kernel* from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R}^{\rho-1})$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}^{\rho-1}) \to [0,1]$ such that $x \mapsto K(x,B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R}^{\rho-1})$ and $B \mapsto K(x,B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}^{\rho-1}) \to [0,1]$ is called *regular conditional distribution of* $\mathbf{Y}$ *given* X ($\mathbf{Y} : \Omega \to \mathbb{R}^{\rho-1}$ and $X : \Omega \to \mathbb{R}$ random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$) if for every $B \in \mathcal{B}(\mathbb{R}^{\rho-1})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ \mathbf{Y}|X)(\omega) \tag{1}$$

holds $\mathcal{P}$-a.s. It is well known that for each random vector $(X, \mathbf{Y})$ a regular conditional distribution $K(\cdot,\cdot)$ of $\mathbf{Y}$ given X exists, that $K(\cdot,\cdot)$ is unique $\mathcal{P}^X$-a.s. (i.e. unique for $\mathcal{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot,\cdot)$ only depends on $\mathcal{P}^{X\otimes\mathbf{Y}}$. Thereby $\mathcal{P}^X$ and $\mathcal{P}^{X\otimes\mathbf{Y}}$ denote the distribution of X and $(X,\mathbf{Y})$ respectively. Hence, given $A \in \mathcal{C}_\rho$, we will denote (a version of) the regular conditional distribution of $\mathbf{Y}$ given X by $K_A(\cdot,\cdot)$ and refer to $K_A(\cdot,\cdot)$ simply as *regular conditional distribution* or *Markov kernel of A*. Note that for every $A \in \mathcal{C}_\rho$, its Markov kernel $K_A(\cdot,\cdot)$, and a Borel set $F \in \mathcal{B}([0,1]^\rho)$ we have (with $F_x = \{\mathbf{y} \in [0,1]^{\rho-1} : (x,\mathbf{y}) \in F\}$)

$$\int_{[0,1]} K_A(x, F_x)\, d\lambda(x) = \mu_A(F), \tag{2}$$

so in particular

$$\int_{[0,1]} K_A(x, G)\, d\lambda(x) = \lambda(G_{i_0}) \tag{3}$$

in case $G = \times_{i=1}^{\rho-1} G_i$ and $G_i = [0,1]$ for all $i \neq i_0$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}(\mathbb{R}^{\rho-1}) \to [0,1]$ fulfilling (3) is the regular conditional distribution of a copula $A \in \mathcal{C}_\rho$. As a slightly more general notion we will call a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}^{\rho-1}) \to \mathbb{R}$ *signed Markov kernel* if $x \mapsto K(x,B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R}^{\rho-1})$ and $B \mapsto K(x,B)$ is a finite signed measure (see Rudin (1987)) fulfilling $K(x,[0,1]^{\rho-1}) = 1$ for every fixed

$x \in \mathbb{R}$. Every signed Markov kernel induces a signed measure on $[0,1]^\rho$ via equation (2). For more details and properties of conditional expectation and regular conditional distributions see Kallenberg (1997); Klenke (2007).

3. The two-dimensional case

Given an increasing homeomorphism $h : [0,1] \to [0,1]$ fulfilling $h(x) < x$ for every $x \in (0,1)$ the (compact) set $\Gamma^\star(h) := \Gamma(h) \cup \Gamma(h^{-1})$ will be called two-dimensional *hairpin*. The class of all these homeomorphisms will be denoted by $\mathcal{H}$. For $h \in \mathcal{H}$ the inverse h^{-1} is strictly increasing and fulfills $h^{-1}(x) > x$ for every $x \in (0,1)$. As usual we will write h^j (h^{-j}) for the j-times composition of h (h^{-1}) with itself for every $j \in \mathbb{N}$ and set $h^0 := id_{[0,1]}$. Following Nelsen and Fredricks (1997); Seethoff and Shiflett (1977/78) we will call $A \in \mathcal{C}_2$ *two-dimensional hairpin copula* if $Supp(A) \subseteq \Gamma^\star(h)$ for some $h \in \mathcal{H}$. The objective of this section is to recall some results on hairpin copulas going back to Nelsen and Fredricks (1997); Seethoff and Shiflett (1977/78) before studying the general multivariate setting.

We start with the following theorem collecting some results going back to Seethoff and Shiflett (1977/78) (also see Kamiński et al. (1990, 1987/88)) and give an alternative very simple proof using Markov kernels:

Theorem 1. *For every $h \in \mathcal{H}$ there is at most one copula $A \in \mathcal{C}_2$ with $Supp(A) \subseteq \Gamma^\star(h)$. If such a copula exists it is necessarily symmetric and for every $x \in (0,1)$ the set O_x, defined by $O_x = \bigcup_{n\in\mathbb{Z}}[h^{2n+2}(x), h^{2n+1}(x)]$, fulfills $\lambda(O_x) = 1/2$. Furthermore the diagonal δ of A fulfills*

$$2h(x) = \delta(x) + \delta(h(x)) \tag{4}$$

for every $x \in [0,1]$.

Proof: Suppose that A is a copula whose support is contained in $\Gamma^\star(h)$. Then its kernel K_A has to be of the form

$$K_A(x,F) = w(x)\mathbf{1}_F(h(x)) + (1-w(x))\mathbf{1}_F(h^{-1}(x)) \tag{5}$$

for some Borel-measurable weight function $w : [0,1] \to [0,1]$. Hence for every $y \in [0,1]$ we have $y = \int_{[0,1]} K_A(x,[0,y])d\lambda(x) = \int_{[0,h^{-1}(y)]} wd\lambda + \int_{[0,h(y)]}(1-w)d\lambda$, so

$$\int_{[h(y),h^{-1}(y)]} wd\lambda = y - h(y). \tag{6}$$

In particular, for every $x \in (0,1)$ and $n \in \mathbb{Z}$, considering $y = h^{2n+1}(x)$ we get

$$\int_{[h^{2n+2}(x),h^{2n}(x)]} wd\lambda = h^{2n+1}(x) - h^{2n+2}(x). \tag{7}$$

Summing up over all $n \in \mathbb{N}_0$ therefore directly yields

$$\int_{[0,x]} wd\lambda = \sum_{n=1}^{\infty}(-1)^{n+1}h^n(x) \tag{8}$$

for $x \in [0,1)$, implying that w is determined uniquely by h, i.e. A is unique. If A was not symmetric, then A^t would be another copula with $Supp(A^t) \subseteq \Gamma^\star(h)$. Summing up (7) over all $n \in \mathbb{Z}$ and using symmetry directly yields $\lambda(O_x) = 1/2$. Since A is symmetric we have

$$\frac{\delta(x)}{2} = \int_{[0,x]} w d\lambda = \int_{[0,h(x)]} (1-w) d\lambda. \tag{9}$$

for every $y \in [0,1]$, from which equation (4) follows immediately ■

As direct consequence of Theorem 1 every hairpin copula is extreme in $\mathcal{C}_2$. Interestingly, all hairpin copulas are diagonal copulas since the following result holds (see Nelsen and Fredricks (1997)):

Theorem 2. *Suppose that $A \in \mathcal{C}$ fulfills $Supp(A) \subseteq \Gamma^\star(h)$ for some $h \in \mathcal{H}$ and let δ_A denote the corresponding diagonal. Then A coincides with E_{δ_A}.*

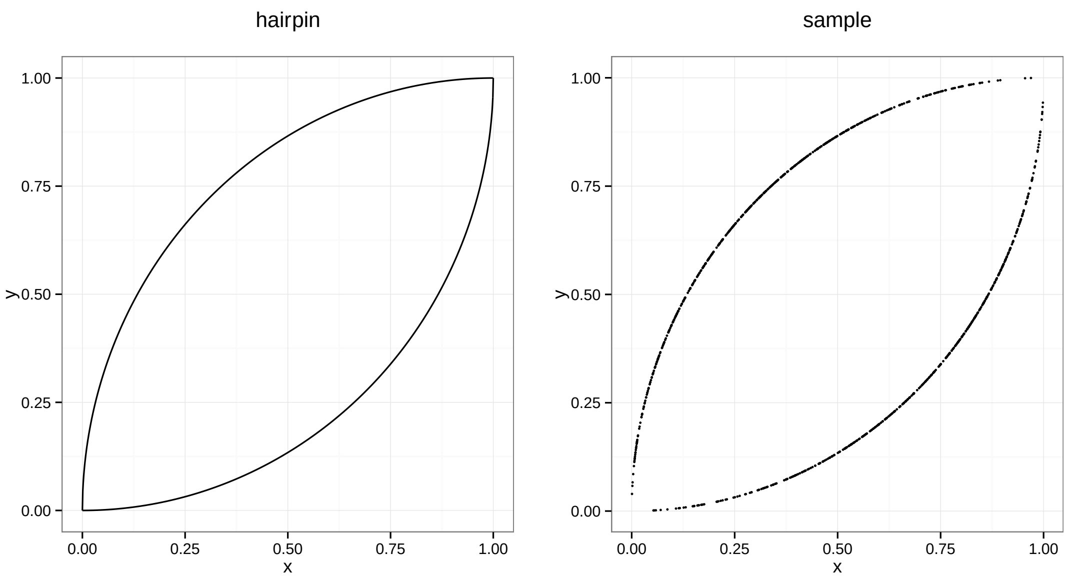

Figure 1: $\Gamma^\star(h)$ for h from Example 3 (left) as well as a sample of size $n = 1000$ from the diagonal copula E_δ (right).

We will call a diagonal *strict* if, firstly, $\delta(x) < x$ for all $x \in (0,1)$ and, secondly, δ and $\tilde{\delta} : [0,1] \to [0,1]$, defined by

$$\tilde{\delta}(x) = 2x - \delta(x) \tag{10}$$

are strictly increasing (hence homeomorphisms). It has been shown in Nelsen and Fredricks (1997) that E_δ is a hairpin copula if and only if δ is strict.

Example 3. Consider the strict diagonal $\delta(x) = x^2$. Then $A := E_\delta$ is a hairpin copula and the corresponding homeomorphism h is given by $h(x) = 1 - \sqrt{1-x^2}$, i.e. $\Gamma^\star(h)$ is the union of two quarter-circles (see Nelsen and Fredricks (1997)). Figure 1 shows perspective plots of A and the function $(x,y) \mapsto K_A(x,[0,y])$.

We close this section with the following example showing that hairpin copulas may be proper generalized shuffles of (the upper Fréchet-Hoeffding bound) M in the sense of Durante et al. (2009) (also see Trutschnig and Fernández Sánchez (2013)), so in particular, (mutually) completely dependent copulas (see Trutschnig (2011) and the references therein). Necessary and sufficient conditions for a shuffle of M to be a diagonal copula can be found in Nelsen and Fredricks (1997) and in Fernández Sánchez and Trutschnig (2014a). In the latter article, the following result was proved:

Theorem 4. *Suppose that δ is a two-dimensional diagonal. Then the diagonal copula E_δ is a generalized shuffle of M whose support is contained in the graph of a λ-preserving bijection $S : [0,1] \to [0,1]$ fulfilling $S \circ S = id_{[0,1]}$ if and only if for almost every $x \in [0,1]$ either $\delta'(x) \in \{0,2\}$ or $\delta(x) = x$ holds.*

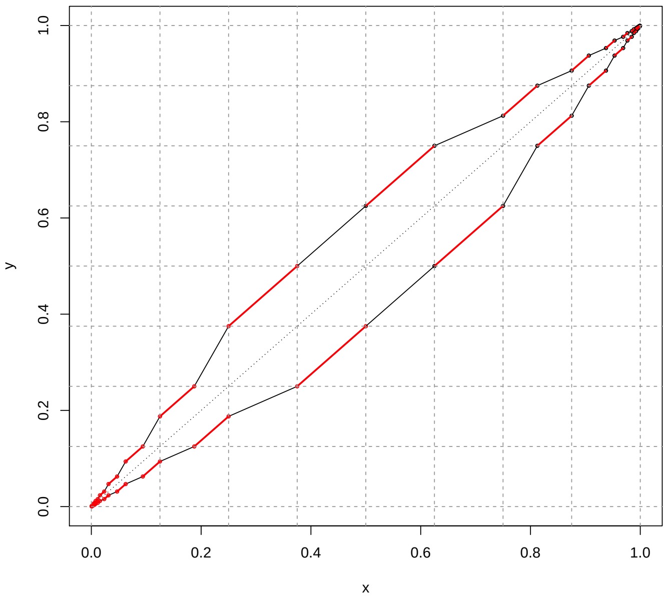

Figure 2: Completely dependent copula A from Example 5 whose support (red) is strictly contained in a hairpin (black).

Example 5. Consider the partition $\mathfrak{Z} = \{I_1, J_1, I_2, J_2, \ldots\}$ of $[0,1]$, whereby

$$I_n = \left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right] \quad \text{and} \quad J_n = \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$$

for every $n \in \mathbb{N}$, and the ordinal sum (see Nelsen (2006)) of the copula E_{δ_W} with respect to $\mathfrak{Z}$, whereby δ_W denotes the diagonal of the lower Fréchet-Hoeffding bound W. Figure 2 depicts a hairpin $\Gamma^\star(h)$ in black and the support of the corresponding ordinal sum in red. It follows immediately from the construction that A is a generalized shuffle of M. As hairpin copula A is an extreme point in $\mathcal{C}_2$ - the corresponding diagonal is an extreme point in the convex set $\mathcal{D}_2$ (also see Jaworski and Rychlik (2008)).

4. The general multivariate case

We will now concentrate on the case of dimension $\rho \geq 3$ and start with the first possible extension of the notion of hairpins (the alternative will be studied afterwards): For every $h \in \mathcal{H}$ and every $i \in \{1, \dots, \rho\}$ define

$$\Gamma^i(h) := \big\{(x, \dots, x, h(x), x, \dots, x) : \, x \in [0,1]\big\},$$

whereby $h(x)$ occupies the i-th position. Then the (compact) set $\Gamma^\star(h) := \bigcup_{i=1}^{\rho} \Gamma^i(h)$ will be called a *ρ-dimensional sub-hairpin*. As in the two-dimensional case we will show in the sequel that for each ρ-dimensional sub-hairpin $\Gamma^\star(h)$ there exists at most one copula $A \in \mathcal{C}_\rho$ with $Supp(A) \subseteq \Gamma^\star(h)$. We will call $A \in \mathcal{C}_\rho$ a *ρ-dimensional sub-hairpin copula* if $Supp(A) \subseteq \Gamma^\star(h)$ for some $h \in \mathcal{H}$.
Whenever $A \in \mathcal{C}_\rho$ with $Supp(A) \subseteq \Gamma^\star(h)$ exists we will write

$$F_i(x) := \mu_A\big([0,x]^\rho \cap \Gamma^i(h)\big) \tag{11}$$

for every $x \in [0,1]$ and $i \in \{1, \dots, \rho\}$, whereby μ_A denotes the corresponding ρ-stochastic measure.

Lemma 6. *Suppose that $h \in \mathcal{H}$ and that there exists a copula $A \in \mathcal{C}_\rho$ with $Supp(A) \subseteq \Gamma^\star(h)$. Then all functions F_i, defined according to equation (11), coincide.*

Proof: Since A is a copula, each distribution function F_i is Lipschitz continuous, hence absolutely continuous, and fulfills $F_i(0) = 0$. For each $x \in [0,1]$ we have

$$\begin{aligned} h(x) &= \mu_A\Big([0,h(x)] \times [0,1]^{\rho-1}\Big) = \sum_{j=1}^{\rho} \mu_A\Big(\big([0,h(x)] \times [0,1]^{\rho-1}\big) \cap \Gamma^j(h)\Big) \\ &= \mu_A\Big(\underbrace{\big([0,h(x)] \times [0,1]^{\rho-1}\big) \cap \Gamma^1(h)}_{[0,x]^\rho \cap \Gamma^1(h)}\Big) + \sum_{j=2}^{\rho} \mu_A\Big(\underbrace{\big([0,h(x)] \times [0,1]^{\rho-1}\big) \cap \Gamma^j(h)}_{[0,h(x)]^\rho \cap \Gamma^j(h)}\Big) \\ &= F_1(x) + \sum_{j=2}^{\rho} F_j(h(x)). \end{aligned}$$

Analogous arguments show that for each $i \in \{1, \dots, \rho\}$ and $x \in [0,1]$ we have

$$h(x) = F_i(x) + \sum_{j \neq i}^{\rho} F_j(h(x)). \tag{12}$$

Considering Equation (12) for i_1, i_2 with $i_1 \neq i_2$ yields $F_{i_2}(x) - F_{i_1}(x) = F_{i_2}(h(x)) - F_{i_1}(h(x))$ for every $x \in [0,1]$. Taking into account that $\lim_{n\to\infty} h^n(x) = 0$ holds for every $x \in [0,1)$ and using continuity of all F_i therefore shows $F_{i_2} = F_{i_1}$. ■

Theorem 7. *Suppose that $h \in \mathcal{H}$. Then there exists at most one copula $A \in \mathcal{C}_\rho$ with $Supp(A) \subseteq \Gamma^\star(h)$. Furthermore the function $F := F_i$ from equation (11) can be expressed in terms of h as*

$$F(x) = \sum_{j=0}^{\infty} (-1)^j \frac{h^{-j}(x)}{(\rho-1)^{j+1}}. \tag{13}$$

Proof: Since we have $F = F_i$ for all $i \in \{1, \ldots, \rho\}$ equation (12) reduces to

$$h(z) = F(z) + (\rho - 1)F(h(z)). \tag{14}$$

for all $z \in [0,1]$. Considering $h(z) = x$ yields

$$F(x) = \frac{x}{\rho - 1} - \frac{F(h^{-1}(x))}{\rho - 1} \tag{15}$$

from which equation (13) directly follows. Having this, uniqueness of the copula A is clear. ■

Corollary 8. *If for $h \in \mathcal{H}$ a copula A with $Supp(A) \subseteq \Gamma^\star(h)$ exists, then it is an extreme point of $\mathcal{C}_\rho$.*

Corollary 8 assures that, for given $h \in \mathcal{H}$, there is at most one copula fulfilling $Supp(A) \subseteq \Gamma^\star(h)$ but gives no conditions (h has to fulfill) under which such a copula really exists. In Theorem 13, however, we will state a simple sufficient condition, based on which we show the existence of sub-hairpin copulas.

Remark 9. As a direct consequence of Theorem 7, we will write A^h for the unique copula whose support is contained in $\Gamma^\star(h)$ (if it exists). Since all F_i coincide, the diagonal δ_{A^h} of A^h can be expressed as

$$\delta_{A^h}(x) = \rho F(x) = \rho \sum_{j=0}^{\infty} (-1)^j \frac{h^{-j}(x)}{(\rho - 1)^{j+1}}. \tag{16}$$

Note that, contrary to the two-dimensional case described in equation (8), the series in (16) converges absolutely and uniformly on $[0,1]$.

As in the previous section we can easily calculate the Markov kernel K_{A^h} of the sub-hairpin copula A^h.

Theorem 10. *Suppose that $h \in \mathcal{H}$, that $\delta \in \mathcal{D}_\rho$ and that $F := \frac{1}{\rho}\delta$ fulfills equation (14) for every $z \in [0,1]$. Furthermore let $w : [0,1] \to [0,1]$ be a measurable function fulfilling $w = F'$ a.e. Then $K : [0,1] \times \mathcal{B}([0,1]^{\rho-1}) \to \mathbb{R}$, defined by*

$$K(x,G) = \mathbf{1}_G(h^{-1}(x), \ldots, h^{-1}(x))\big(1 - (\rho-1)w(x)\big) + \sum_{i=2}^{\rho} \mathbf{1}_G(x, \ldots, x, \underbrace{h(x)}_{(i-1)\text{-th coordinate}}, x, \ldots, x)w(x), \tag{17}$$

is a signed Markov kernel whose corresponding signed measure μ is ρ-stochastic. The support (of the Jordan decomposition) of μ is contained in $\Gamma^\star(h)$.

Proof: Obviously K is a signed Markov kernel that concentrates the negative mass (if any) on $\Gamma^1(h)$. It suffices to prove that the induced signed measure μ is ρ-stochastic. In fact, if $G := [0,y] \times [0,1]^{\rho-2}$ for some $y \in [0,1]$, then, using equations (14) and (15), we have

$$\begin{aligned}
\int_{[0,1]} K(x,G)d\lambda(x) &= \int_{[0,1]} \mathbf{1}_{[0,h(y)]}(x)\big(1 - (\rho-1)w(x)\big)d\lambda(x) + \int_{[0,1]} \mathbf{1}_{[0,h^{-1}(y)]}(x)w(x)d\lambda(x) \\
&\quad + (\rho - 2)\int_{[0,1]} \mathbf{1}_{[0,y]}(x)w(x)d\lambda(x) \\
&= h(y) - (\rho-1)F(h(y)) \;+\; F(h^{-1}(y)) \;+\; (\rho-2)F(y) \\
&= (\rho-1)F(y) + F(h^{-1}(y)) = y.
\end{aligned}$$

Sets G of the form $G := [0,1] \times \cdots \times [0,y] \times [0,1] \times \cdots \times [0,1]$ can be handled analogously. ■

In case of $F' \leq \frac{1}{\rho-1}$ a.e., K according to equation (17) is a Markov kernel and the induced (probability) measure is ρ-stochastic. We will come back to this condition after proving that ρ-dimensional sub-hairpin copulas may be expressed in a similar (but not identical) way as diagonal copulas studied by Jaworski (2009). In the sequel τ will denote the cyclic permutation $\tau : \{1,\ldots,\rho\} \to \{1,\ldots,\rho\}$, defined by $\tau(i) = i+1 \,(\mathrm{mod}\, \rho)$.

Theorem 11. *Suppose that $h \in \mathcal{H}$ and that $A^h \in \mathcal{C}_\rho$ exists. Then A^h is given by*

$$A^h(x_1,\ldots,x_\rho) = \frac{1}{\rho}\sum_{i=1}^{\rho} \min\Big\{\delta_{A^h}(x_{\tau^i(1)}),\ldots,\delta_{A^h}(x_{\tau^i(\rho-1)}),\, \rho x_{\tau^i(\rho)} - (\rho-1)\delta_{A^h}(x_{\tau^i(\rho)})\Big\} \quad (18)$$

for all $x_1,\ldots,x_\rho \in [0,1]$.

Proof: For every $i \in \{1,\ldots,\rho\}$ and $G \in \mathcal{B}([0,1]^\rho)$ define $\mu_i(G) := \mu_{A^h}(G \cap \Gamma^i(h))$ and set $\hat{\mu_i} := \rho\mu_i$. Then $\hat{\mu_i}$ is a probability measure. We will calculate the marginals of $\hat{\mu_i}$. Letting $\pi_j : [0,1]^\rho \to [0,1]$ denote the projection onto the j-th coordinate for every $j \in \{1,\ldots,\rho\}$ we have the following: If $j = i$ then

$$\hat{\mu_i}^{\pi_i}([0,x]) \;=\; \rho\mu_A\Big(\big([0,1]^{i-1} \times [0,x] \times [0,1]^{\rho-i}\big) \cap \Gamma^i(h)\Big) = \rho F(h^{-1}x) = \rho x - \rho(\rho-1)F(x),$$

and in case of $j \neq i$ we get

$$\hat{\mu_i}^{\pi_j}([0,x]) \;=\; \rho\mu_A\Big(\big([0,1]^{j-1} \times [0,x] \times [0,1]^{\rho-j}\big) \cap \Gamma^i(h)\Big) = \rho F(x).$$

Since $\hat{\mu_i}$ has mass only on $\Gamma^i(h)$, Sklar's theorem (also see Scarsini (1989)) implies

$$\hat{\mu_i}\big(\times_{j=1}^{\rho} [0,x_j]\big) = \min\big\{\rho F(x_{\tau^i(1)}),\ldots,\rho F(x_{\tau^i(\rho-1)}),\, \rho x_{\tau^i(\rho)} - \rho(\rho-1)F(x_{\tau^i(\rho)})\big\}.$$

Considering $\mu_{A^h} = \frac{1}{\rho}\sum_{i=1}^{\rho} \hat{\mu_i}$ as well as $\delta_{A^h} = \rho F$ completes the proof. ■

As a consequence, for A^h to be a copula, the condition $\delta' \leq \frac{\rho}{\rho-1}$ a.e. must hold. Conversely, if $\delta' \leq \frac{\rho}{\rho-1}$ a.e. holds, then the right-hand-side of equation (18) is a copula:

Theorem 12. *Suppose that δ is a ρ-dimensional diagonal fulfilling $\delta' \leq \frac{\rho}{\rho-1}$ a.e., then*

$$H_\delta(x_1,\ldots,x_\rho) = \frac{1}{\rho}\sum_{i=1}^{\rho} \min\Big\{\delta(x_{\tau^i(1)}),\ldots,\delta(x_{\tau^i(\rho-1)}),\, \rho x_{\tau^i(\rho)} - (\rho-1)\delta(x_{\tau^i(\rho)})\Big\}$$

is a copula with diagonal δ.

Proof: Since the functions $x \mapsto \delta(x)$ and $x \mapsto \rho x - (\rho-1)\delta(x)$ are absolutely continuous distribution functions the theorem directly follows from Sklar's theorem. ■

Based on Theorem 12 we can finally prove the existence of copulas whose support are ρ-dimensional hairpins: It is straightforward to verify that $\mathcal{D}_\rho$ contains diagonals δ such that firstly, $\delta(x) < x$ for all $x \in (0,1)$, and, secondly, δ and the function $\tilde{\delta} : [0,1] \to [0,1]$, defined by

$$\tilde{\delta}(x) = \rho x - (\rho-1)\delta(x) \quad (19)$$

are strictly increasing (hence homeomorphisms). We will refer to such δ as *strict* in the sequel. Suppose now that $\delta \in \mathcal{D}_\rho$ is strict and set $h^{-1} = \delta^{-1} \circ \tilde{\delta}$. Then h^{-1} is a homeomorphism and we have

$$\delta(h^{-1}(x)) = \rho x - (\rho - 1)\delta(x) = \tilde{\delta}(x)$$

as well as $h^{-1}(x) = \delta^{-1} \circ \tilde{\delta}(x) > \tilde{\delta}(x) > x$ for every $x \in (0, 1)$. Hence $h \in \mathcal{H}$ follows. For this h the copula A^h exists, is given by equation (18), and even fulfills $Supp(A^h) = \Gamma^\star(h)$ since $F_i = F = \frac{\delta}{\rho}$ and δ is strictly increasing. In other words, we have shown the following existence result for sub-hairpin copulas:

Theorem 13. *Suppose that δ is a strict ρ-dimensional diagonal, let $\tilde{\delta}$ be defined according to equation (19) and set $h := \tilde{\delta}^{-1} \circ \delta$. Then $\Gamma^\star(h)$ is the support of a (unique) copula.*

Remark 14. As mentioned in the Introduction, to the best of the authors' knowledge, taking apart multivariate shuffles of M, sub-hairpin copulas are the first examples of extreme point of $\mathcal{C}_\rho$ appearing in the literature. Shuffles of M are easily seen to be extreme points of $\mathcal{C}_\rho$ that concentrate their mass on finitely many lines and the lines may even be chosen in such a way that there is a hyperplane containing all of them. We refer to Mikusiński and Taylor (2010) and Fernández Sánchez and Trutschnig (2014b) for more details and remark that a full characterization of all extreme points of $\mathcal{C}_\rho$ still seems a very difficult task.

Thanks to their very simple explicit form and their simple Markov kernel (17) samples of sub-hairpin copulas may easily be generated. One may proceed as in the following example.

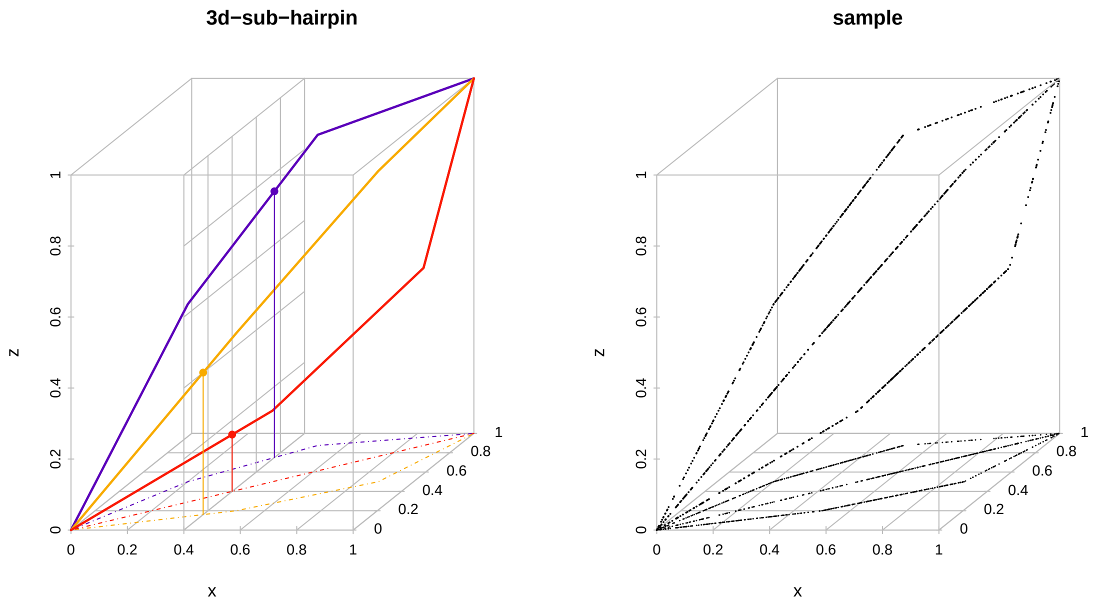

Figure 3: $\Gamma^\star(h)$ for h from Example 15 (left) as well as a sample of size $n = 1000$ from the copula A^h (right). The three points on the plane $x = \frac{2}{5}$ in the left plot depict the point masses of the probability measure $G \mapsto K_{A^h}(\frac{2}{5}, G)$ according to Theorem 10. The lines/points in the planes $z = 0$ are the corresponding projections.

Example 15. We consider the case $\rho = 3$ and the diagonal $\delta \in \mathcal{D}_3$ given by

$$\delta(x) = \begin{cases} \frac{2}{3}x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{1}{3} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then δ fulfills the properties of Theorem 13 and the corresponding homeomorphism h is given by

$$h(x) = \begin{cases} \frac{2}{5}x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{4}{5}x - \frac{1}{5} & \text{if } x \in (\frac{1}{2}, \frac{7}{8}] \\ 4x - 3 & \text{if } x \in (\frac{7}{8}, 1]. \end{cases}$$

Figure 3 depicts the hairpin $\Gamma^\star(h)$ as well as a sample of size $n = 1000$ from the copula A^h, whereby the sample was generated in two steps using the simple form of the Markov kernel (17): First a sample $x_1, \ldots, x_n$ from $X \sim \mathcal{U}(0,1)$ is generated, then for given $x = x_i$, we randomly draw one of the values

$$\big\{(h^{-1}(x), h^{-1}(x)), (h(x), x), (x, h(x))\big\} \in [0,1]^2$$

according to the probabilities $\big(1 - \frac{2\delta'(x)}{3}, \frac{\delta'(x)}{3}, \frac{\delta'(x)}{3}\big)$.

So far in this section we have only studied copulas having support in sub-hairpins $\Gamma^\star(h)$ with $h \in \mathcal{H}$. Starting with a homeomorphism h fulfilling $h^{-1} \in \mathcal{H}$ the corresponding set $\Gamma^\star(h) = \bigcup_{i=1}^{\rho} \Gamma^i(h)$ will be called *ρ-dimensional super-hairpin*. Note that in this case we have $h(x) > x$ for all $x \in (0,1)$, so $h \notin \mathcal{H}$. We will call $A \in \mathcal{C}_\rho$ *ρ-dimensional super-hairpin copula* if $Supp(A) \subseteq \Gamma^\star(h)$ for some h with $h^{-1} \in \mathcal{H}$. Given a super-hairpin $\Gamma^\star(h)$ we may ask the same questions about existence and uniqueness of ρ-dimensional copulas A with $Supp(A) \subseteq \Gamma^\star(h)$ and proceed analogously as for sub-hairpins. We will state the corresponding results for super-hairpins now (the proofs are analogous to the sub-hairpin setting) and then shortly discuss a very simple transformation mapping super-hairpin copulas to sub-hairpin copulas and vice versa. To simplify notation $\mathcal{H}^{-1}$ will denote the family of all h with $h^{-1} \in \mathcal{H}$.
Suppose that $h \in \mathcal{H}^{-1}$ and that A is a ρ-dimensional copula fulfilling $Supp(A) \subseteq \Gamma^\star(h)$. Again we will write

$$F_i(x) := \mu_A\big([0,x]^\rho \cap \Gamma^i(h)\big). \tag{20}$$

Analogously to the sub-hairpin setting it can shown that all F_i coincide, that $F := F_i$ is absolutely continuous and that the following result holds:

Theorem 16. *Suppose that $h \in \mathcal{H}^{-1}$. Then there exists at most one copula $A \in \mathcal{C}_\rho$ with $Supp(A) \subseteq \Gamma^\star(h)$. Furthermore the function $F := F_i$ from equation (20) can be expressed in terms of h as*

$$F(x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{h^{-j}(x)}{(\rho-1)^j} \tag{21}$$

and the following interrelation between F and h holds for every $x \in [0,1]$:

$$x = F(x) + (\rho - 1)F(h(x)) \tag{22}$$

Corollary 17. *If for $h \in \mathcal{H}$ a copula A with $Supp(A) \subseteq \Gamma^\star(h)$ exists, then it is an extreme point of $\mathcal{C}_\rho$.*

Remark 18. As a direct consequence of Theorem 16 we will write A^h for the unique copula with $Supp(A) \subseteq \Gamma^\star(h)$ if it exists. Additionally, the diagonal δ_{A^h} of A^h can be expressed as

$$\delta_{A^h}(x) = \rho F(x) = \rho \sum_{j=1}^{\infty} (-1)^{j+1} \frac{h^{-j}(x)}{(\rho-1)^j}. \tag{23}$$

As in the sub-hairpin setting, the series in (23) converges absolutely and uniformly on $[0,1]$.

Theorem 19. *Suppose that $h \in \mathcal{H}^{-1}$, that $\delta \in \mathcal{D}_\rho$ and that $F := \frac{1}{\rho}\,\delta$ fulfills equation (22) for every $x \in [0,1]$. Furthermore let $w : [0,1] \to [0,1]$ be a measurable function fulfilling $w = F'$ a.e. Then $K : [0,1] \times \mathcal{B}([0,1]^{\rho-1}) \to [0,1]$, defined by*

$$K(x,G) = \mathbf{1}_G(h^{-1}(x),\dots,h^{-1}(x))\, w(x) + \sum_{i=2}^{\rho} \mathbf{1}_G(x,\dots,x,\underbrace{h(x)}_{(i-1)\text{-th coordinate}},x,\dots,x)\, \frac{1-w(x)}{\rho-1}, \tag{24}$$

is the Markov kernel of a ρ-stochastic measures whose support is contained in $\Gamma^\star(h)$.

Note that, contrary to the sub-hairpin case we do not have any restriction on the derivative δ' of the diagonal δ here. Moreover super-hairpin copulas even coincide with diagonal copulas studied by Jaworski (2009) since the following result holds:

Theorem 20. *Suppose that $h \in \mathcal{H}^{-1}$ and that $A^h \in \mathcal{C}_\rho$ exists. Then A^h is given by*

$$A^h(x_1,\dots,x_\rho) = \frac{1}{\rho}\sum_{i=1}^{\rho} \min\left\{ f(x_{\tau^i(1)}),\dots,f(x_{\tau^i(\rho-1)}),\, \delta_{A^h}(x_{\tau^i(\rho)}) \right\} \tag{25}$$

for all $x_1,\dots,x_\rho \in [0,1]$, whereby $f(x) = \frac{\rho x - \delta_{A^h}(x)}{\rho-1}$ for every $x \in [0,1]$.

It has already been shown in Proposition 2.1 in Jaworski (2009) that, given a ρ-dimensional diagonal δ, all functions A of the form (25) are copulas. As a straightforward consequence we may therefore easily construct copulas whose supports are super-hairpins: Suppose that δ is a strictly increasing diagonal fulfilling $\delta(x) < x$ for all $x \in (0,1)$ and for which the function $\tilde{\delta} : [0,1] \to [0,1]$, defined by

$$\tilde{\delta}(x) = \frac{\rho\, x}{\rho-1} - \frac{\delta(x)}{\rho-1}, \tag{26}$$

is strictly increasing (hence a homeomorphism) too. Then, using $\delta = \rho F$ and equation (22), we have

$$\delta(h(x)) = \tilde{\delta}(x),$$

implying $h = \delta^{-1} \circ \tilde{\delta}$. Considering $h(x) = \delta^{-1} \circ \tilde{\delta}(x) > \tilde{\delta}(x) > x$ for every $x \in (0,1)$, $h \in \mathcal{H}^{-1}$ follows. For this $h \in \mathcal{H}^{-1}$ the copula A^h exists, is given by equation (20), and even fulfills $Supp(A^h) = \Gamma^\star(h)$. In other words, we have shown the following existence result for super-hairpin copulas:

Theorem 21. *Suppose that δ is a strictly increasing ρ-dimensional diagonal fulfilling $\delta(x) < x$ for all $x \in (0,1)$, and for which the function $\tilde{\delta} : [0,1] \to [0,1]$, defined by (26), is strictly increasing too. Then for $h := \tilde{\delta}^{-1} \circ \delta$, the super-hairpin $\Gamma^\star(h)$ is the support of a copula.*

Example 22. We consider again the case $\rho = 3$ and the diagonal $\delta \in \mathcal{D}_3$ given by

$$\delta(x) = \begin{cases} \frac{2}{3}x & \text{if } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{1}{3} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then δ fulfills the properties of Theorem 21 and the corresponding homeomorphism $h \in \mathcal{D}^{-1}$ is given by

$$h(x) = \begin{cases} \frac{7}{4}x & \text{if } x \in [0, \frac{2}{7}] \\ \frac{7}{8}x - \frac{1}{4} & \text{if } x \in (\frac{2}{7}, \frac{1}{2}] \\ \frac{5}{8}x - \frac{3}{8} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Figure 4 depicts the hairpin $\Gamma^\star(h)$ as well as a sample of size $n = 1000$ from the copula A^h.

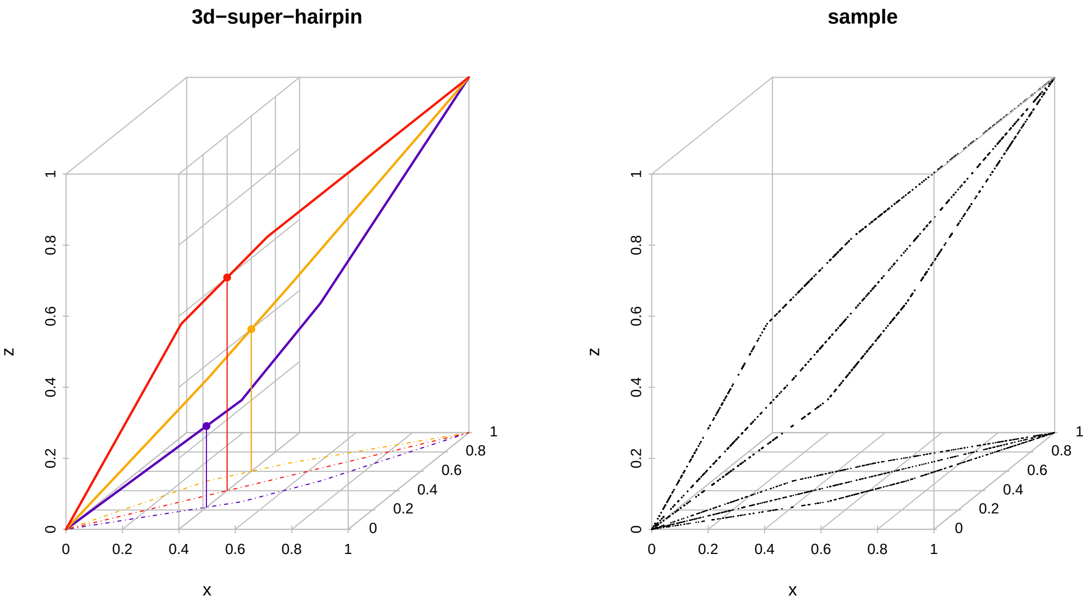

Figure 4: $\Gamma^\star(h)$ for h from Example 22 (left) as well as a sample of size $n = 1000$ from the copula A^h (right). The three points on the plane $x = \frac{2}{5}$ in the left plot depict the point masses of the probability measure $G \mapsto K_{A^h}(\frac{2}{5}, G)$ according to Theorem 19. The lines/points in the planes $z = 0$ are the corresponding projections.

Not surprisingly super-hairpin copulas and sub-hairpin copulas are strongly connected. The isometry $T : [0,1]^\rho \to [0,1]^\rho$, defined by

$$T(x_1, \ldots, x_\rho) = (1 - x_1, 1 - x_2, \ldots, 1 - x_\rho),$$

obviously maps $\mathcal{C}_\rho$ into $\mathcal{C}_\rho$. In fact, the push forward μ^T of any ρ-stochastic measure μ is ρ-stochastic again - if $(U_1, \ldots, U_\rho)$ has distribution function A then $(1 - U_1, \ldots, 1 - U_\rho)$ has distribution function $T(A)$. Furthermore considering

$$\begin{aligned} T(\Gamma^j(h)) &= \big\{(1 - z, \ldots, 1 - z, 1 - h(z), 1 - z, \ldots, 1 - z) : z \in [0,1]\big\} \\ &= \big\{(x, \ldots, x, 1 - h(1 - x), x, \ldots, x) : x \in [0,1]\big\} \end{aligned}$$

and the fact that the mapping Φ, defined by $\Phi(h)(x) := 1 - h(1-x)$, maps $\mathcal{H}^{-1}$ into $\mathcal{H}$ (and vice versa) it follows immediately that for every super-hairpin $\Gamma^\star(h)$ the set $T(\Gamma^\star(h)) = \Gamma^\star(\Phi(h))$ is a sub-hairpin. Consequently there is a one-to-one correspondence between copulas whose support is contained in a super-hairpin and copulas whose support is contained in a sub-hairpin. Considering this, all existence and uniqueness results for super-hairpins follow immediately from the corresponding ones from sub-hairpins. Since, however, the Markov kernel and the explicit form (25) do not follow directly we stated the results for super-hairpins explicitly.

Example 23. Consider the homeomorphism $\Phi(h)$ for h from Example 15, i.e.

$$\Phi(h)(x) = \begin{cases} 4x & \text{if } x \in [0, \frac{1}{8}] \\ \frac{4}{5}x + \frac{2}{5} & \text{if } x \in (\frac{1}{8}, \frac{1}{2}] \\ \frac{2}{5}x + \frac{3}{5} & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Then $\Gamma^\star(\Phi(h))$ is a 3-dimensional super-hairpin and $A := T(A^h)$ is the unique copula fulfilling $Supp(A) = \Gamma^\star(\Phi(h))$.

Remark 24. Considering Theorem 13 and Theorem 21 it is somehow surprising that the condition for the corresponding ρ-dimensional diagonal δ is much more restrictive in the case of sub-hairpins. In fact, we need $\delta' \le \frac{\rho}{\rho-1}$ a.e. which especially for high dimensions ρ is a very strong restriction. Considering the afore-mentioned transformations T, Φ we may easily check how the diagonals $\delta_{A^{\Phi h}}$ and δ_{A^h} of $A^{\Phi h}$ and A^h are interrelated: It is straightforward to verify that for every $h \in \mathcal{H}^{-1}$, $n \in \mathbb{N}$ and $x \in [0,1]$

$$(\Phi h)^{-n}(x) = \Phi(h^{-n})(x)$$

holds. Having this it follows that for $h \in \mathcal{H}^{-1}$ we have

$$\begin{aligned} \delta_{A^{\Phi h}}(x) &= \rho \sum_{j=0}^{\infty} (-1)^j \frac{(\Phi(h))^{-j}(x)}{(\rho-1)^{j+1}} = \rho \sum_{j=0}^{\infty} (-1)^j \frac{1 - h^{-j}(1-x)}{(\rho-1)^{j+1}} \\ &= 1 + \rho \sum_{j=0}^{\infty} (-1)^{j+1} \frac{h^{-j}(1-x)}{(\rho-1)^{j+1}} \\ &= 1 + \frac{\rho}{\rho-1}\Big(-(1-x) + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{h^{-j}(1-x)}{(\rho-1)^j} \Big) \\ &= 1 - \frac{\rho}{\rho-1}(1-x) + \frac{1}{\rho-1}\, \delta_{A^h}(1-x). \end{aligned}$$

In particular, for a.e. $x \in (0,1)$ we have

$$\delta'_{A^{\Phi h}}(x) = \frac{\rho}{\rho-1} - \frac{1}{\rho-1}\, \delta'_{A^h}(1-x)$$

from which we again deduce the restrictive condition $\delta'_{A^{\Phi h}} \le \frac{\rho}{\rho-1}$ a.e. for sub-hairpins.

We close the paper with the following remark.

Remark 25. In case we start with a homeomorphism h fulfilling only that there exist finitely or infinitely many (non-degenerated) pairwise disjoint intervals (a_i, b_i) such that on each interval we either have (i) $h(x) > x$ or (ii) $h(x) < x$ or (iii) $h(x) = x$ for all $x \in (a_i, b_i)$ then the problem of finding a copula whose support is contained in $\Gamma^\star(h)$ reduces to finding copulas A_i whose support is contained in $\Gamma^\star(h_i)$ for all i, whereby h_i is given by

$$h_i(x) = \frac{h(a_i + (b_i - a_i)x) - h(a_i)}{h(b_i) - h(a_i)}.$$

Notice that there exists a unique copula A whose support is contained in $\Gamma^\star(h)$ if (and only if) all such A_i exist (in the positive case the copula A is then the ordinal sum of all A_i).

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