Baire category results for exchangeable copulas

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Abstract

Considering two different metrics on the space of two-dimensional copulas $\mathscr{C}$ we prove some Baire category results for important subclasses of copulas, including the families of exchangeable, associative, and Archimedean copulas. From the point of view of Baire categories, with respect to the uniform metric d_∞, a typical copula is not symmetric and a typical symmetric copula is not associative, whereas a typical associative copula is Archimedean and a typical Archimedean copula is strict. The results in particular answer the open question posed in [1] whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

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1. Introduction

Copulas are (the restriction to $[0,1]^d$ of) distribution functions of probability measures on $[0,1]^d$ ($d \geq 2$) whose one–dimensional marginals are uniformly distributed on $[0,1]$. Considering that, according to Sklar's theorem [6, 7, 16, 17], every distribution function of a random vector can be expressed as composition of a suitable copula and the corresponding marginal distribution functions, copulas are the natural building blocks of modern multivariate analysis. Having a variety of copulas at one's disposal may help in building different stochastic models that possibly differ in features being of essential importance in applications (e.g. tail behavior). Nevertheless, taking into account numerical and analytic aspects, in practice copulas are chosen from few well–studied standard (parametric or semi–parametric) families and it is arguable whether they actually form "small" or "large" families of copulas.

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In order characterize the relative size of subclasses of copulas, we will use a topological approach (as also suggested in [15]) and work with Baire categories (see, e.g. [14]). Doing so we consider the topologies induced by two different metrics on the space $\mathscr{C}$ of two-dimensional copulas: the standard uniform metric d_∞ as well as the stronger metric D_1 introduced and studied in [9, 18]. In both cases the resulting metric spaces are complete (in case of d_∞ even compact). A subset N of a metric space (Ω, d) is called *nowhere dense* if it is not dense in any non-degenerate open ball $B(x, r)$ of radius $r > 0$ (equivalently, if its closure has empty interior). A set $A \subseteq \Omega$ is called *meager* or of *first category* in (Ω, d) if it can be expressed as (or covered by) a countable union of nowhere dense sets. A is called of *second category* if it is not meager. Finally, A is called *co-meager* (or residual) if $A^c = \Omega \setminus A$ is meager. Following [3], in complete metric spaces, first category sets are the "small" sets, co-meager sets are the "large" sets and second-category sets are merely "not small". Loosely speaking (and following conventions in the literature), we will therefore refer to the elements of a co-meager set as (topologically) *typical* and to the elements of a meager set as (topologically) *atypical* in Ω.

In the sequel $\mathscr{C}_e$ will denote the family of all exchangeable (i.e. symmetric) copulas, $\mathscr{C}_a$ the family of all associative copulas, $\mathscr{C}_{ar}$ the family of all Archimedean copulas, and $\mathscr{C}_{ar}^s$ the family of all strict Archimedean copulas, i.e. Archimedean copulas whose generator φ fulfills $\varphi(0) = \infty$ (see [13]). It is well known that we have $\mathscr{C}_{ar} \subset \mathscr{C}_a \subset \mathscr{C}_e$ and that $\mathscr{C}_a$ and $\mathscr{C}_e$ are closed in $(\mathscr{C}, d_\infty)$ (see, for instance, [10]). Since convergence w.r.t. D_1 implies convergence w.r.t. d_∞ (see again [18]) the families $\mathscr{C}_a$ and $\mathscr{C}_e$ are also closed in $(\mathscr{C}, D_1)$.

We will prove the following results:

- The family of exchangeable copulas $\mathscr{C}_e$ is nowhere dense (hence of first category) in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.
- The family $\mathscr{C}_a$ of associative copulas is nowhere dense in $(\mathscr{C}_e, d_\infty)$ as well as in $(\mathscr{C}_e, D_1)$.
- The family $\mathscr{C}_{ar}$ of Archimedean copulas is co-meager (hence of second category) in $(\mathscr{C}_a, d_\infty)$.
- The family $\mathscr{C}_{ar}^s$ of all strict Archimedean copulas is co-meager in $(\mathscr{C}_{ar}, d_\infty)$.

As a byproduct, we give an affirmative answer to the open problem posed in [1, Problem 10] (also see [2]) asking whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

Remark 1.1. The results presented in this paper are not intended to suggest any families of copulas to be used in practice but merely to give a purely topological characterization of some well-known classes. In fact, as pointed out before, (in complete metric spaces) Baire categories establish a rough classification of subsets as "small", "large", or "not small" but do not allow for a more accurate quantification of size, implying that Baire categories are not useful for deciding which parametric classes of copulas should (or should not) be used in practice.

2. The results

For basic definitions and properties of copulas we refer to [8, 13], for the metric D_1 to [18], and directly start with the following result.

Theorem 2.1. *$\mathscr{C}_e$ is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$.*

Proof. For every $A \in \mathscr{C}$ there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of non–exchangeable copulas that converges to A w.r.t. D_1 (hence w.r.t. d_∞). In fact, if A itself is not exchangeable we may choose $A_n = A$ and if A is exchangeable we may consider $A_n := (1 - 1/n)A + (1/n)\,E$ with $E \in \mathscr{C}$ being an arbitrary asymmetric copula. As immediate consequence $\mathscr{C}_e$ cannot contain any nonempty open subset of $(\mathscr{C}, D_1)$ or of $(\mathscr{C}, d_\infty)$. $\square$

Having Theorem 2.1 we immediately get that each subclass of $\mathscr{C}_e$ (including the family of all Gaussian copulas, Gumbel copulas, Clayton copulas, symmetric Bernstein copulas, symmetric checkerboard copulas, and many more) is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$ too. Additionally, considering that, according to [5, 19], all idempotent copulas (idempotent with respect to the star product introduced in [4]) are symmetric, and, letting $\mathscr{C}_i$ the family of all idempotent copulas, we also get that $\mathscr{C}_i$ is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$. The subsequent corollary serves to gather some important consequences - in particular it gives a positive answer to Problem 10 in [1] asking whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

Corollary 2.2. *$\mathscr{C}_{ar}$, $\mathscr{C}_a$ and $\mathscr{C}_i$ are nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$. In particular, all three families are of first category in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.*

Next we look at the family of associative copulas as subset of the family exchangeable copulas and prove the following result.

Theorem 2.3. *$\mathscr{C}_a$ is nowhere dense in $(\mathscr{C}_e, d_\infty)$ and in $(\mathscr{C}_e, D_1)$.*

Proof. For every $A \in \mathscr{C}_e$ there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of non–associative copulas that uniformly converges to A. In fact, if A itself is not associative we may choose $A_n = A$, and if A is associative we may consider A_n being the ordinal sum of (E, A) with respect to the partition $([0, 1/(n + 1)], [1/(n + 1), 1])$, where $E \in \mathscr{C}_e$ is a non–associative copula. Note that, by the representation theorem of associative copulas in [12], A_n is not associative. As immediate consequence $\mathscr{C}_a$ cannot contain any nonempty open subset of $(\mathscr{C}_e, d_\infty)$, which completes the proof for the metric d_∞.
The fact that $\mathscr{C}_a$ is nowhere dense in $(\mathscr{C}_e, D_1)$ follows from the first part of the proof since the ordinal sum A_n even converges to A w.r.t. the metric D_1, which can be shown as follows. As *first step* suppose that A is absolutely continuous with density k_A, that k_A is continuous on $[0, 1]^2$ and that we have $\int_{[0,1]} k_A(x, s)d\lambda(s) = 1$ for every $x \in [0, 1]$. Using Lebesgue's theorem on dominated convergence it follows that the Markov kernel $K_A(x, [0, y]) = \int_{[0,y]} k_A(x, s)d\lambda(s)$ of A fulfills that $(x, y) \mapsto K_A(x, [0, y])$ is continuous on

$[0,1]^2$. For the Markov kernel K_{A_n} of the ordinal sum A_n of (E, A) with respect to the partition $([0, 1/(n+1)], [1/(n+1), 1])$ and $x, y \in (\frac{1}{n+1}, 1]$ we obviously have

$$K_{A_n}(x, [0, y]) = K_A(\psi_n^{-1}(x), [0, \psi_n^{-1}(y)]),$$

where the mapping $\psi_n : [0,1] \to [\frac{1}{n+1}, 1]$ is given by $\psi_n(x) = \frac{1}{n+1} + x\,\frac{n}{n+1}$ and ψ_n^{-1} denotes its inverse. Using the notation from [18], for $\frac{1}{n+1} < y$ we get

$$\begin{aligned}\Phi_{A_n,A}(y) &= \int_{[0,1]} |K_{A_n}(x, [0, y]) - K_A(x, [0, y])| d\lambda(x) \\ &\leq \frac{1}{n+1} + \int_{(\frac{1}{N+1},1]} |K_A(\psi_n^{-1}(x), [0, \psi_n^{-1}(y)]) - K_A(x, [0, y])| d\lambda(x).\end{aligned}$$

Having that, again using Lebesgue's theorem on dominated convergence, $\lim_{n\to\infty} \Phi_{A_n,A}(y) = 0$ for every $y \in (0, 1]$ follows, which implies $\lim_{n\to\infty} D_1(A_n, A) = 0$ (see [18]).
As *second step* we prove the assertion for arbitrary associative A by considering Bernstein approximations of A. For every copula A and every $m \in \mathbb{N}$ the Bernstein approximation $\mathfrak{B}_m(A)$ of A with bandwidth (or degree) $m \in \mathbb{N}$ is defined by (see [11])

$$\mathfrak{B}_m(A)(x, y) = \sum_{i,j=0}^{m} A\Big(\frac{i}{m}, \frac{j}{m}\Big)\, b_{i,m}(x)\, b_{j,m}(y) \tag{1}$$

where for $j \in \{0, 1, \ldots, m\}$ and $x \in [0, 1]$ the Bernstein polynomial $b_{j,m} : [0, 1] \to [0, 1]$ is defined by $b_{j,m}(x) = \binom{m}{j} x^j (1 - x)^{m-j}$. Combining the results from [11] and [18] we have $\lim_{m\to\infty} D_1(\mathfrak{B}_m(A), A) = 0$. Now, the Bernstein approximation $\mathfrak{B}_m(A)$ is, firstly, absolutely continuous with continuous density $k_{\mathfrak{B}_m(A)}$ on $[0,1]^2$ and, secondly, symmetric provided that A is symmetric. Thus it follows that every associative copula A can be approximated arbitrary well with respect to the metric D_1 by copulas fulfilling the regularity condition used in the first step. Letting B_n^m denote the ordinal sum of $(E, \mathfrak{B}_m(A))$ w.r.t. the same partition $([0, 1/(n+1)], [1/(n+1), 1])$, using the triangle inequality and the contractivity results in [18, Section 6] we finally get

$$\begin{aligned}D_1(A_n, A) &\leq D_1(A_n, B_n^m) + D_1(B_n^m, \mathfrak{B}_m(A)) + D_1(\mathfrak{B}_m(A), A) \\ &\leq D_1(B_n^m, \mathfrak{B}_m(A)) + 2D_1(\mathfrak{B}_m(A), A),\end{aligned}$$

from which the desired $\lim_{n\to\infty} D_1(A_n, A) = 0$ follows immediately. □

Interestingly, the situation changes when we consider the class of Archimedean copulas as subset of $\mathscr{C}_a$.

Theorem 2.4. *$\mathscr{C}_{ar}$ is co–meager (hence of second category) in $(\mathscr{C}_a, d_\infty)$.*

Proof. We show that the set $\mathscr{C}_a \setminus \mathscr{C}_{ar}$ is of first category in $(\mathscr{C}_a, d_\infty)$. For every $n \in \mathbb{N}$, $n \geq 2$, let $\mathscr{A}_n$ be the subset of $\mathscr{C}_a$ given by

$$\mathscr{A}_n = \{C \in \mathscr{C}_a \colon C(x, x) = x \text{ for some } x \in [1/n, 1 - 1/n]\ \}.$$

Every $\mathscr{A}_n$ is closed in $\mathscr{C}_a$. In fact, if $(C_j)_{j\in\mathbb{N}}$ is a sequence of copulas in $\mathscr{A}_n$ converging to a copula C with respect to d_∞, then we have $C \in \mathscr{C}_a$ and there exists a sequence $(x_j)_j$ in $[1/n, 1-1/n]$ with $C_j(x_j, x_j) = x_j$. By compactness we can find a subsequence $(x_{j_l})_{l\in\mathbb{N}}$ that converges to some $x \in [1/n, 1-1/n]$ as $l \to \infty$. Thus, using the triangle inequality we get $C(x,x) = x \in [1/n, 1-1/n]$.

Since $\mathscr{C}_{ar}$ is dense in $\mathscr{C}_a$ (see [10]) it follows in the same way as in the previous proofs that $\mathscr{A}_n$ cannot contain any non-degenerate open ball in $(\mathscr{C}_a, d_\infty)$, implying that $\mathscr{A}_n$ is nowhere dense in $(\mathscr{C}_a, d_\infty)$. The result now follows from the fact that $\cup_{n\geq 2}\mathscr{A}_n = \mathscr{C}_a \setminus \mathscr{C}_{ar}$. $\square$

Theorem 2.5. *$\mathscr{C}_{ar}^s$ is co–meager in $(\mathscr{C}_{ar}, d_\infty)$.*

Proof. In order to achieve a one-to-one correspondence between Archimedean copulas and their generators we will only consider *normalized generators*, i.e. generators $\varphi : [0,1] \to [0,\infty]$ fulfilling $\varphi(1/2) = 1$. It suffices to prove that the family $\mathscr{C}_{ar}^n$ of all non-strict Archimedean copulas is of first category in $(\mathscr{C}_{ar}, d_\infty)$, which can be done in the following two steps.
(i) We show that for every Archimedean copula A_φ whose generator φ fulfills $\varphi(0) = b \in [2,\infty)$ the following inequality holds for every $x \in [0, \frac{1}{2}]$:

$$A_\varphi(x,x) \leq \max\Big\{x\Big(1 - \frac{2-2x}{b+2x-2}\Big), 0\Big\} \tag{2}$$

Let $x \in [0, \frac{1}{2}]$ be arbitrary but fixed. Note that convexity of φ and the normalization $\varphi(1/2) = 1$ imply $2 - 2x \leq \varphi(x)$. In fact, in case $\varphi(x_0) < 2 - 2x_0$ holds for some $x_0 \in [0, \frac{1}{2})$, setting $t := \frac{1}{2-2x_0}$ we get

$$\varphi\big(tx_0 + (1-t)1\big) = \varphi\big(\tfrac{1}{2}\big) = 1 = \tfrac{2-2x_0}{2-2x_0} > \tfrac{\varphi(x_0)}{2-2x_0} = t\varphi(x_0) + (1-t)\varphi(1),$$

which contradicts convexity. Furthermore, in case of $\varphi(x) \geq \frac{b}{2}$ we obviously have $A_\varphi(x,x) = \varphi^{[-1]}(2\varphi(x)) = 0$. Suppose now that $\varphi(x) \leq \frac{b}{2}$ and that $y \in [0,x]$. Convexity of φ implies $\varphi(y) \leq \frac{y}{x}\varphi(x) + (1 - \frac{y}{x})b$, implying that for every $z \in [\varphi(x), \frac{b}{2}]$ we have $\varphi^{[-1]}(z) \leq x\frac{z-b}{\varphi(x)-b}$. As direct consequence, if $y \in [0,x]$ fulfills $\varphi(y) \geq \frac{b}{2}$ then we get $A_\varphi(y,y) = 0$ and if $\varphi(y) \leq \frac{b}{2}$ then

$$A_\varphi(y,y) \leq \varphi^{[-1]}(2\varphi(y)) \leq x\ \frac{b - 2\varphi(y)}{b - \varphi(x)} \tag{3}$$

follows. Since $y \in [0,x]$ was arbitrary inequality (3) also holds for y replaced by x, i.e. we get

$$A_\varphi(x,x) \leq \varphi^{[-1]}(2\varphi(x)) \leq x\ \frac{b - 2\varphi(x)}{b - \varphi(x)}.$$

Considering that the function $t \mapsto \frac{b-2t}{b-t}$ is decreasing in t together with $\varphi(x) \geq 2 - 2x$ finally yields the desired inequality (2).
(ii) We can now prove the fact that $\mathscr{C}_{ar}^n$ is of first category in $(\mathscr{C}_{ar}, d_\infty)$. For every $b \geq 4$ define the family $\mathscr{A}_b \subset \mathscr{C}_{ar}^n$ by

$$\mathscr{A}_b = \big\{A \in \mathscr{C}_{ar} : \varphi(0) \leq b\big\}.$$

Since we obviously have $\mathscr{C}_{ar}^n = \bigcup_{b=4}^\infty \mathscr{A}_b$ it suffices to proof that each $\mathscr{A}_b$ is nowhere dense in $(\mathscr{C}_{ar}, d_\infty)$. Assume, on the contrary, that $\overline{\mathscr{A}}_b$ contains a non-empty open subset of $\mathscr{C}_{ar}$. Then we can find $A_\varphi \in \mathscr{A}_b$ and $r > 0$ such that the open ball $B(A_\varphi, r)$ in $\mathscr{C}_{ar}$ fulfills $B(A_\varphi, r) \subseteq \overline{\mathscr{A}}_b$. Since the right hand side of equation (2) is monotonically increasing as a function of b we must have $A_\varphi(x,x) \leq x(1 - \frac{2-2x}{b+2x-2})$ for every $x \in [0, \frac{1}{2}]$. Using convexity of φ we can find another generator $\varphi_r : [0,1] \to [0, \infty)$ that coincides with φ on $[\frac{r}{2}, 1]$ and fulfills

$$A_{\varphi_r}\Big(\frac{r}{2}, \frac{r}{2}\Big) = \varphi_r^{-1}\Big(2\varphi_r\Big(\frac{r}{2}\Big)\Big) \geq \frac{r}{2}\Big(1 - \frac{1}{2b}\Big) > \frac{r}{2}\Big(1 - \frac{2-r}{b+2-r}\Big).$$

On the other hand, it follows directly from the construction of φ_r that we must have $d_\infty(A_{\varphi_r}, A_\varphi) \leq \frac{r}{2}$, which implies $A_{\varphi_r} \in B(A_\varphi, r) \subseteq \overline{\mathscr{A}}_b$. Consequently, the assumption has to be wrong and we conclude that $\mathscr{A}_b$ is nowhere dense in $(\mathscr{C}_{ar}, d_\infty)$. $\square$

Since $(\mathscr{C}_{ar}, d_\infty)$ is not complete, we cannot directly use Theorem 2.5 to deduce that $\mathscr{C}_{ar}^s$ is of second category in $(\mathscr{C}_{ar}, d_\infty)$. However, using the fact that $(\mathscr{C}_a, d_\infty)$ is a compact metric space of second category offers an alternative simple proof.

Theorem 2.6. *$\mathscr{C}_{ar}^s$ is of second category in $(\mathscr{C}_{ar}, d_\infty)$.*

Proof. Since every nowhere dense subset T of $\mathscr{C}_{ar}$ is also nowhere dense in $\mathscr{C}_a$ it follows that $\mathscr{C}_{ar}^n$ is of first category in $(\mathscr{C}_a, d_\infty)$. If $\mathscr{C}_{ar}^s$ were of first category in $(\mathscr{C}_{ar}, d_\infty)$ then it would also be of first category in $(\mathscr{C}_a, d_\infty)$. As finite union $\mathscr{C}_a = \mathscr{C}_a \setminus \mathscr{C}_{ar} \cup \mathscr{C}_{ar}^s \cup \mathscr{C}_{ar}^n$ of first category sets $\mathscr{C}_a$ would be of first category too, which contradicts the compactness of $(\mathscr{C}_a, d_\infty)$ and completes the proof. $\square$

In short, with respect to d_∞, a typical copula is not symmetric and a typical symmetric copula is not associative but a typical associative copula is Archimedean and a typical Archimedean copula is strict.

Remark 2.7. *It remains an open question if $\mathscr{C}_{ar}$ is co–meager (hence of second category) in $(\mathscr{C}_a, D_1)$ and if $\mathscr{C}_{ar}^s$ is co–meager and/or of second category in $(\mathscr{C}_{ar}, D_1)$.*

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