A typical copula is singular*

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Abstract

We present Baire category results in the class of bivariate copulas (or, equivalently, doubly stochastic probability measures) endowed with two different metrics under which the space is complete. Main content of the paper is that, in the sense of Baire categories with respect to the topology induced by the uniform metric, the family of absolutely continuous copulas is of first category, whereas the family of purely singular copulas is co-meager and, hence, of second category. Moreover, several other popular dense sub-classes of copulas are considered, like shuffles of Min and checkerboard copulas.

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1 Introduction

Starting with the seminal paper by A. Sklar [28] copulas have gathered a lot of interest because of their relevance to model stochastic dependence and, as such, they constitute a relevant tool in applied probability and statistics. The interested reader may refer, for instance, to the recent monographs [5, 8, 13, 14, 15, 20, 23, 27] and reference therein.

The growing interest of copulas has also generated a number of different ways to provide families of copulas (e.g., shuffles, Bernstein, etc.) that are dense (in a given topology) in the class of copulas. However, little attention has been devoted to the question whether such families are big (or small) in a specific sense, i.e. whether they cover a sufficiently large spectrum of the copula space.

For a given metric space, topology offers a natural way of distinguishing small and big sets through Baire categories (see, e.g. [24]). Roughly speaking, a subset of a metric space (S, d) is considered to be "small" if it is *nowhere dense*, i.e. if it is not dense in any non-degenerated open ball $B_r(x, d)$ of radius $r > 0$ (equivalently, if its closure has empty interior). A set $A \subseteq S$

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is called *meager* or of *first category* in (S, d) if it is expressible as a countable union of nowhere dense sets. A is called of *second category* if it is not meager. Finally, A is called *co-meager* if $A^c = S \setminus A$ is meager. Loosely speaking, we will also refer to the elements of a co-meager set as *typical* and to the elements of a meager set as *atypical* in S.

In statistical theory, as stressed in [25], many asymptotic results do not hold for all underlying distributions $P \in \mathscr{P}$, but for all $P \in \mathscr{P} \setminus \mathscr{D}$, where $\mathscr{D}$ is a "small" compared to $\mathscr{P}$. In this context, small set is usually intended in the sense of Baire categories. For instance, in [17] the authors used Baire category arguments to motivate their interest in developing kernel density estimators that work in a wider class of densities. Analogously, in [25] (see also [26]) some bootstrap procedures are shown to work outside a set of first category (in the class of all possible underlying probability distributions).

A first study of Baire category results for copulas was presented in [18], where it is proved that the set of copula models that are dynamically consistent (with respect to a specific joint default model) and satisfy some technical regularity conditions, is a set of the first category in the Baire sense in a certain space of copulas with finitely many parameters. Recently, category results for exchangeable copulas and related families (associative, Archimedean) have been investigated in [6], motivated by an open problem proposed in [1].

Following this general framework, here we concentrate on the study of classes of copulas characterized by their measure-theoretic properties, namely absolutely continuous and purely singular copulas.

The main results are the following: the class of absolutely continuous copulas is of first category, while the class of (purely) singular copulas is of second category in the space $(\mathscr{C}, d_\infty)$ of all two-dimensional copulas with the uniform metric. Related results are also established for the strong metric D_1 introduced in [29] and special sub-classes of copulas like checkerboard copulas and shuffles of Min. For the sake of simplicity, all the results will be presented in the bivariate case. However, they can be also formulated in d–dimension $(d \geq 3)$ with simple and obvious modifications.

2 Notation and preliminaries

In the sequel Ω will denote a square of the form $\Omega := [e, e + \delta] \times [f, f + \delta] \subseteq [0, 1]^2$ with $\delta > 0$. $\mathscr{B}(\Omega)$ will denote the Borel σ-field in Ω, $\mathscr{M}(\Omega)$ the family of all finite (positive) measures on $(\Omega, \mathscr{B}(\Omega))$, and $\mathscr{M}_m(\Omega)$ the family of all $\mu \in \mathscr{M}(\Omega)$ fulfilling $\mu(\Omega) = m$, whereby $m \in (0, \infty)$. For every $\mu \in \mathscr{M}(\Omega)$ the corresponding measure-generating function (see, e.g., [2]) will be denoted by F_μ and is defined, for every $(x, y) \in \Omega$, by $F_\mu(x, y) = \mu([e, x] \times [f, y])$. We will use the symbols $\mathscr{F}(\Omega)$ and $\mathscr{F}_m(\Omega)$ for the families of the measure-generating functions corresponding to elements of $\mathscr{M}(\Omega)$ and $\mathscr{M}_m(\Omega)$ respectively. Given $\mu \in \mathscr{M}(\Omega)$ we will write $\mu = \mu_s + \mu_a$ for the Lebesgue decomposition of μ, where μ_a (respectively, μ_s) is absolutely continuous with respect to the Lebesgue measure λ_2; in particular, k_μ will denote the density of μ_a. For the sake of simplicity we will also write k_F instead of k_μ if F is the measure-generating function corresponding to μ. For every $L \in \mathbb{N}$ and $\mu \in \mathscr{M}(\Omega)$ the set B_μ^L is formed by all points in Ω such that the density of μ_a is upper bounded by L and strictly positive, namely

$$B_\mu^L := \{(x,y) \in \Omega :\ 0 < k_\mu(x,y) \leq L\}. \tag{2.1}$$

The symbol $\mathscr{C}$ will denote the family of all two-dimensional copulas, $\mathscr{P}_\mathscr{C} \subset \mathscr{M}_1([0,1]^2)$ the family of all doubly stochastic measures. It is well-known that there is a one-to-one correspondence between $\mathscr{C}$ and $\mathscr{P}_\mathscr{C}$. Moreover, $\mathscr{C}_{abs}$ (respectively, $\mathscr{C}_{sing}$) denotes the subclass of all absolutely continuous (respectively, singular) copulas, i.e. those copulas whose induced measure is absolutely continuous (respectively, singular) with respect to λ_2. Notice that $\mathscr{C}$ equipped with the d_∞ metric (respectively, D_1 metric by [29]) is complete and, hence, is a Baire space (see, e.g., [29]).

In the sequel we will also work with slightly generalized checkerboard-like constructions induced by so-called transformation matrices (see [11, 29]): A $n \times m$ matrix $\tau = (\tau_{ij}) \in [0,1]^{n\times m}$ is called *transformation matrix* if no row or column has all entries 0 and $\sum_{i,j} \tau_{i,j} = 1$. We will let $\mathscr{T}$ denote the family of all transformation matrices and $\mathscr{T}_{n,m}$ the subfamily of all transformation matrices with n rows and m columns. Given $\tau \in \mathscr{T}_{n,m}$ we define the vectors $(a_j)_{j=0}^m, (b_i)_{i=0}^n$ of cumulative column and row sums by $a_0 = b_0 = 0$ and

$$\begin{aligned} a_j &= \sum_{j_0 \leq j} \sum_{i=1}^n \tau_{ij}, \quad j \in \{1, \dots, m\}, \\ b_i &= \sum_{i_0 \leq i} \sum_{j=1}^m \tau_{ij}, \quad i \in \{1, \dots, n\}. \end{aligned} \tag{2.2}$$

Since both $(a_j)_{j=0}^m$ and $(b_i)_{i=0}^n$ are strictly increasing $R_{ji} := [a_{j-1}, a_j] \times [b_{i-1}, b_i]$ is a compact non-empty rectangle for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$. For every i, j, letting $w_{ji} : [0,1]^2 \to R_{ji}$ denote the affine contraction

$$w_{ji}(x,y) = \big(a_{j-1} + x(a_j - a_{j-1})\,,\, b_{i-1} + y(b_i - b_{i-1})\big),$$

it is straightforward to verify that the operator $\mathscr{V}_\tau$, defined by

$$\mathscr{V}_\tau(\mu) = \sum_{j=1}^m \sum_{i=1}^n \tau_{ij}\, \mu^{w_{ji}}$$

maps $\mathscr{P}_\mathscr{C}$ into itself. Hence we can also view it as operator on $\mathscr{C}$. In particular, for every copula $C \in \mathscr{C}$, we can refer to $\mathscr{V}_\tau(C)$ as the *τ-checkerboard of C*, while $\mathscr{V}_\tau(\Pi)$, where Π is the independence copula, will simply be called *τ-checkerboard*. The family of all checkerboard copulas will be denoted by $\mathscr{C}_{\text{CB}}$ (see, e.g., [22]). Note that, for a given checkerboard copula $A \in \mathscr{C}_{\text{CB}}$, there is not necessarily a unique transformation matrix τ with $\mathscr{V}_\tau(\Pi) = A$, e.g. for $A = \Pi$ both $\tau^1 = (1)$ and $\tau^2 = (\frac{1}{4})^{2\times 2}$ fulfill $\mathscr{V}_{\tau^1}(\Pi) = \mathscr{V}_{\tau^2}(\Pi) = \Pi$. In order to be able to assign each checkerboard copula a unique transformation matrix we will always choose the one having minimal dimensions. For more information about such copulas and their uses, see for instance [12, 19, 22].

Finally, notice that in case $\tau \in \mathscr{T}_{n,n}$ fulfills that $n\tau \in [0,1]^{n\times n}$ is a doubly stochastic matrix, then the rectangles R_{ji} defined above are actually squares and the maps w_{ji} are similarities. Given the comonotonicity copula M, we will call $\mathscr{V}_\tau(M)$ the τ-*checkmin* and refer to all τ-checkmins with $n \times n$-dimensional τ as *checkmins of order* n. In particular, if $\tau \in (\mathbb{Q} \cap [0,1])^{n\times n}$, the copula $\mathscr{V}_\tau(M)$ will be called *rational checkmin of order* n. In other words, if the random pair (U,V) is distributed according to a checkmin copula, then the conditional distribution of (U,V) given $(U,V) \in R_{ji}$ has copula M. For more details about checkmins, see [22, 32].

3 Category results for absolutely continuous copulas

In the sense of Baire category, $\mathscr{C}_{abs}$ constitutes a "small" set in $(\mathscr{C}, d_\infty)$, as the following result holds.

Theorem 3.1. *$\mathscr{C}_{abs}$ is of first category in $(\mathscr{C}, d_\infty)$.*

Proof. For every $A \in \mathscr{C}_{abs}$ and $b \in \mathbb{N}$ the set G_A^b is defined by

$$G_A^b := \{(x,y) \in [0,1]^2 : k_A(x,y) > b\}\big).$$

Furthermore, for every $b \in \mathbb{N}$ set

$$\mathscr{C}_{abs}^b := \Big\{ A \in \mathscr{C}_{abs} : \mu_A\big(G_A^b\big) \leq 1/4 \Big\}.$$

Applying Lebesgue's theorem on dominated convergence we get $\lim_{b\to\infty} \mu_A(G_A^b) = 0$ for arbitrary $A \in \mathscr{C}_{abs}$. Hence we have $\mathscr{C}_{abs} \subseteq \bigcup_{b\in\mathbb{N}} \mathscr{C}_{abs}^b$ and it suffices to prove that $\mathscr{C}_{abs}^b$ is nowhere dense for every $b \in \mathbb{N}$ in order to prove the assertion.
To this end, suppose that $B \in \mathscr{C}$ is a checkmin copula of order N. Then the support of B consists of at most N^2 line segments of length $\frac{1}{N}\sqrt{2}$. Cover each line segment with a strip of width $w = \frac{1}{4bN\sqrt{2}}$ like sketched in Figure 1. Letting $S \subseteq [0,1]^2$ denote the union of the resulting strips we therefore have $\lambda_2(S) \leq \frac{1}{4b}$. Finally, let $f : [0,1]^2 \to [0,1]$ be a continuous function that is one on the support of B and zero outside S. It follows directly from the construction that $\int_{[0,1]^2} f d\mu_B = 1$. Considering, however, $A \in \mathscr{C}_{abs}^b$ we get

$$\begin{aligned}
\int_{[0,1]^2} f\, d\mu_A &= \int_{G_A^b} f\, d\mu_A + \int_{(G_A^b)^c} f\, d\mu_A \\
&\leq \mu_A(G_A^b) + \int_{(G_A^b)^c \cap S} f\, k_A\, d\lambda_2 \leq \frac{1}{4} + b\lambda_2(S) \leq \frac{1}{2},
\end{aligned}$$

which, using the fact that d_∞ is a metrization of weak convergence on $\mathscr{C}$, implies that $B \notin \overline{\mathscr{C}_{abs}^b}$. The desired result now follows from the fact that the family of all checkmin copulas are dense in $(\mathscr{C}, d_\infty)$ (see [22]). □

Replacing d_∞ by the metric D_1 introduced in [10, 29] and mimicking the proof of Theorem 3.1 we get the following result.

Corollary 3.1. *The family $\mathscr{C}_{abs}$ is of first category in $(\mathscr{C}, D_1)$.*

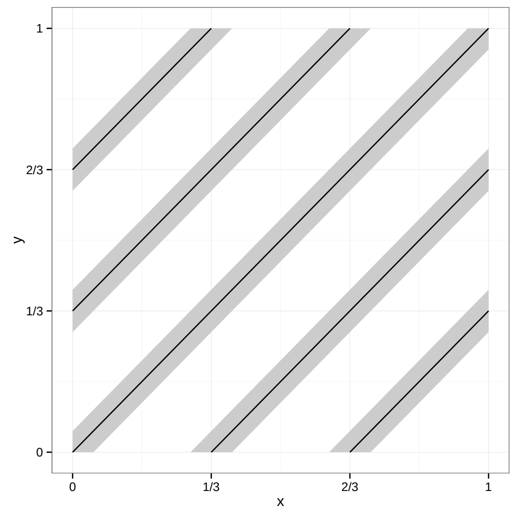

Figure 1: Checkmin construction in the proof of Theorem 3.1.

Then, any subset of $\mathscr{C}_{abs}$ is of first category in $(\mathscr{C}, d_\infty)$ (respectively, $(\mathscr{C}, D_1)$). For instance, Bernstein copulas, which form a dense set in $(\mathscr{C}, d_\infty)$ (respectively, $(\mathscr{C}, D_1)$), are of first category in $(\mathscr{C}, d_\infty)$ (respectively, $(\mathscr{C}, D_1)$).

Another subclass of absolutely continuous copulas is formed by the family $\mathscr{C}_{\mathrm{CB}}$ of all checkerboard copulas, which include the multilinear extensions of empirical copulas (see [12]). As said, this family is of first category in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$. Here we would like to prove these results in a different way (without using Theorem 3.1) in order to underline some interesting aspects of this family.

We start with a simple, but useful, lemma, which states that, if a sequence of transformation matrices converges to a given matrix τ, then the corresponding sequence of checkerboard copulas converges to the checkerboard copula generated by τ.

Lemma 3.1. *Suppose that $(\tau^n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{T}_{m_t,m_s}$ that converges (coordinate-wise) to $\tau \in \mathscr{T}_{m_t,m_s}$. Then we have*

$$\lim_{n\to\infty} D_1\big(\mathscr{V}_{\tau^n}(\Pi), \mathscr{V}_\tau(\Pi)\big) = 0 = \lim_{n\to\infty} d_\infty\big(\mathscr{V}_{\tau^n}(\Pi), \mathscr{V}_\tau(\Pi)\big).$$

Proof. Let R_{ji}, R_{ji}^n denote the rectangles induced by τ and τ^n respectively. Obviously, $\mathscr{V}_{\tau^n}(\Pi)$ and $\mathscr{V}_\tau(\Pi)$ are absolutely continuous copulas with densities

$$k_n(x,y) = \sum_{j=1}^{m_s}\sum_{i=1}^{m_t} \frac{\tau_{ij}^n}{\lambda_2(R_{ji})} \mathbf{1}_{R_{ji}^n}(x,y) \quad \text{and} \quad k(x,y) = \sum_{j=1}^{m_s}\sum_{i=1}^{m_t} \frac{\tau_{ij}}{\lambda_2(R_{ji})} \mathbf{1}_{R_{ji}}(x,y)$$

respectively. Since $\lim_{n\to\infty} \tau_{ij}^n = \tau_{ij}$, it follows from the checkerboard construction that we also have $\lim_{n\to\infty} \mathbf{1}_{R_{ji}^n} = \mathbf{1}_{R_{ji}}$ λ_2–almost everywhere as well as $\lim_{n\to\infty} \lambda_2(R_{ji}^n) = \lambda_2(R_{ji})$. Thus we get $\lim_{n\to\infty} k_n = k$ λ_2–almost everywhere. Considering that the family $\{k, k_1, k_2, \dots\}$

is uniformly bounded $\lim_{n\to\infty} D_1\big(\mathscr{V}_{\tau^n}(\Pi), \mathscr{V}_\tau(\Pi)\big) = 0$ follows immediately. Finally, since convergence with respect to D_1 metric implies uniform convergence, the proof is complete. □

Now, for any arbitrary $N \in \mathbb{N}$, let $\mathscr{C}_{\mathrm{CB}}^N$ denote the family of all checkerboard copulas A for which there exists a transformation matrix τ with row and column sums greater than or equal to $1/N$. Note that for each $B \in \mathscr{C}_{\mathrm{CB}}^N$ the corresponding (minimal) transformation matrix $\tau \in [0,1]^{m_t \times m_s}$ with $\mathscr{V}_\tau(\Pi) = B$ fulfills $m_t, m_s \in \{1, \ldots, N\}$.

Lemma 3.2. *For every $N \in \mathbb{N}$, the set $\mathscr{C}_{\mathrm{CB}}^N$ is closed in $(\mathscr{C}, D_1)$ as well as in $(\mathscr{C}, d_\infty)$.*

Proof. It suffices to prove that $\mathscr{C}_{\mathrm{CB}}^N$ is closed in $(\mathscr{C}, d_\infty)$. To this end, suppose that $(\mathscr{V}_{\tau_n}(\Pi))_{n\in\mathbb{N}}$ is a sequence in $\mathscr{C}_{\mathrm{CB}}^N$ that converges uniformly to a copula $A \in \mathscr{C}$. We can find a pair $(m_t, m_s) \in \{1, \ldots, N\}^2$ such that, for infinitely many $n \in \mathbb{N}$, the transformation matrix τ_n has exactly m_t rows and m_s columns. Let $(n_k)_{k\in\mathbb{N}}$ denote a strictly increasing sequence of integers fulfilling $\tau^{n_k} \in [0,1]^{m_t \times m_s}$ for every $k \in \mathbb{N}$. Compactness of $[0,1]^{m_t \times m_s}$ implies the existence of a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ and of a transformation matrix $\tau \in \mathscr{T}_{m_t, m_s}$ such that $\lim_{l\to\infty} \tau_{ij}^{n_{k_l}} = \tau_{ij}$. Applying Lemma 3.1 yields $\lim_{l\to\infty} d_\infty\big(\mathscr{V}_{\tau^{n_{k_l}}}(\Pi), \mathscr{V}_\tau(\Pi)\big)$, from which we get $A = \mathscr{V}_\tau(\Pi)$. □

Theorem 3.2. *The family $\mathscr{C}_{CB}$ is of first category in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.*

Proof. Since $\mathscr{C}_{\mathrm{CB}} = \bigcup_{N\in\mathbb{N}} \mathscr{C}_{\mathrm{CB}}^N$ it suffices to prove that $\mathscr{C}_{\mathrm{CB}}^N$ is nowhere dense. Let τ be a transformation matrix with row and column sums greater than or equal to $1/N$. Consider an open ball of the form $B_r(\mathscr{V}_\tau(\Pi), D_1)$ with radius $r > 0$ in $(\mathscr{C}, D_1)$. Then, choosing an arbitrary check-min copula A with $D_1(A, \Pi) < r$ (this is guaranteed by [22, Theorem 6]), we get (see [29]) $D_1(\mathscr{V}_\tau(\Pi), \mathscr{V}_\tau(A)) \le D_1(\Pi, A) < r$, which implies $\mathscr{V}_\tau(A) \in B_r(\mathscr{V}_\tau(\Pi), D_1)$. Since $\mathscr{V}_\tau(A)$ is not absolutely continuous, $\mathscr{C}_{\mathrm{CB}}^N$ cannot contain $B_r(\mathscr{V}_\tau(\Pi), D_1)$, showing that $\mathscr{C}_{\mathrm{CB}}^N$ is nowhere dense. The fact that $\mathscr{C}_{\mathrm{CB}}^N$ is nowhere dense in $(\mathscr{C}, d_\infty)$ can be proved analogously. □

4 Category results for singular copulas

From Theorem 3.1 and from the compactness of $(\mathscr{C}, d_\infty)$ it follows that the family of all copulas having a non-degenerated singular component are of second category. Surprisingly, even the family $\mathscr{C}_{sing}$ of all (purely) singular copulas is of second category (in fact even co-meager). We will prove this result in several steps and start with some notation and two lemmas.

In the sequel, $\Delta = \{(x,x) : x \in [0,1]\}$ will denote the diagonal of $[0,1]^2$. For every natural number $N \ge 2$, we set $\Delta_{1/N} := [0,1/N]^2 \cap \Delta$ and, for every $m \in [1/N^3, 1/N]$ let $\mathscr{A}_m^N$ be defined by

$$\mathscr{A}_m^N = \left\{ \mu \in \mathscr{M}_m\big([0,1/N]^2\big) : \int_{B_\mu^N} k_\mu d\lambda_2 \ge \frac{1}{N^3} \right\}. \tag{4.1}$$

The set $\mathscr{A}_m^N$ is formed by all finite positive measures μ in $[0,1/N]^2$ that have total measure equal to m and which fulfill that $\mu_a(B_\mu^N)$ is bounded from below by $1/N^3$, whereby μ_a denotes the absolutely continuous component of μ.

Lemma 4.1. *Suppose that $N \geq 2$ is a natural number, that $m \in [1/N^3, 1/N]$, and let $\mathscr{A}_m^N$ be defined according to Equation (4.1). Furthermore let $\Upsilon_m \in \mathscr{M}_m([0,1/N]^2)$ denote the uniform distribution on $\Delta_{1/N}$. Then no sequence in $\mathscr{A}_m^N$ converges weakly to Υ_m.*

Proof. Fix an arbitrary $\mu \in \mathscr{A}_m^N$. Considering

$$\frac{1}{N^3} \leq \int_{B_\mu^N} k_\mu d\lambda_2 \leq N\lambda_2(B_\mu^N)$$

we get $1/N^4 \leq \lambda_2(B_\mu^N)$ as well as $\mu_s([0,1/N]^2) + \int_{(B_\mu^N)^c} k_\mu d\,\lambda_2 \leq m - \frac{1}{N^3}$. Cover $\Delta_{1/N}$ with a strip of width $w = \frac{1}{2\sqrt{2}N^3}$, let S denote the resulting subset of $[0,1/N]^2$ and choose a continuous function $f : [0,1/N]^2 \to [0,1]$ which is one on $\Delta_{1/N}$ and zero outside S. Then, on the one hand, we have $\int_{[0,1/N]^2} f d\Upsilon_m = m$, and on the other hand, using $\lambda_2(S) \leq \frac{1}{2N^4}$, we get

$$\begin{aligned}\int_{[0,1/N]^2} f d\mu &= \int_{[0,1/N]^2} f d\mu_s + \int_{(B_\mu^N)^c \cap S} f k_\mu d\lambda_2 + \int_{B_\mu^N \cap S} f k_\mu d\lambda_2 \\ &\leq m - \frac{1}{N^3} + N\lambda_2(S) \leq m - \frac{1}{N^3} + \frac{1}{2N^3} = m - \frac{1}{2N^3}.\end{aligned}$$

Since f is continuous and $[0,1/N]^2$ is compact the result follows immediately. □

We will now focus on the family of all elements $\mu \in \mathscr{A}_m^N$ for which F_μ is continuous and use the fact that for measures μ on $[0,1/N]^2$ with continuous F_μ weak convergence is equivalent to uniform convergence of the respective measure-generating functions. Set

$$\hat{\mathscr{A}}_m^N = \left\{\mu \in \mathscr{M}_m\big([0,1/N]^2\big) : \int_{B_\mu^N} k_\mu d\lambda_2 \geq \frac{1}{N^3} \text{ and } F_\mu \text{ is continuous}\right\} \tag{4.2}$$

and let $\hat{\mathscr{F}}_m^N$ denote the family of all measure-generating functions F_μ with $\mu \in \hat{\mathscr{A}}_m^N$. Obviously, $\Upsilon_m \in \hat{\mathscr{A}}_m^N$. In the sequel we will write $U_m := F_{\Upsilon_m}$ for every $m \in [1/N^3, 1/N]$.

Lemma 4.2. *Suppose that $N \geq 2$ is a natural number. Then the following inequality holds:*

$$\varphi(N) := \inf_{m \in \left[\frac{1}{N^3}, \frac{1}{N}\right]} \inf_{H \in \hat{\mathscr{F}}_m^N} d_\infty(U_m, H) > 0. \tag{4.3}$$

Proof. As consequence of Lemma 4.1 we have $\varphi_m(N) := \inf_{H \in \hat{\mathscr{F}}_m^N} d_\infty(U_m, H) > 0$. Suppose now that $\inf_{m \in \left[\frac{1}{N^3}, \frac{1}{N}\right]} \varphi_m(N) = 0$. Then there exist a sequence $(m_n)_{n\in\mathbb{N}}$ in $[1/N^3, 1/N]$ and a sequence $(H_{m_n})_{n\in\mathbb{N}}$ fulfilling $H_{m_n} \in \hat{\mathscr{F}}_{m_n}^N$ for every $n \in \mathbb{N}$ such that

$$d_\infty(U_{m_n}, H_{m_n}) < 2\varphi_{m_n}(N) \quad \text{and} \quad \lim_{n\to\infty} \varphi_{m_n}(N) = 0.$$

Let $(m_{n_j})_{j\in\mathbb{N}}$ denote a subsequence of $(m_n)_{n\in\mathbb{N}}$ that converges to some $m_0 \in \left[\frac{1}{N^3}, \frac{1}{N}\right]$. Then obviously we have $\lim_{j\to\infty} d_\infty(U_{m_{n_j}}, U_{m_0}) = 0 = \lim_{j\to\infty} d_\infty(H_{m_{n_j}}, U_{m_0})$. Considering the same stripe S and the same function f as in the proof of Lemma 4.1 and letting $\mu_{m_{n_j}}$ denote the measure corresponding to $H_{m_{n_j}}$ we get

$$\lim_{j\to\infty}\int_{[0,1/N]^2} f d\mu_{m_{n_j}} \leq \lim_{j\to\infty} m_{n_j} - \frac{1}{2N^3} = m_0 - \frac{1}{2N^3} < m_0 = \int_{[0,1/N]^2} f d\kappa_{m_0},$$

contradicting $\lim_{j\to\infty} d_\infty(H_{m_{n_j}}, U_{m_0}) = 0$. Hence we must have $\inf_{m\in\left[\frac{1}{N^3},\frac{1}{N}\right]} \varphi_m(N) > 0$. □

In the sequel, $C_1, C_2, \ldots$ will denote an enumeration of all rational checkmin copulas. According to [29] the set of all rational checkerboard copulas is dense in $(\mathscr{C}, D_1)$, so it is also dense in $(\mathscr{C}, d_\infty)$. It is straightforward to verify that the same holds for $\{C_1, C_2, \ldots\}$.

Theorem 4.1. *Suppose that $C_1, C_2, \ldots$ is an enumeration of all rational checkmin copulas, let $N_t \geq 2$ denote the order of C_t for every $t \in \mathbb{N}$ and suppose that ψ is a function mapping $\mathbb{N}$ into $(0, 1)$. For every pair of integers (t, n) set*

$$\mathscr{S}_{t,n} := \left\{A \in \mathscr{C} : d_\infty(A, C_t) < \frac{\psi(N_t)}{8\,(N_t)^n}\right\} \tag{4.4}$$

as well as $\mathscr{S} := \bigcap_{n=1}^\infty \bigcup_{t=1}^\infty \mathscr{S}_{t,n}$. Then $\mathscr{S}$ is co-meager (and, hence, of second category) in $(\mathscr{C}, d_\infty)$. Furthermore, if we choose $\psi = \varphi$ with φ given by (4.3) then $\mathscr{S}$ only contains singular copulas.

Proof. Since $\{C_1, C_2, C_3, \ldots\}$ is dense in $(\mathscr{C}, d_\infty)$ the set $\mathscr{S}_n := \bigcup_{t=1}^\infty \mathscr{S}_{t,n}$ is open and dense, so $\mathscr{S}_n^c$ is nowhere dense. Considering $\mathscr{S}^c = \bigcup_{n=1}^\infty \mathscr{S}_n^c$ it follows that $\mathscr{S}^c$ is of first category, which implies that $\mathscr{S}$ is of second category because $(\mathscr{C}, d_\infty)$ is compact.

To prove the second assertion we assume that $A \in \mathscr{C}$ is not purely singular and proceed in three steps:

(S1) If $A \notin \mathscr{C}_{sing}$, then $\int_{[0,1]^2} k_A \, d\lambda_2 > 0$. Thus, we can define $L_A \in \mathbb{N}$ by

$$L_A := \min\left\{L \in \mathbb{N} : \int_{B_A^L} k_A \, d\lambda_2 \geq \frac{1}{L}\right\}.$$

For every $N \geq L_A$ we have $\int_{B_A^N} k_A \, d\lambda_2 \geq \frac{1}{N}$, so there exists at least one square Q of the form $Q = [\frac{i-1}{N}, \frac{i}{N}] \times [\frac{j-1}{N}, \frac{j}{N}]$ with $i, j \in \{1, \ldots, N\}$ fulfilling

$$\int_{B_A^N \cap Q} k_A \, d\lambda_2 \geq \frac{1}{N^3}. \tag{4.5}$$

(S2) $A \in \mathscr{S}_{t,n}$ implies $N_t < L_A$.
Suppose, on the contrary, that $N_t \geq L_A$ holds. Letting $Q = [\frac{i-1}{N_t}, \frac{i}{N_t}] \times [\frac{j-1}{N_t}, \frac{j}{N_t}]$ denote one square fulfilling inequality (4.5) we obviously have $m := \mu_A(Q) \in [1/N_t^3, 1/N_t]$. We can construct a new probability measure $\mu_{A^\star}$ as follows: on $Q^c \cap [0, 1]^2$ the measure

$\mu_{A^\star}$ coincides with μ_A but on Q $\mu_{A^\star}$ distributes all of its mass $m = \mu_A(Q) = \mu_{A^\star}(Q)$ uniformly on the diagonal $\{(\frac{i-1+z}{N_t}, \frac{j-1+z}{N_t}) : z \in [0,1]\}$. The corresponding measure-generating function $A^\star : [0,1]^2 \to [0,1]$ is continuous but not necessarily a copula. For $(x,y) \in Q$ we have the following:

$$\begin{aligned} A(x,y) =& A\Big(x, \frac{j-1}{N_t}\Big) + A\Big(\frac{i-1}{N_t}, y\Big) - A\Big(\frac{i-1}{N_t}, \frac{j-1}{N_t}\Big) \\ &+ \mu_A\Big(\Big[\frac{i-1}{N_t}, x\Big] \times \Big[\frac{j-1}{N_t}, y\Big]\Big) \\ A^\star(x,y) =& A\Big(x, \frac{j-1}{N_t}\Big) + A\Big(\frac{i-1}{N_t}, y\Big) - A\Big(\frac{i-1}{N_t}, \frac{j-1}{N_t}\Big) \\ &+ mN_t \min\Big\{x - \frac{i-1}{N_t}, y - \frac{j-1}{N_t}\Big\} \\ C_t(x,y) =& C_t\Big(x, \frac{j-1}{N_t}\Big) + C_t\Big(\frac{i-1}{N_t}, y\Big) - C_t\Big(\frac{i-1}{N_t}, \frac{j-1}{N_t}\Big) \\ &+ \mu_{C_t}(Q) N_t \min\Big\{x - \frac{i-1}{N_t}, y - \frac{j-1}{N_t}\Big\} \end{aligned}$$

Applying Lemma 4.2 yields $\max_{(x,y)\in Q} |A(x,y) - A^\star(x,y)| \geq \varphi(N_t) > 0$, and, using the triangle inequality, we get

$$\begin{aligned} \max_{(x,y)\in Q} |A^\star(x,y) - C_t(x,y) &\leq 3\, d_\infty(A, C_t) + |\mu_{C_t}(Q) - \mu_A(Q)| \\ &\leq 3\, \frac{\varphi(N_t)}{8(N_t)^n} + 4\, \frac{\varphi(N_t)}{8(N_t)^n} = 7\, \frac{\varphi(N_t)}{8(N_t)^n} < \frac{\varphi(N_t)}{(N_t)^n}. \end{aligned}$$

Altogether, again using the triangle inequality, this yields

$$\frac{\varphi(N_t)}{8(N_t)^n} > \max_{(x,y)\in Q} |A(x,y) - C_t(x,y)| \geq \varphi(N_t) - 7\, \frac{\varphi(N_t)}{8(N_t)^n},$$

implying $\frac{\varphi(N_t)}{(N_t)^n} > \varphi(N_t)$, which cannot hold for any n. Hence we conclude that $A \in \mathscr{S}_{t,n}$ implies $N_t < L_A$.

(S3) Since A has non-degenerated absolutely continuous component it is impossible to find a sequence of checkmin copulas of order smaller than L_A which converges to A uniformly. Hence we can find an integer $n_0(A) \in \mathbb{N}$ such that for every $n \geq n_0(A)$ and every C_t with $N_t < L_A$ we have

$$d_\infty(A, C_t) \geq \frac{1}{2^{n_0}} \geq \frac{1}{2^n} > \frac{\varphi(N_t)}{8(N_t)^n}.$$

As direct consequence, $A \in \mathscr{S}_{t,n}$ also implies $n < n_0(A)$, from which we directly get the desired result $A \notin \mathscr{S}$.

This concludes the proof. $\square$

In the proof of Theorem 4.1, it is proved that $\mathscr{S}$ is co–meager (and, hence, of second category) in $(\mathscr{C}, d_\infty)$. Thus, since every superset of a set of second category is itself of second category, we finally get the following corollary:

Corollary 4.1. *$\mathscr{C}_{sing}$ is of second category (and co-meager) in $(\mathscr{C}, d_\infty)$.*

Remark 4.1. As a consequence of the proof of Theorem 4.1, it also follows that the class of copulas having non-degenerated absolutely continuous component is of first category in $(\mathscr{C}, d_\infty)$. In particular, this generalizes Theorem 3.1 about absolutely continuous copulas.

Now, we consider the subclass of singular copulas given by shuffles of Min, which constitutes another set that is dense in $(\mathscr{C}, d_\infty)$ (see, e.g., [7, 21, 30]). Since $(\mathscr{C}_{sing}, d_\infty)$ is of second category in itself (see, e.g., [3, Corollary 8.17]), it is not straightforward to check whether shuffles of Min are of first category in the class of singular copulas.

To this end, let $\mathscr{S}_N$ denote the family of all shuffles of Min (generated by a partition not necessarily equidistant) for which the corresponding interval exchange transformation T has at most N jumps in $(0,1)$. The following result holds.

Lemma 4.3. *For every $N \in \mathbb{N}_0$ the set $\mathscr{S}_N$ is closed in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.*

Proof. Suppose that $(A_n)_{n\in\mathbb{N}}$ is a sequence in $\mathscr{S}_N$ converging to a copula $A \in \mathscr{C}$ with respect to d_∞. Without loss of generality we choose the transformation T_n corresponding to A_n to be right-continuous on $(0,1)$ for every n. Considering subsequences if necessary, we may furthermore assume that each T_n has exactly $M \in \{0,\dots,N\}$ jumps in $(0,1)$. For every $n \in \mathbb{N}$ let $\mathbf{a}_n = (a_n^1,\dots,a_n^M)$ denote the strictly increasing vector of jumps of T_n in $(0,1)$ and set $\mathbf{b}_n := (T_n(a_n^1),\dots,T_n(a_n^M))$. Using compactness of $[0,1]^{2M}$ we may find a subsequence $(n_j)_{j\in\mathbb{N}}$ and vectors $\mathbf{a} = (a^1,\dots,a^M)$, $\mathbf{b}$ such that $\lim_{j\to\infty} \|\mathbf{a}_{n_j} - \mathbf{a}\|_2 = 0 = \lim_{j\to\infty} \|\mathbf{b}_{n_j} - \mathbf{b}\|_2$. Note that the vector $\mathbf{a}$ is not necessarily strictly increasing and may also contain 0 or 1. Nevertheless $\mathbf{a}, \mathbf{b}$ induce a piecewise linear map $T : [0,1] \to [0,1]$ with slope one on each segment which is right-continuous on $(0,1)$. For this map we have $\lim_{j\to\infty} T_{n_j} = T$ a.e., so T is also λ-preserving and we get $\lim_{j\to\infty} D_1(A_{n_j}, C_T) = 0$ whereby C_T denote the copula whose mass is concentrated on the graph of T. Since convergence in the metric D_1 implies convergence in the metric d_∞ we finally get $A = C_T$, which completes the proof. □

Since for every shuffle of Min A we can find sequences $(A_n)_{n\in\mathbb{N}}$ of singular copulas whose mass is not (fully) concentrated on the graph of one transformation for which it holds that $\lim_{n\to\infty} D_1(A_n, A) = 0$, it follows that $\mathscr{S}_N$ is nowhere dense in $(\mathscr{C}_{sing}, d_\infty)$. As direct consequence we get the following result:

Theorem 4.2. *The family $\mathscr{S}$ of all (not necessarily equidistant) shuffles of Min is of first category in $(\mathscr{C}_{sing}, d_\infty)$.*

Remark 4.2. It is easy to show that the family $\mathscr{S}$ of all shuffles of Min is of first category also in $(\mathscr{C}_{sing}, D_1)$. In fact, even the closed set $\mathscr{C}_d$ of all completely dependent copulas has empty interior in $(\mathscr{C}_{sing}, D_1)$, as a straightforward consequence of the results in [29].

5 Conclusions

We have provided Baire category results for bivariate copulas. The main achievement is that the class of absolutely continuous copulas is of first category in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$, while the class of purely singular copulas is co-meager in $(\mathscr{C}, d_\infty)$. It remains an open question whether $\mathscr{C}_{sing}$ is of second category in $(\mathscr{C}, D_1)$.

To conclude, we would like to discuss briefly a possible analogy of our findings with classical statements of real analysis. For continuous real–valued functions, in the sense of category, it is known that a typical continuous functions is nowhere differentiable; in fact, it is exceptional for a continuous function to have a finite one-sided derivative (see, e.g., [4] where even stronger similar results are formulated). Analogously, by means of Kolmogorov representation of continuous functions of several variables, typical continuous functions do not admit partial derivatives (see, e.g., [16]).

Here, we have shown that it is exceptional for a copula to be absolutely continuous (or even have an absolutely continuous component), while it is typical that a copula is purely singular. Notice, however, that the elements of this class cannot be fully characterized in terms of the properties of the derivatives. In fact, proceeding similarly as in [31] one can prove the existence of singular copulas A (possibly with full support) such that the family $(F_x^A)_{x\in[0,1]}$ of all corresponding conditional distribution functions satisfying

$$A(x, y) = \int_{[0,x]} F_t(y) d\lambda(t)$$

for all $x, y \in [0, 1]$ also has the following properties: (i) $x \mapsto F_x^A(y)$ is continuous on $[0, 1]$ for every $y \in [0, 1]$, implying $\frac{\partial A(x,y)}{\partial x} = F_x^A(y)$ for all $x, y \in [0, 1]$; (ii) $y \mapsto F_x^A(y)$ is continuous on $[0, 1]$ for every $x \in [0, 1]$; (iii) for every $x \in [0, 1]$ we have $(F_x^A)' = 0$ almost everywhere.

In statistical theory, the obtained result about absolutely continuous copulas suggests that this assumption is somehow too strong. Novel procedures are called for (especially in various goodness-of-fit tests [9]) to extend their setting to a larger set of copulas.

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