Nested Square Roots of 2 Revisited

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Abstract

We show how various recently established results on finite and infinite nested square roots of 2 can be derived elegantly via a topological conjugacy linking the tent map and the logistic map. We also prove an additional new and striking consequence of this relationship.

1 Introduction.

In [2, 3] the author proved the existence of the limit of nested square roots of 2 of the form

$$\lim_{n\to\infty} a_0\sqrt{2-a_1\sqrt{2-a_2\sqrt{2-a_3\sqrt{\cdots-a_n\sqrt{2}}}}} \tag{1}$$

for arbitrary $\underline{a}=(a_0,a_1,a_2,\ldots)\in\{-1,1\}^{\mathbb{N}_0}$ and showed that for periodic $\underline{a}$ the limit can be expressed in terms of sine and cosine. His results extended those presented some years earlier in [6] and, partially, those in [4].

The main purpose of our article is to show, first, that the aforementioned results can be derived more easily by utilizing a topological conjugacy argument (explained in Section 2) and, second, that the conjugacy also allows us to prove additional surprising results. The remainder of this article is organized as follows. Section 2 gathers some notation and preliminaries that will be used in the sequel, including the aforementioned conjugacy. Finite nested square root of 2 representations will be derived in Section 3, infinite representations and a striking new result based on these representations are the main content of Section 4. Finally, Section 5 provides a quick summary and sketch for how to obtain additional results.

2 Notation and preliminaries.

Define the tent map $T:[-\frac{\pi}{2},\frac{\pi}{2}]\longrightarrow[-\frac{\pi}{2},\frac{\pi}{2}]$ and the logistic map $L:[-2,2]\longrightarrow[-2,2]$ (see Figure 1) by

$$T(x)=\begin{cases} 2x+\frac{\pi}{2} & \text{if } x\in[-\frac{\pi}{2},0],\\ -2x+\frac{\pi}{2} & \text{if } x\in(0,-\frac{\pi}{2}] \end{cases}$$

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and $L(x) = -x^2 + 2$. Then the dynamical systems $([-\frac{\pi}{2}, \frac{\pi}{2}], T)$ and $([-2, 2], L)$ are

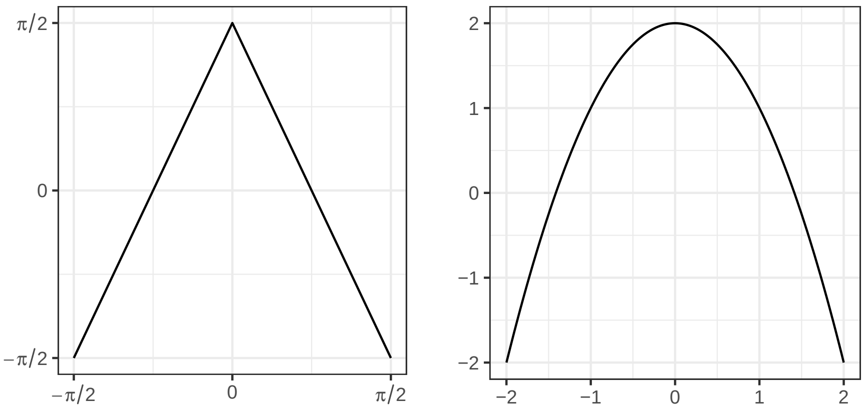

Figure 1: The tent map T (left panel) and logistic map L (right panel).

topologically conjugate via the homeomorphism $h : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-2, 2]$, defined by $h(x) = 2\sin(x)$, i.e., the following diagram, in which the vertical two-headed arrows symbolize bijectivity, is commutative (via straightforward calculation):

$$\begin{array}{ccc} [-\frac{\pi}{2}, \frac{\pi}{2}] & \xrightarrow{T} & [-\frac{\pi}{2}, \frac{\pi}{2}] \\ h \big\downarrow & & \big\downarrow h \\ [-2, 2] & \xrightarrow[L]{} & [-2, 2] \end{array} \tag{2}$$

In other words, $h(T(x)) = L(h(x))$ holds for every $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. As a direct consequence, the dynamical systems $([-\frac{\pi}{2}, \frac{\pi}{2}], T)$ and $([-2, 2], L)$ have the same dynamical properties, e.g., the same number of fixed points, the same number of points with period $p \in \{2, 3, \ldots\}$, the transformation T is topologically transitive if and only if L is, etc. (see [1, 8] for further properties).

The two intervals $I_{-1} = [-\frac{\pi}{2}, 0], I_1 = (0, \frac{\pi}{2}]$ form a partition of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ based on which the T-orbit of every point $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ can be assigned a unique element $\underline{a}$ in $\Sigma_2 := \{-1, 1\}^{\mathbb{N}_0}$ by setting $C_T(x) = \underline{a} = (a_0, a_1, a_2, \ldots) \in \Sigma_2$ if and only if $T^i(x) \in I_{a_i}$ for every $i \in \mathbb{N}$. Analogously, $J_{-1} = [-2, 0], J_1 = (0, 2]$ form a partition of $[-2, 2]$ and the L-orbit of every point $x \in [-2, 2]$ can be encoded by setting $C_L(x) = \underline{a} = (a_0, a_1, a_2, \ldots) \in \Sigma_2$ if and only if $L^i(x) \in J_{a_i}$ for every $i \in \mathbb{N}$. In the sequel we will refer to $C_T(x)$ and $C_L(y)$ as the T-*coding* of x and the L-*coding* of y, respectively. Considering that h bijectively maps I_i to J_i for every $i \in \{-1, 1\}$, diagram (2) implies that

$$C_T(x) = C_L(h(x)) \tag{3}$$

holds for every $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

3 Finite nested square roots of 2.

Defining $w_{-1} : [-2, 2] \to [-2, 0]$ and $w_1 : [-2, 2] \to [0, 2]$ by $w_{-1}(x) = -\sqrt{2-x}$ and $w_1(x) = \sqrt{2-x}$, we have

$$x = \begin{cases} w_{-1}(L(x)) & \text{if } x \in J_{-1}, \\ w_1(L(x)) & \text{if } x \in J_1, \end{cases} \tag{4}$$

Equation (4) is key in what follows and has been the sole reason for considering the tent map and the logistic map on the intervals $[\frac{\pi}{2}, \frac{\pi}{2}]$ and $[-2, 2]$, respectively (usually they are considered on the interval $[0, 1]$). In fact, if $C_T(x) = (a_0, a_1, a_2, \ldots) \in \Sigma_2$, then applying equation (4) once yields $h(x) = w_{a_0}(L(h(x)))$, applying it twice yields $h(x) = w_{a_0} \circ w_{a_1}(L^2(h(x)))$. Proceeding completely analogously it follows that

$$\begin{aligned} h(x) &= w_{a_0} \circ w_{a_1} \circ \cdots \circ w_{a_n}\big(L^{n+1}(h(x))\big) \\ &= a_0\sqrt{2 - a_1\sqrt{2 - a_2\sqrt{2 - a_3\sqrt{\cdots - a_n\sqrt{2 - L^{n+1}(h(x))}}}}} \end{aligned} \tag{5}$$

holds for every $n \in \mathbb{N}_0$. Equation (5) can be used to derive finite nested square root of 2 representations very easily; we start with a simple example.

Example 1. For $x = \frac{\pi}{16} \in I_1$, we have $T(x) = \frac{3\pi}{8} \in I_1, T^2(x) = -\frac{\pi}{4} \in I_{-1}$ and $T^3(x) = 0 \in I_{-1}$. Applying equation (5) for $n = 2$ directly yields

$$\begin{aligned} 2\sin\left(\frac{\pi}{16}\right) &= w_1 \circ w_1 \circ w_{-1} \circ \underbrace{L^3 \circ h(x)}_{=h\circ T^3(x)=h(0)=0} = w_1 \circ w_1 \circ w_{-1}(0) \\ &= \sqrt{2 - \sqrt{2 + \sqrt{2}}}. \end{aligned}$$

Simplification 1. All formulas in the right column of Example 1 in [6] (finite nested square root of 2 expressions for $2\sin\left(\frac{k\pi}{32}\right)$ with $k \in \{1, \ldots, 15\}$) follow in the same manner since (as direct consequence of the conjugacy h) $T^n(x) = 0$ if and only if $L^n(h(x)) = 0$. In fact, setting

$$Z_n = \frac{\pi}{2} \cdot \Big\{ \pm\frac{1}{2^n}, \pm\frac{3}{2^n}, \pm\frac{5}{2^n}, \ldots, \pm\frac{2^n - 1}{2^n} \Big\}, \tag{6}$$

we have that $T^n(x) = 0$ and $L^n(h(x)) = 0$ holds for every $x \in Z_n$, and for each such x the number $h(x)$ can be expressed as a finite composition of the functions w_{-1}, w_1, i.e.,

$$2\sin(x) = w_{a_0} \circ w_{a_1} \circ \cdots \circ w_{a_{n-1}}(0), \tag{7}$$

where $C_T(x) = (a_0, a_1, a_2, \ldots, a_{n-1}, -1, 1, \overline{-1})$ and $\overline{-1}$ denotes an infinite string of -1s.

Example 2. For $n \geq 3$ and $x_n = \frac{\pi}{2}\frac{1}{2^n} \in Z_n$ it is straightforward to verify that $C_T(x_n)$ is given by

$$C_T(x_n) = (1, 1, \underbrace{-1, -1, \ldots, -1}_{n-2}, -1, 1, \overline{-1}).$$

Considering $T^n(x_n) = 0$ and applying equation (7) yields $2\sin(x_n) = w_1^2 \circ w_{-1}^{n-3}(0)$. Since obviously $\lim_{n\to\infty} x_n = 0$ holds, the identity

$$1 = \lim_{n\to\infty} \frac{2\sin(x_n)}{2x_n} = \lim_{n\to\infty} \frac{w_1^2 \circ w_{-1}^{n-2}(0)}{\frac{\pi}{2^n}} = \frac{1}{\pi} \lim_{n\to\infty} 2^n w_1^2 \circ w_{-1}^{n-2}(0)$$

and, using the explicit expressions for w_{-1} and w_1, the famous representation of π in terms of nested square roots of 2, i.e.,

$$\pi = \lim_{n\to\infty} 2^n \cdot \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}}_{n \text{ square roots}},$$

follows immediately.

4 Infinite nested square roots of 2.

To derive infinite (periodic and nonperiodic) nested square root of 2 representations, we again use the conjugacy h and proceed as follows: Defining the functions $f_{-1}, f_1 : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ (analogously to w_{-1} and w_1 above) by $f_{-1}(x) = \frac{x}{2} - \frac{\pi}{4}$ and $f_1(x) = \frac{\pi}{4} - \frac{x}{2}$ yields

$$x = \begin{cases} f_{-1}(T(x)) & \text{if } x \in I_{-1}, \\ f_1(T(x)) & \text{if } x \in I_1. \end{cases}$$

Additionally, a straightforward calculation shows that

$$h \circ f_{a_0}(x) = w_{a_0} \circ h(x) \tag{8}$$

holds for every $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and every $a_0 \in \{-1, 1\}$. For every $\underline{a} \in \Sigma_2$ and arbitrary $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, define $G_T(\underline{a}) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ by

$$G_T(\underline{a}) = \lim_{n\to\infty} f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_n}(z). \tag{9}$$

Notice that the limit exists and does not depend on z since f_{-1}, f_1 are contractions on the complete metric space $[-\frac{\pi}{2}, \frac{\pi}{2}]$; the map G_T is therefore well-defined. We will refer to G_T as the *T-address map* in the sequel. Trying to proceed in the same way and defining $G_L(\underline{a}) \in [-2, 2]$ for arbitrary $z \in [-2, 2]$ by

$$G_L(\underline{a}) = \lim_{n\to\infty} w_{a_0} \circ w_{a_1} \circ \cdots \circ w_{a_n}(z), \tag{10}$$

it is a priori not clear if $G_L(\underline{a})$ exists and, in the positive case, if is independent of z since, unlike f_{-1}, f_1, the maps w_{-1}, w_1 are not contractions on $[-2, 2]$ (they are, in fact, not even Lipschitz-continuous). The following lemma clarifies this aspect and states some additional useful facts for deriving infinite nested square roots of 2 representations.

Lemma 1. *The L-address map G_L is well-defined. Additionally, the following assertions hold:*

(i) For every $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with $C_T(x) = \underline{a}$ we have $G_T(\underline{a}) = x$ and $G_L(\underline{a}) = h(x)$.

(ii) Both address maps $G_T : \Sigma_2 \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $G_L : \Sigma_2 \longrightarrow [-2, 2]$ are onto.

Proof. Well-definedness of G_L is a direct consequence of equation (8) and the fact that G_T is well-defined. In fact, according to equation (8) we have $h \circ f_i \circ h^{-1}(y) = w_i(y)$ for every $y \in [-2, 2]$ and every $i \in \{-1, 1\}$, which implies

$$\begin{aligned} h \circ G_T(\underline{a}) &= h\Big(\lim_{n\to\infty} f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_n}(h^{-1}(y))\Big) \\ &= \lim_{n\to\infty} h \circ f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_n} \circ h^{-1}(y) \\ &= \lim_{n\to\infty} w_{a_0} \circ w_{a_1} \circ \cdots \circ w_{a_n}(y) = G_L(\underline{a}). \end{aligned}$$

To show assertion (i) notice that

$$\bigcap_{n=1}^{\infty} f_{a_0} \circ f_{a_1} \circ \cdots \circ f_{a_n} \left([-\tfrac{\pi}{2}, \tfrac{\pi}{2}]\right)$$

contains exactly one point z and this point has to satisfy $T^n(z) \in I_{a_n}$ for every $n \in \mathbb{N}_0$. Since x is such a point and since this implies $h(x) = h \circ G_T(C_T(x)) = G_L(C_L(h(x)))$, it only remains to prove that G_T and G_L are surjective, which, however, is a straightforward consequence of the already implicitly proved identities $G_T \circ C_T = id_{[-\frac{\pi}{2}, \frac{\pi}{2}]}$ and $G_L \circ C_L = id_{[-2,2]}$. □

Simplification 2. Despite its simplicity and straightforward proof, Lemma 1 immediately implies the existence of the limit of the continued radicals as studied in [4] and proved in [3] using different methods. In fact, for every $\underline{a} \in \Sigma_2$, we have

$$\begin{aligned} G_L(\underline{a}) &= \lim_{n\to\infty} w_{a_0} \circ w_{a_1} \circ \cdots \circ w_{a_n}(0) \\ &= \lim_{n\to\infty} a_0 \sqrt{2 - a_1 \sqrt{2 - a_2 \sqrt{2 - a_3 \sqrt{\cdots - a_n \sqrt{2}}}}} \end{aligned} \tag{11}$$

Simplification 3. Notice that the simple form of the functions f_{-1}, f_1 allows us to express $G_T(\underline{a})$ as an infinite sum (see [3, 5] and the references therein): In the case of $C_T(x) = \underline{a}$ we obtain

$$x = \frac{\pi}{2} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \prod_{i=0}^{n-1} a_i = \frac{\pi}{2} \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \left(-\frac{a_i}{2}\right), \tag{12}$$

as well as

$$\begin{aligned} h(x) &= 2\sin\left(\frac{\pi}{2} \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \left(-\frac{a_i}{2}\right)\right) \\ &= \lim_{n\to\infty} a_0 \sqrt{2 - a_1 \sqrt{2 - a_2 \sqrt{2 - a_3 \sqrt{\cdots - a_n \sqrt{2}}}}}. \end{aligned} \tag{13}$$

We now turn to periodic nested square root of 2 representations as derived in [2] and provide alternative short proofs based merely on Lemma 1. For every $n \in \mathbb{N}$ let P_n denote the set of all $x \in [0, \frac{\pi}{2}]$ such that $T^n(x) = x$. Since the graph of T^n restricted to $[0, \frac{\pi}{2}]$ consists of 2^{n-1} line segments with slope $\pm 2^n$ that cover $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it is straightforward to show that P_n contains exactly 2^{n-1} points and is given by

$$P_n = \frac{\pi}{2} \cdot \left\{ \frac{2i-1}{2^n + (-1)^i} : i \in \{1, \ldots 2^{n-1}\} \right\}.$$

For each $x \in P_n$ we have $L^n(h(x)) = h(x)$ and the T-coding $\underline{a} = C_T(x) \in \Sigma_2$ is n-periodic, i.e., it is of the form $\underline{a} = (\overline{1, a_1, \ldots, a_{n-1}})$ for some $a_1, \ldots, a_{n-1} \in \{-1, 1\}$. By Lemma 1, for every $x \in P_n$ we get

$$h(x) = 2\sin(x) = \lim_{n\to\infty} (w_1 \circ w_{a_1} \circ w_{a_2} \ldots w_{a_{n-1}})^n(0),$$

so $h(x)$ can be expressed as periodic nested square root of 2. Since each $x \in P_n$ corresponds to a unique element $(\overline{1, a_1, \ldots, a_{n-1}})$ and vice versa, the main result in [2] (and consequently the identities in Tables 1–3 therein) follows.

The representation just mentioned can easily be extended to all eventually periodic $\underline{a} \in \Sigma_2$, i.e., all $\underline{a}$ of the form $\underline{a} = (b_0, b_1, \ldots, b_k, \overline{a_1, a_2, \ldots, a_n})$ for some $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. In fact, if we set $y = G_T(\overline{1, a_1, \ldots, a_{n-1}})$ and define x by $x = f_{b_0} \circ \cdots \circ f_{b_k}(y)$, then, according to Lemma 1, $h(x)$ has the (eventually periodic) nested square root of 2 representation

$$h(x) = \lim_{n \to \infty} w_{b_0} \circ \ldots \circ w_{b_k} \circ (w_{a_1} \circ \ldots w_{a_n})^n(0).$$

Simplification 4. It has already been mentioned that (as a direct consequence of the conjugacy h) zeros and fixed points of T and L are strongly related: (i) $T^n(x) = 0$ is equivalent to $L^n(h(x)) = 0$, and (ii) $T^n(x) = x$ if and only if $L^n(h(x)) = h(x)$. As mentioned in [7] and the references therein, the zeros of L^n are sorted according to the so-called Gray code. This property is yet another direct consequence of the conjugacy h linking of L and T. In fact, if for every $\underline{a} \in \Sigma_2$ and every $n \in \mathbb{N}_0$ we set

$$\begin{aligned} J_n(\underline{a}) &:= J_{(a_0,\ldots,a_n)} = \left\{x \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] : T^i(x) \in J_{a_i} \text{ for every } i \in \{0, 1, \ldots, n\}\right\}, \\ I_n(\underline{a}) &:= I_{(a_0,\ldots,a_n)} = \left\{x \in [-2, 2] : L^i(x) \in I_{a_i} \text{ for every } i \in \{0, 1, \ldots, n\}\right\}, \end{aligned}$$

then $J_n(\underline{a}), I_n(\underline{a})$ are intervals whose interiors $int(J_n(\underline{a})), int(I_n(\underline{a}))$ satisfy

$$\begin{aligned} int(J_n(\underline{a})) &= f_{a_0} \circ f_{a_1} \circ \ldots \circ f_{a_n} \left((-\tfrac{\pi}{2}, \tfrac{\pi}{2})\right), \\ int(I_n(\underline{a})) &= w_{a_0} \circ w_{a_1} \circ \ldots \circ w_{a_n} \left((-2, 2)\right). \end{aligned}$$

More important, the intervals are sorted according to the Gray code (if -1 is substituted for 0): Writing $E \prec F$ for intervals E, F, if for all $(e, f) \in E \times F$ we have $e < f$, it follows that

$$\begin{aligned} J_{(-1)} &\prec J_{(1)} \\ J_{(-1,-1)} \prec J_{(-1,1)} &\prec J_{(1,1)} \prec J_{(1,-1)} \\ J_{(-1,-1,-1)} \prec J_{(-1,-1,1)} \prec J_{(-1,1,1)} \prec J_{(-1,1,-1)} &\prec J_{(1,1,-1)} \prec J_{(1,1,1)} \\ &\prec J_{(1,-1,1)} \prec J_{(1,-1,-1)} \\ \cdots &\prec \cdots \end{aligned}$$

holds, and considering $I_n(\underline{a}) = h(J_n(\underline{a}))$ the same if true for the intervals $I_n(\underline{a})$. Notice that our conjugacy-based approach also allows for a simplified derivation of some of the formulas involving Gray code as presented in [7].

We conclude this article by showing how Lemma 1 can also be used to derive an additional surprising result which (to the best of the authors' knowledge) has not yet appeared in the literature. As usual we will call an element $\underline{a} \in \Sigma_2$ *simply normal (in base 2)* if

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{1}_1(a_i) = \frac{1}{2}$$

holds. The map T is ergodic (in fact even exact and strongly mixing, see [8]), i.e., every Borel set B in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that satisfies $T^{-1}(B) = B$ has either measure 0 or full measure. As a direct consequence of Birkhoff's ergodic theorem (see [8]), it follows that λ-almost every

$x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ has a normal T-coding $C_T(x)$ (thereby λ denotes the Lebesgue measure). This simple and well-known fact has the following consequence (in the theorem "randomly" is to be interpreted in the sense of "randomly from the uniform distribution on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$"):

Theorem 1. *If we randomly select a point $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then with probability one $C_L(2\sin(x)) = \underline{a} \in \Sigma_2$ is simply normal (in base 2), and we have*

$$2\sin(x) = \lim_{n\to\infty} a_0\sqrt{2 - a_1\sqrt{2 - a_2\sqrt{2 - a_3\sqrt{\cdots - a_n\sqrt{2}}}}}.$$

Proof. Let Λ_T denote the Borel subset of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ containing all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for which $C_T(x)$ is normal. Setting $\Lambda = h(\Lambda_T)$, considering that h is absolutely continuous, using equation (3), and applying Lemma 1 then completes the proof. $\square$

5 Conclusion.

We have derived finite and infinite nested square root of 2 representations of $2\sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ by using the fact that the dynamical systems $([-\frac{\pi}{2}, \frac{\pi}{2}], T)$ and $([-2, 2], L)$ are topologically conjugate via the homeomorphism $h(x) = 2\sin(x)$. Replacing T by the map $\hat{T} : [0, \pi] \longrightarrow [0, \pi]$, defined by

$$\hat{T}(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{\pi}{2}], \\ -2x + 2\pi & \text{if } x \in (\frac{\pi}{2}, \pi] \end{cases}$$

and L by the reflected logistic map $\hat{L} = -L$, working with the homeomorphism $\hat{h}(x) = 2\cos(x)$, and proceeding completely analogously would directly yield finite and infinite nested square root of 2 representations of $2\cos(x)$ for every $x \in [0, \pi]$.

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