LINEABILITY AND INTEGRABILITY IN THE SENSE OF RIEMANN, LEBESGUE, DENJOY, AND KHINTCHINE

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Abstract. In this paper, we continue the ongoing research on lineability related questions. On this occasion, we shall consider (among others) the classes of integrable functions (in the sense of Riemann, Lebesgue, Denjoy, and Khintchine), improving some already known results and expanding the study of lineability to other famous integrable classes never considered before.

1. Introduction

Since the beginning of this century many mathematicians have shown interest in the search for large algebraic structures within nonlinear sets of a topological vector space. This area of research commonly receives the name of lineability. This terminology was coined by V.I. Gurariy and it was first introduced in [6, 26]. There has been plenty of work in this direction since its appearance about a decade ago. As a matter of fact, this notion was (just recently) introduced by the American Mathematical Society under the MSC2020 15A03 and 46B87 classification references.

This area of research has shown to be extremely fruitful, and it is not a topic just linked to Mathematical Analysis, it has shown to be related and have applications to many other areas such as Set Theory and Foundations of Mathematics [10, 13, 20], Linear and Multilinear Algebra [19], Linear Dynamics [4], or even Algebraic Geometry [9]. A comprehensive description of theses concepts (as well as numerous examples and some general techniques) can be found in, e.g., [3, 5, 11, 12, 14, 15].

Let us now recall some definitions which, nowadays, have become usual terminology. We say that a subset A of a vector space V is:

- **lineable** in V if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subseteq A$.
- **κ-lineable** in V (where κ is a cardinal number) if there exists a vector space M of dimension κ and $M \setminus \{0\} \subseteq A$. Thus, $\aleph_0$-lineability means lineability.

If, in addition, V is a topological vector space over then A is said to be:

- **spaceable** in V if there exists a closed infinite dimensional vector space M such that $M \setminus \{0\} \subseteq A$.

Key words and phrases. Lineability, Algebrability, integrable function, Denjoy integral, Riemann Integral, Lebesgue Integral, Khintchine integral.

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Finally, and following [5], if V is a vector space contained in a (not necessarily unital) algebra and if κ is any (finite or infinite) cardinal number, then A is called:

- **algebrable** if there is an algebra M so that $M \setminus \{0\} \subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.
- **κ-algebrable** if there is a κ-generated algebra M with $M \setminus \{0\} \subset A$.
- **strongly κ-algebrable** if there is a κ-generated free algebra M such that $M \setminus \{0\} \subset A$.

Note that if V is contained in a commutative algebra, then a set $B \subset X$ is a system of generators of some free algebra M with $M \setminus \{0\} \subset A$ if and only if for any $n \in \mathbb{N}$, any nonzero polynomial P in n variables without constant term and any $b_1, \dots, b_n \in B$, we have $P(b_1, \dots, b_n) \in A \setminus \{0\}$.

We shall also need some typical definitions and notations from Real Analysis. If f is Riemann integrable on a compact interval $[a, b]$, we let $\int_a^b f(t)dt$ denote the Riemann integral of f on $[a, b]$ and, if f is Lebesgue integrable on a measurable set A, we denote $\int_A f(x)d\lambda$ the Lebesgue integral on A.

As it is commonly defined, we call a function $f\colon [a, \infty) \to \mathbb{R}$ Riemann integrable if f is Riemann integrable on $[a, x]$ for every $x \in (a, \infty)$ and the limit $\lim_{x\to\infty} \int_a^x f(t)dt$ exists and is finite. This limit will be denoted by $\int_a^\infty f(t)dt$. In the same terminology, given a function $f\colon [a, b] \to \mathbb{R}$ unbounded on b (respectively a), we shall also call f Riemann integrable if f is Riemann integrable on $[a, x]$ for every $x \in (a, b)$ and the limit $\lim_{x\to b} \int_a^x f(t)dt$ exists and is finite. Given a function defined on $\mathbb{R}$, we say that f is Riemann integrable on $\mathbb{R}$ if f and $g(x) = f(-x)$ are Riemann integrable on $[0, \infty)$. This will be denoted by $\int_{-\infty}^\infty f(t)dt$.

Let us also present the following notations for some classes of functions from I to $\mathbb{R}$ (where I is non-degenerate interval):

$\mathcal{R}(I)$: functions that are Riemann integrable on I.
$\mathcal{R}_b(I)$: bounded functions that are Riemann integrable on I.
$\mathcal{R}_u(I)$: unbounded functions that are Riemann integrable on I.
$\mathcal{NR}(I)$: nowhere Riemann integrable functions on I, that is, the family of functions f such that $f \restriction [a, b]$ is not Riemann integrable for every non-degenerate interval $[a, b] \subset I$, where $f \restriction [a, b]$ denotes the restriction of f to $[a, b]$.
$\mathcal{L}_p(I)$: Lebesgue p-integrable functions on I, that is, the functions f such that $\int_I |f|^p d\lambda$ exists, where λ denotes the Lebesgue measure.
$\mathcal{C}_b(I)$: continuous bounded functions on I.
$\mathcal{C}_u(I)$: continuous unbounded functions on I.

Moreover in the case $I = \mathbb{R}$, since no confusion arises, we shall simply write $\mathcal{R}$, $\mathcal{R}_b$, $\mathcal{R}_u$, $\mathcal{NR}$, $\mathcal{L}_p$, $\mathcal{C}_b$ and $\mathcal{C}_u$.

Furthemore, let us also denote the following families of functions from a non-degenerate interval I into $\mathbb{C}$:

$\mathcal{R}(I, \mathbb{C})$: functions that are Riemann integrable on I.

$\mathcal{L}_1(I,\mathbb{C})$: Lebesgue 1-integrable functions on I, that is, the functions f such that $\int_I |f| d\lambda$ exists.

Let us consider the following definitions which have been taken from [23]. We begin with the well known concept of absolute continuity.

Definition 1.1. *A function $f\colon [a,b]\to\mathbb{R}$ is absolutely continuous (AC) on $E\subseteq [a,b]$ if for every $\varepsilon>0$, there exists $\delta>0$ such that whenever a finite sequence of pairwise subintervals $\{[c_i,d_i]\colon i\in\{1,\ldots,n\}\}$ that have endpoints in E satisfies $\sum_{i=1}^n (d_i-c_i)<\delta$ we have $\sum_{i=1}^n |f(d_i)-f(c_i)|<\varepsilon$.*

The concept of absolute continuity can be tightened in the following way:

Take $f\colon [a,b]\to\mathbb{R}$ and $[c,d]\subset[a,b]$. We denote the oscillation of f on the interval $[c,d]$ by

$$\omega(f;[c,d]) := \sup\{|f(y)-f(x)|\colon c\le x<y\le d\}.$$

Definition 1.2. *A function $f\colon [a,b]\to\mathbb{R}$ is absolutely continuous in the restricted sense (AC_*) on $E\subseteq[a,b]$ if for every $\varepsilon>0$, there exists $\delta>0$ such that whenever a finite sequence of pairwise subintervals $\{[c_i,d_i]\colon i\in\{1,\ldots,n\}\}$ that have endpoints in E satisfies $\sum_{i=1}^n (d_i-c_i)<\delta$ we have $\sum_{i=1}^n \omega(f;[c_i,d_i])<\varepsilon$.*

Notice that AC_* implies AC.

Definition 1.3. *A function $f\colon [a,b]\to\mathbb{R}$ is:*

(i) *generalized absolutely continuous (ACG) on E if $f\restriction E$ is continuous on E and E can be written as the countable union of sets E_n such that $f\restriction E_n$ is AC;*

(ii) *generalized absolutely continuous in the restricted sense (ACG_*) on E if $f\restriction E$ is continuous on E and E can be written as the countable union of sets E_n such that $f\restriction E_n$ is AC_*.*

Let us consider the following functions from $I=[a,b]$ to $\mathbb{R}$.

Definition 1.4. *A function $f\colon I\to\mathbb{R}$ is Denjoy integrable if there exists an ACG_* function $F\colon I\to\mathbb{R}$ such that $F'=f$ a.e. We denote the family of Denjoy integrable functions from I to $\mathbb{R}$ by $\mathcal{D}(I)$.*

Finally we will also consider Khintchine integrable functions from $I=[a,b]$ to $\mathbb{R}$. To do so, let us introduce some basic definitions.

Definition 1.5. *Let A be a Lebesgue measurable set and let $x\in\mathbb{R}$. The Lebesgue density of A at x is defined by*

$$d_xA := \lim_{h\to 0+}\frac{\lambda(A\cap(x-h,x+h))}{2h}.$$

Notice that such limit exists a.e. (consider the Lebesgue points of χ_A). Moreover $0\le d_xA\le 1$. We say that x is a point of density of A if $d_xA=1$.

Definition 1.6. *Let $f\colon I\to\mathbb{R}$ and $x\in I$. We say that f is approximately differentiable at x if there exists $A\subset I$ measurable set such that x is a point of density of A and $f\restriction A$ is differentiable at x. We denote the approximate derivative of f at x by $f'_{ap}(x)$.*

It is easy to see that given $f\colon [a,b]\to\mathbb{R}$ differentiable at $x\in(a,b)$, then f is approximately differentiable at x.

Definition 1.7. *A function $f\colon I \to \mathbb{R}$ is Khintchine integrable if there exists an ACG function $F\colon I \to \mathbb{R}$ such that $F'_{ap} = f$ a.e. We denote the family of Khintchine integrable functions from I to $\mathbb{R}$ by $\mathcal{K}(I)$.*

The following construction of sets will be used throughout this work. Let us begin by recalling the following well known result of set theory (see, for instance, [25]).

Lemma 1.8. *There exists a family $\mathcal{N}$ of $\mathfrak{c}$-many distinct subsets of $\mathbb{N}$ such that*

$$N_1 \cap \ldots \cap N_n \cap N_{n+1}^c \cap \ldots \cap N_m^c \neq \varnothing$$

for every finite collection $\{N_1, \ldots, N_n, N_{n+1}, \ldots, N_m\} \subset \mathcal{N}$ with $N_i \neq N_j$ if $i \neq j$ and where N^c denotes $\mathbb{N} \setminus N$ provided that $N \subset \mathbb{N}$.

Fix a family $\mathcal{N}$ that satisfies Lemma 1.8 and partition $(0,1)$ as follows: for every positive integer n, let

$$I_n = \begin{cases} \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] & \text{if } n > 1, \\ \left(\frac{1}{2}, 1\right) & \text{otherwise.} \end{cases}$$

Now, since $|\mathcal{N}| = \mathfrak{c}$, let $\{N_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration, without repetition, of $\mathcal{N}$. For every $\xi < \mathfrak{c}$, we define

$$M_{N_\xi} := \bigcup_{n \in N_\xi} I_n,$$

and

$$M_{N_\xi^c} := \bigcup_{n \in N_\xi^c} I_n,$$

By Lemma 1.8 notice that $M_{N_\xi} \neq M_{N_\zeta}$ and $M_{N_\xi^c} \neq M_{N_\zeta^c}$ for every $\zeta < \xi < \mathfrak{c}$. Furthermore, for every finite collection $\{N_1, \ldots, N_n, N_{n+1}, \ldots, N_m\} \subset \mathcal{N}$ with $N_i \neq N_j$ if $i \neq j$, we have that $M_{N_1} \cap \ldots \cap M_{N_n} \cap M_{N_{n+1}^c} \cap \ldots M_{N_m^c} \supset I_k$ for some $k \in \mathbb{N}$.

After the previous battery of notations, concepts, and definitions we now proceed with Section 2, which deals with lineability within the framework of Riemann and Lebesgue integrability. Section 3 focuses on the interaction among the classes of Denjoy, Khintchine, Lebesgue, and Riemann integrable functions within the lineability point of view. A final section shall consider some remarks

2. On Riemann and Lebesgue integrability

In this section we will be concerned primarily on improving some results of [7] or [21].

Theorem 2.1. *There exist a Hamel basis $\{f_\xi \colon \xi < 2^{\mathfrak{c}}\}$ of cardinality $2^{\mathfrak{c}}$ with $span\{f_\xi \colon \xi < 2^{\mathfrak{c}}\} \subset \mathcal{R}_b$ and $span\{f_\xi \colon \xi < \mathfrak{c}\} \setminus \{0\} \subset \mathcal{R}_b \setminus \bigcup_{p>0} \mathcal{L}_p$, and a vector space W of dimension $\mathfrak{c}$ of continuous functions on $[0,1]$ such that for every $f \in span\{f_\xi \colon \mathfrak{c} \leq \xi < 2^{\mathfrak{c}}\} \setminus \{0\}$ and $g \in W \setminus \{0\}$ we have $f \circ g \notin \mathcal{R}([0,1])$.*

Proof. For every $\xi < \mathfrak{c}$, let us define the auxiliar function $h_\xi \colon [0,1] \to \mathbb{R}$ in the following way (see Figure 1)

$$h_\xi(x) = \begin{cases} 1 & \text{if } x \in M_{N_\xi}, \\ -1 & \text{if } x \in M_{N_\xi^c}, \\ 0 & \text{if } x \in \{0,1\}. \end{cases}$$

Using function h_ξ, consider $\widetilde{f}_\xi\colon \mathbb{R}\to\mathbb{R}$ as (see Figure 1)

$$\widetilde{f}_\xi(x)=\begin{cases}\frac{(-1)^n h_\xi(|x|-n+1)}{\ln n} & \text{if } |x|\in(n-1,n) \text{ for some } n\in\mathbb{N} \text{ with } n>2,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that all functions $\widetilde{f}_\xi$ are distinct by Lemma 1.8. Since the space $\mathcal{B}$ (of all bounded functions from $\mathbb{R}$ to $\mathbb{R}$) has dimension $2^{\mathfrak{c}}$ (see [7]), let $\mathcal{G}$ be a Hamel basis of $\mathcal{B}$ of dimension $2^{\mathfrak{c}}$. Now, as $\operatorname{card}(\mathcal{G})=2^{\mathfrak{c}}$, let $\{g_\xi\colon \xi<2^{\mathfrak{c}}\}$ be an enumeration, without repetition, of $\mathcal{G}$. For every $\xi<2^{\mathfrak{c}}$, we define $f_\xi\colon\mathbb{R}\to\mathbb{R}$, if $\xi<\mathfrak{c}$, as follows

$$f_\xi(x)=\begin{cases}\widetilde{f}_\xi(x) & \text{if } |x|\notin\bigcup_{n\in\mathbb{N}}C_n,\\ (g_\xi\circ\tau_n)(|x|) & \text{if } |x|\in C_n \text{ for some } n\in\mathbb{N},\end{cases}$$

and, if $\xi\geq\mathfrak{c}$,

$$f_\xi(x)=\begin{cases}0 & \text{if } |x|\notin\bigcup_{n\in\mathbb{N}}C_n,\\ (g_\xi\circ\tau_n)(|x|) & \text{if } |x|\in C_n \text{ for some } n\in\mathbb{N},\end{cases}$$

where, for every $n\in\mathbb{N}$, C_n is a Cantor set of Lebesgue measure 0 contained in $\left(\frac{1}{n+1},\frac{1}{n}\right)$ and $\tau_n\colon C_n\to\mathbb{R}$ is a bijection. Notice that f_ξ is an even mapping for every $\xi<2^{\mathfrak{c}}$. The set $\{f_\xi\colon\xi<2^{\mathfrak{c}}\}$ forms a Hamel basis such that the result is satisfied with W, the following vector space which can be found in [7].

Let $g_0\colon[0,1]\to[0,1]$ be a continuous function such that g_0 is not the zero function on any subinterval of $[0,1]$ and the total Lebesgue measure of all intervals in which g_0 is not identically zero is $\frac{1}{2}$. Furthermore, if $[a,b]\subset[0,1]$ and there exists $x_0\in(a,b)$ such that $g_0(x_0)=0$, then there exists $[c,d]\subset(a,b)$ such that $g_0([c,d])=\left[0,\frac{1}{2^n}\right]$ for some $n\in\mathbb{N}$ (see [7, proposition 2.1] for the existence of such function g_0). Take $W=\operatorname{span}\{g_0^\alpha\colon\alpha>0\}$, which is an algebra of dimension $\mathfrak{c}$ of continuous functions on $[0,1]$ (see [7, proposition 2.2]).

The proof is structured into 5 steps: in *Step 1* it is shown that all functions are dinstinct; in *Step 2* that $f_\xi\in\mathcal{R}_b$ for every $\xi<\mathfrak{c}$ and, if $\xi<\mathfrak{c}$, we have $f_\xi\notin\bigcup_{p>0}\mathcal{L}_p$; in *Step 3* we show the linear independency of $\{f_\xi\colon\xi<2^{\mathfrak{c}}\}$; *Step 4* deals with the composition argument; and *Step 5* tackles the problem that $\operatorname{span}\{f_\xi\colon\xi<\mathfrak{c}\}\setminus\{0\}\subset\mathcal{R}_b\setminus\bigcup_{p>0}\mathcal{L}_p$.

Step 1: For every $\alpha<\beta<2^{\mathfrak{c}}$, it is enough to show that $f_\alpha\neq f_\beta$. Fix $n\in\mathbb{N}$. Notice that the functions f_α and f_β are uniquely determined in C_n by g_α and g_β, respectively. Hence, $f_\alpha\neq f_\beta$. Indeed, since $g_\alpha\neq g_\beta$, there exists $x\in\mathbb{R}$ such that $g_\alpha(x)\neq g_\beta(x)$. Now, as τ_n is a bijection from C_n to $\mathbb{R}$, there exists $\bar{x}\in C_n$ such that $\tau_n(\bar{x})=x$. Thus, $f_\alpha(\bar{x})=(g_\alpha\circ\tau_n)(\bar{x})=g_\alpha(x)\neq g_\beta(x)=(g_\beta\circ\tau_n)(\bar{x})=f_\beta(\bar{x})$.

Step 2: It is easy to see that f_ξ is bounded. To prove that f_ξ is Riemann integrable, it is enough to show that $\int_{-\infty}^{\infty}f_\xi(t)dt=\int_{-1}^{1}f_\xi(t)dt+\int_{-\infty}^{-1}f_\xi(t)dt+\int_{1}^{\infty}f_\xi(t)dt$ exists. Notice that the set of discontinuities of f_ξ, when $\xi<\mathfrak{c}$, has Lebesgue measure 0 since it is contained in the union of the set of descontinuities of $\widetilde{f}_\xi$ (which is countable) and the set $\bigcup_{n\in\mathbb{N}}((-C_n)\cup C_n)$ (which has measure 0). If $\xi\geq\mathfrak{c}$, then the set of discontinuities of f_ξ has also measure 0 since it is contained in $\bigcup_{n\in\mathbb{N}}((-C_n)\cup C_n)$. Then, we have that $\int_{-1}^{1}f_\xi(t)dt$ exists by the Lebesgue criterion for Riemann integrability. Now, if $\xi\geq\mathfrak{c}$, then it is easy to see that $\int_{-\infty}^{-1}f_\xi(t)dt=\int_{1}^{\infty}f_\xi(t)dt=0$. Thus, $f_\xi\in\mathcal{R}_b$ for every $\xi\geq\mathfrak{c}$. Assume that $\xi<\mathfrak{c}$. For every $x>1$, the function $F_\xi(x)=\int_1^x f_\xi(t)dt$ is well defined by the Lebesgue criterion for

Riemann integrability. Now, notice that $\int_0^1 h_\xi(t)dt = \lambda(M_{N_\xi}) - \lambda(M_{N_\xi^c})$. Then, for every $n \geq 2$, we have

$$\begin{aligned} F_\xi(n) &= \int_1^n f_\xi(t)dt = \sum_{i=2}^n \int_{i-1}^i f_\xi(t)dt = \sum_{i=2}^n \frac{(-1)^i}{\ln i} \int_{i-1}^i h_\xi(t-i+1)dt \\ &= (\lambda(M_{N_\xi}) - \lambda(M_{N_\xi^c})) \sum_{i=2}^n \frac{(-1)^i}{\ln i}. \end{aligned}$$

Notice that $\lambda(M_{N_\xi}) - \lambda(M_{N_\xi^c}) \neq 0$ since N_ξ and N_ξ^c have infinitely many elements and either M_{N_ξ} or $M_{N_\xi^c}$ contains $\left(\frac{1}{2}, 1\right)$ which has Lebesgue measure $\frac{1}{2}$. Thus, by Dirichlet's test, the limit $\lim_{n\to\infty} F_\xi(n)$ exists. Hence, if $x \in (n, n+1)$ for some $n \in \mathbb{N}$, notice that

$$F_\xi(x) = F_\xi(n) + \int_n^x f(t)dt = F_\xi(n) + O\left(\frac{1}{\ln(n+1)}\right),$$

which implies that $\int_1^\infty f(t)dt$ exists. Furthermore, since f_ξ is even, we have that $\int_{-\infty}^{-1} f_\xi(t)dt$ also exists. It remains to show that $f_\xi \notin \bigcup_{p>0} \mathcal{L}_p$ for every $\xi < \mathfrak{c}$. Assume, by way of contradiction, that $\int_\mathbb{R} |f_\xi|^p d\lambda$ exists for some $p > 0$. Hence, $\int_{[1,\infty)} |f_\xi|^p d\lambda$ also exists. But

$$\int_{[1,\infty)} |f_\xi|^p d\lambda = \lim_{n\to\infty} \int_{[1,n]} |f_\xi|^p d\lambda = \lim_{n\to\infty} \sum_{i=2}^n \frac{1}{\ln^p i}$$

diverges.

Step 3: Assume that $\sum_{i=1}^m \alpha_i f_{\xi_i} = 0$ where $f_{\xi_1}, \ldots, f_{\xi_m} \in \{f_\xi \colon \xi < 2^\mathfrak{c}\}$ are distinct, $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and $m \in \mathbb{N}$. We need to prove that $\alpha_i = 0$ for every $i \in \{1, \ldots, m\}$. As $\sum_{i=1}^m \alpha_i f_{\xi_i}(x) = 0$ for every $x \in \mathbb{R}$, we have that, for any $n \in \mathbb{N}$, $0 = \sum_{i=1}^n \alpha_i f_{\xi_i} \restriction C_n = \sum_{i=1}^m \alpha_i (g_{\xi_i} \circ \tau_n) \restriction C_n$. Thus, since the functions $\{g_{\xi_1} \circ \tau_n, \ldots, g_{\xi_m} \circ \tau_n\}$ are linearly independent in $\mathcal{B}(C_n)$ (the family of bounded functions from C_n to $\mathbb{R}$), we have $\alpha_i = 0$ for every $i \in \{1, \ldots, m\}$.

Step 4: Notice that $\mathrm{span}\{f_\xi \colon \xi < 2^\mathfrak{c}\} \subset \mathcal{R}_b$ and, by applying similar techniques to the ones used in [7] to $\mathrm{span}\{f_\xi \colon \mathfrak{c} \leq \xi < 2^\mathfrak{c}\}$ and W, the composition part easily follows.

Step 5: It is obvious that $\mathrm{span}\{f_\xi \colon \xi < \mathfrak{c}\} \subset \mathcal{R}_b$. Thus, it remains to show that $\sum_{i=1}^n \alpha_i f_{\xi_i} \notin \bigcup_{p>0} \mathcal{L}_p$, where $f_{\xi_1}, \ldots, f_{\xi_n} \in \{f_\xi \colon \xi < \mathfrak{c}\}$ are distinct, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}\setminus\{0\}$ and $n \in \mathbb{N}$. Let $A = \{j \in \{1, \ldots, n\} \colon \alpha_j > 0\}$ and $B = \{j \in \{1, \ldots, n\} \colon \alpha_j < 0\}$. By the constuction of the sets M_{N_ξ}'s and $M_{N_\xi^c}$'s, there exists an interval J of the form $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ or $\left(\frac{1}{2}, 1\right)$ such that

$$J \subseteq \left(\bigcap_{j\in A} M_{N_{\xi_j}}\right) \cap \left(\bigcap_{j\in B} M_{N_{\xi_j}^c}\right).$$

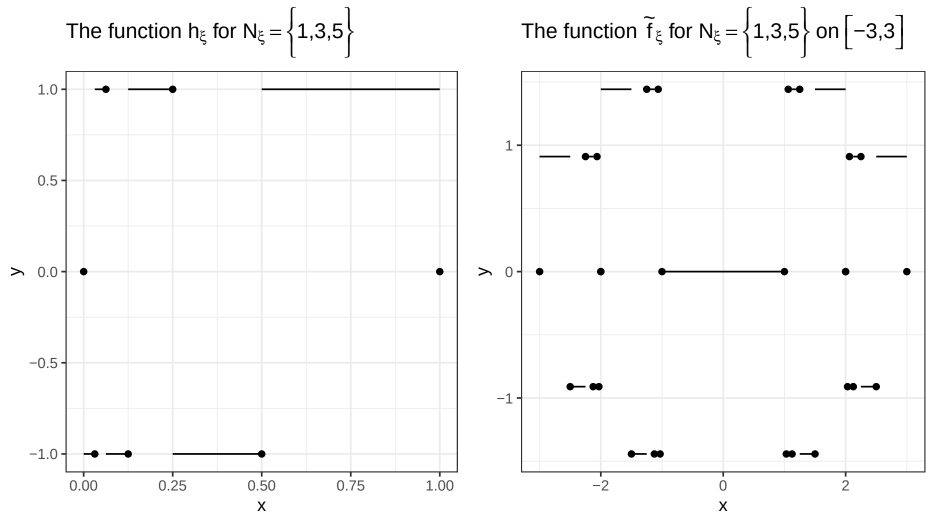

FIGURE 1. Left figure shows an approximate representation of the functions h_ξ and right figure shows an approximate representation of the functions $\widetilde{f}_\xi$ defined in the proof of Theorem 2.1.

Thus, for every odd positive integer $k \geq 3$, we have $\sum_{i=1}^n \alpha_i f_{\xi_i} \upharpoonright (k+J) = \frac{1}{\ln k} \sum_{i=1}^n |\alpha_i|$. Hence,

$$\begin{aligned}\int_{\mathbb{R}} \left| \sum_{i=1}^n \alpha_i f_{\xi_i} \right|^p d\lambda &\geq \int_{\bigcup_{k\in\mathbb{N}}(2k+1+J)} \left| \sum_{i=1}^n \alpha_i f_{\xi_i} \right|^p d\lambda \\ &= \lim_{m\to\infty} \sum_{k=1}^m \int_{2k+1+J} \left| \sum_{i=1}^n \alpha_i f_{\xi_i} \right|^p d\lambda \\ &= \lim_{m\to\infty} \sum_{k=1}^m \frac{1}{\ln^p(2k+1)} \int_{2k+1+J} \left(\sum_{i=1}^n |\alpha_i| \right)^p d\lambda \\ &= \lambda(J) \left(\sum_{i=1}^n |\alpha_i| \right)^p \lim_{m\to\infty} \sum_{k=1}^m \frac{1}{\ln^p(2k+1)}.\end{aligned}$$

However, the series $\sum_{k=1}^\infty \frac{1}{\ln^p(2k)}$ diverges. Thus, $\sum_{i=1}^n \alpha_i f_{\xi_i} \notin \bigcup_{p>0} \mathcal{L}_p$. □

Remark 2.2. *By adapting the proof of Theorem 2.1, the same can be said for any closed interval $[a,b]$ instead of $[0,1]$. Furthermore, by also adapting the proof of Theorem 2.1, the same result is also true for the family $\mathcal{R}_b(I) \setminus \bigcup_{p>0} \mathcal{L}_p(I)$, where I is an unbounded interval and by taking $[a,b] \subset I$ for the composition argument.*

Corollary 2.3. *Let I be a non-degenerate interval. The family $\mathcal{R}_b(I) \setminus \bigcup_{p>0} \mathcal{L}_p(I)$ is $\mathfrak{c}$-lineable.*

Theorem 2.4. *Let $I = [a,b]$. Then the family $\mathcal{R}_u(I) \setminus \mathcal{L}_1(I)$ is $\mathfrak{c}$-lineable.*

Proof. We will find a family of linearly independent functions $\mathcal{F}$ such that $\operatorname{span}\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{R}_u(I) \setminus \mathcal{L}_1(I)$ and $\operatorname{card}(\mathcal{F}) = \mathfrak{c}$.

Assume, without loss of generality, that $I = [0,1]$. For every $\beta \in (1,2)$ we define a function $f_\beta \colon [0,1] \to \mathbb{R}$ as follows

$$f_\beta(x) = \begin{cases} (-1)^n n^\beta & \text{if } x \in I_n = \left[\frac{n-1}{n}, \frac{n}{n+1}\right) \text{ where } n \in \mathbb{N}, \\ 0 & \text{if } x = 1. \end{cases}$$

First notice that f_β is unbounded. Now, for every $\beta \in (1,2)$, we have that

$$\int_0^1 f_\beta(t)dt = \sum_{n=1}^{\infty} \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} (-1)^n n^\beta dt = \sum_{n=1}^{\infty} (-1)^n \frac{n^{\beta-1}}{n+1},$$

converges. Furthermore, for every $\beta \in (1,2)$ we have that

$$\int_{[0,1]} |f_\beta(x)| d\lambda = \sum_{n=1}^{\infty} \int_{I_n} n^\beta d\lambda = \sum_{n=1}^{\infty} \frac{n^{\beta-1}}{n+1},$$

diverges. Thus, we have $\mathcal{F} \subset \mathcal{R}_u(I) \setminus \mathcal{L}_1(I)$. It is easy to see that any linear combination of functions in $\mathcal{F}$ is Riemann integrable. It therefore suffices to show that the functions in $\mathcal{F}$ are linearly independent and any linear combination which is not 0 is unbounded and not in $\mathcal{L}_1(I)$. Let $f = \sum_{i=1}^m \alpha_i f_{\beta_i}$, where $f_{\beta_1}, \ldots, f_{\beta_m} \in \mathcal{F}$ are distinct, $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{N}$. Assume, without loss of generality, that $\beta_1 > \cdots > \beta_m$. Then,

$$\lim_{x \to 1^-} \frac{f(x)}{f_{\beta_1}(x)} = \alpha_1. \tag{2.1}$$

Thus, there exists $\frac{1}{k}$ with $k \in \mathbb{N}$ and a non-degenerate interval $[s,1]$ such that $\left|\frac{1}{k} f_{\beta_1}\right| \leq |f|$ on $[s,1]$. If f would be in $\mathcal{L}_1(I)$, then so would f_{β_1}, a contradiction, hence $f \notin \mathcal{L}_1(I)$. Finally, if f were identically 0, then the limit in equation (2.1) is equal to 0 and we have a contradiction. The same argument proves that f is unbounded. $\square$

Remark 2.5. *In the family $\mathcal{R}(I) \setminus \mathcal{L}_1(I)$, the vector space obtained in the proof of Theorem 2.4 is not the only vector space of dimension $\mathfrak{c}$. Indeed, assume that $I = [0,1]$ and let S be a Hamel basis of a vector space of dimension $\mathfrak{c}$ of conditionally convergent series of the form $s = \left\{\sum_{n=1}^k c_n\right\}_{k \in \mathbb{N}}$ (see [1]). For every conditionally convergent series s and $n \in \mathbb{N}$, we will denote by $s(n) = c_n$ the n-th coefficient of the series s. Let $\{s_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration, without repetition, of S. For every $\xi < \mathfrak{c}$, let us define the function $f_\xi \colon [0,1] \to \mathbb{R}$ as*

$$f_\xi(n) = \begin{cases} s_\xi(n) 2^n & \text{if } x \in I_n = \left(\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to prove that $\mathcal{F} = \{f_\xi \colon \xi < \mathfrak{c}\}$ is a family of linearly independent functions such that span$\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{R}(I) \setminus \mathcal{L}_1(I)$.

Theorem 2.6. *Let I be an unbounded interval. Then the family $\mathcal{R}_u(I) \setminus \mathcal{L}_p(I)$ is $\mathfrak{c}$-lineable for every $p > 1$.*

Proof. Assume, without loss of generality, that $I = \mathbb{R}$ and fix $p > 1$. It is enough to find a family of linearly independent functions $\mathcal{F}$ such that $\operatorname{span}\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{R}_u \setminus \mathcal{L}_p$ and $\operatorname{card}(\mathcal{F}) = \mathfrak{c}$.

For every $\beta \in (0,1)$ let us define the function f_β as follows

$$f_\beta(x) = \begin{cases} \frac{1}{(x-\beta)^{\frac{1}{p}}} & \text{if } x \in (\beta, \beta+1), \\ 0 & \text{otherwise.} \end{cases}$$

We will show that the family $\mathcal{F} = \{f_\beta \colon \beta \in (0,1)\}$ is as needed.

First, notice that all the functions are distinct. Indeed, take distinct $\beta_1, \beta_2 \in (0,1)$. Assume, without loss of generality, that $\beta_1 < \beta_2$. Then, for every $x \in (\beta_1, \beta_2)$, we have $f_{\beta_2}(x) = 0 \neq \frac{1}{(x-\beta_1)^{\frac{1}{p}}} = f_{\beta_1}(x)$. Furthermore, the functions f_β's are linearly independent. Assume that $f = \sum_{i=1}^n \alpha_i f_{\beta_i} = 0$, where $f_{\beta_1}, \ldots, f_{\beta_n} \in \mathcal{F}$ are distinct, $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $n \in \mathbb{N}$. It is enough to show that $\alpha_i = 0$ for every $i \in \{1, ..., n\}$. Assume, without loss of generality, that $\beta_1 < \cdots < \beta_n$. Then, for every $x \in (\beta_1, \beta_2)$, we have $0 = f(x) = \alpha_1 f_{\beta_1}(x) = \alpha_1 \frac{1}{(x-\beta_1)^{\frac{1}{p}}}$. Thus, $\alpha_1 = 0$. Repeating the same arguments for $\sum_{i=2}^n \alpha_i f_{\beta_i}$ we have the linear independency.

Now we will show that $f_\beta \in \mathcal{R}_u \setminus \mathcal{L}_p$ for every $\beta \in (0,1)$. It is easy to see that f_β is unbounded. First, we will show that $f_\beta \in \mathcal{R}_u$:

$$\int_{-\infty}^{\infty} f_\beta(t)dt = \int_\beta^{\beta+1} f_\beta(t)dt = \frac{p}{p-1} < \infty.$$

Also, $f_\beta \notin \mathcal{L}_p$. Indeed,

$$\int_{\mathbb{R}} |f_\beta(x)|^p d\lambda = \int_{(\beta,\beta+1)} |f_\beta(x)|^p d\lambda = \int_{(\beta,\beta+1)} \frac{1}{x-\beta} d\lambda = \infty.$$

Finally, it remains to show that any linear combination of the form $\sum_{i=1}^n \alpha_i f_{\beta_i} = f$, where $f_{\beta_1}, \ldots, f_{\beta_n} \in \mathcal{F}$ are distinct, $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, belongs to $\mathcal{R}_u \setminus \mathcal{L}_p$. It is easy to see that $f \in \mathcal{R}_u$. Thus, it is enough to show that $f \notin \mathcal{L}_p$. Assume, without loss of generality, that $\beta_1 < \cdots < \beta_n$. Then,

$$\int_{\mathbb{R}} |f(x)|^p \, d\lambda \geq \int_{(\beta_1,\beta_2)} |f(x)|^p \, d\lambda = \int_{(\beta_1,\beta_2)} \frac{1}{x-\beta_1} d\lambda = \infty.$$

□

By using similar ideas to the ones in [17] we have the following result.

Theorem 2.7. *Let I be a non-degenerate interval. Then the family $\mathcal{R}(I,\mathbb{C}) \setminus \mathcal{L}_1(I,\mathbb{C})$ is strongly $\mathfrak{c}$-algebrable.*

Theorem 2.8. *The family $\mathcal{L}_1 \cap \mathcal{NR}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let Z_0 be a Lebesgue measurable set such that $\lambda(Z_0 \cap J) > 0$ and $\lambda(Z_0^c \cap J) > 0$ for every non-degenerate interval $J \subset [0,1]$ (see [2] or [18]). We define $Z = \mathbb{Q} \cup Z_0$. Take H a positive Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. The family of functions $f_h(x) = e^{hx}\chi_Z(x)$ with $h \in H$ forms a system of generators of an algebra. Note that, by construction of Z, the function f_h is nowhere continuous for any $h \in H$. Furthermore, it is easy to see that every algebraic combination of the functions f_h's that is not the zero function belongs to $\mathcal{L}_1$ and also to $\mathcal{NR}$ by the Lebesgue criterion for Riemann integrability. Therefore, it suffices to show that given $f_{h_1}, \ldots, f_{h_n}$ distinct and P a polynomial of positive degree, without constant term in $n \in \mathbb{N}$ variables and coefficients $\alpha_i \neq 0$, we have that $P(f_{h_1}, \ldots, f_{h_n})(x) = \left(\sum_{i=1}^k \alpha_i e^{\beta_i x}\right) \chi_Z(x)$ is not identically zero. Notice that $\beta_i > 0$ since H is a positive Hamel basis. By way of contradiction assume that $P(f_{h_1}, \dots, f_{h_n}) \equiv 0$ and also assume, without loss of generality, that $\beta_1 < \cdots < \beta_k$. Then,

$$0 = \lim_{x \in Z,\ x \to +\infty} \frac{P(f_{h_1}, \dots, f_{h_n})(x)}{e^{\beta_k x}} = \alpha_k,$$

which is absurd. Thus, $\mathcal{L}_1 \cap \mathcal{NR}$ is strongly $\mathfrak{c}$-algebrable. $\square$

Remark 2.9. *By restricting the family of functions in Theorem 2.8 to the interval $[0, 1]$, we have that $\mathcal{L}_1([0, 1]) \cap \mathcal{NR}([0, 1])$ is also strongly $\mathfrak{c}$-algebrable. It is easy to that it is $\mathfrak{c}$-algebrable, but also $P(f_{h_1}, \dots, f_{h_n})$ is not the zero function given $f_{h_1}, \dots, f_{h_n}$ distinct and P a polynomial of positive degree, without independent term in $n \in \mathbb{N}$ variables and coefficients $\alpha_i \neq 0$. Indeed, if the function $P(f_{h_1}, \dots, f_{h_n})(x) = \left(\sum_{i=1}^k \alpha_i e^{\beta_i x}\right) \chi_Z(x)$ were identically zero, then for every $x \in Z_0$ we have $\sum_{i=1}^k \alpha_i e^{\beta_i x} = 0$. Thus, by continuity, we have $\sum_{i=1}^k \alpha_i e^{\beta_i x} = 0$ on $[0, 1]$. But this is a contradiction since the functions $\{e^{\beta x} \colon \beta \in \mathbb{R}\}$ are linearly independent. In fact, notice that the same can be said for any non-degenerate interval $I = [a, b]$. Furthermore, these functions still belong to $\mathcal{L}_1(I) \cap \mathcal{NR}(I)$ when they are modified on a measure zero set contained in I.*

Theorem 2.10. *Let I be an unbounded interval. Then the family $\mathcal{C}_b(I) \setminus \mathcal{R}(I)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Assume, without loss of generality, that $I = [0, \infty)$. Take H a positive Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Index the family $\mathcal{N}$ with H, that is, $\mathcal{N} = \{N_h \colon h \in H\}$. For every $h \in H$, we will define f_h first on $[0, 1]$ and then extend it periodically to I. For every $x \in [0, 1]$, take (see Figure 2)

$$f_h(x) = \begin{cases} s_h(1)\left(x - \frac{1}{2}\right)(x-1)\, e^{hx} & \text{if } x \in \left(\frac{1}{2}, 1\right), \\ s_h(n)\left(x - \frac{1}{2^n}\right)\left(x - \frac{1}{2^{n-1}}\right) e^{hx} & \text{if } x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \text{ with } n \geq 2, \\ 0 & \text{if } x \in \{0, 1\}, \end{cases}$$

where $s_h \colon \mathbb{N} \to \{-1, 1\}$ is defined by

$$s_h(n) = \begin{cases} 1 & \text{if } n \in N_h, \\ -1 & \text{if } n \in N_h^c. \end{cases}$$

Notice that the family $\mathcal{F} = \{f_h \colon h \in H\}$ forms a system of generators of an algebra of continuous bounded functions. Indeed, the functions f_h are continuous and bounded by construction and, therefore, every algebraic combination of functions f_h is also continuous. Take $P(f_{h_1}, \dots, f_{h_n})$, where $f_{h_1}, \dots, f_{h_n} \in \mathcal{F}$ are distinct and P is a polynomial of positive degree, without constant term in $n \in \mathbb{N}$ variables and coefficients $\alpha_i \neq 0$. Thus $P(f_{h_1}, \dots, f_{h_n})$ is of the form $P(f_{h_1}, \dots, f_{h_n})(x) = \sum_{i=1}^n \alpha_i e^{\beta_i x} s_i(n)\left(x - \frac{1}{2^n}\right)\left(x - \frac{1}{2^{n-1}}\right)$ for every $x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ and this repeats periodically, where $\beta_i > 0$ and $s_i(n) \in \{-1, 1\}$. Assume, without loss of generality, that $\beta_1 < \cdots < \beta_k$. By way of contradiction assume that $P(f_{h_1}, \dots, f_{h_n}) \equiv 0$. Take J an interval of the form $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$ such that $J \subset M_{h_1} \cap \dots M_{h_n}$ and fix $x_J \in J$. Then,

$$0 = \lim_{\substack{y \to +\infty \\ y \in \{x_J + k \colon k \in \mathbb{N}\}}} \frac{P(f_{h_1}, \dots, f_{h_n})(y)}{e^{\beta_k y}} = \pm\infty,$$

which is absurd.

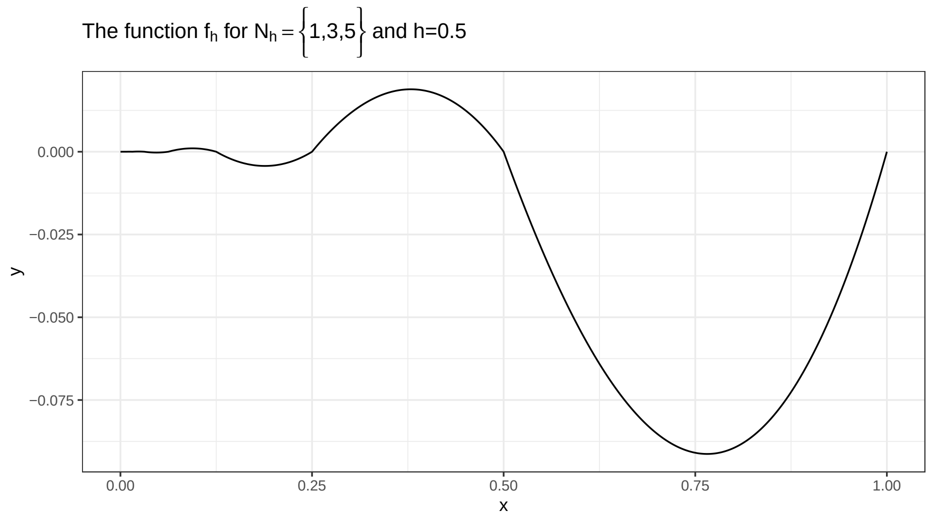

FIGURE 2. Approximate representation of the functions f_h defined in the proof of Theorem 2.10.

Finally, we will show that any algebraic combination of functions f_h (that is not the zero function) does not belong to $\mathcal{R}(I)$. Let $F(x) = \int_0^x P(f_{h_1}, \ldots, f_{h_n})(t)dt$. Since the functions f_h are periodic and non-zero, we have that $F(n) = nF(1)$ for every $n \in \mathbb{N}$. We know that there exists $x \in (0,1)$ fulfilling $F(x) \neq 0$, hence, considering $F(n+x) = F(x)$ for every $n \in \mathbb{N}$, we have that F cannot be Riemann integrable. $\square$

Theorem 2.11. *Let I be an unbounded interval. Then the family $\mathcal{C}_u(I) \cap \mathcal{R}_u(I)$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Assume, without loss of generality, that $I = [0, \infty)$. We define $f \colon I \to \mathbb{R}$ as follows. For every $n \in \mathbb{N}$, let $f \restriction I_n \equiv 0$ and $f \restriction J_n = T_{J_n}$, where $I_n = \left[n-1, n-2^{-2^n}\right]$, $J_n = \left[n - 2^{-2^n}, n\right]$ and $T_{[a,b]}$ is a map defined by

$$T_{[a,b]}(x) = \begin{cases} \frac{2}{b-a}(x-a) & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{2}{b-a}(b-x) & \text{if } x \in \left[\frac{a+b}{2}, b\right], \end{cases}$$

for any non-degenerate interval $[a,b]$. Notice that $T_{[a,b]}(a) = T_{[a,b]}(b) = 0$ and $T_{[a,b]}\left(\frac{a+b}{2}\right) = 1$.

Take H a positive Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$, we define $f_h(x) = e^{hx} f(x)$ (see Figure 3). Notice that the family $\mathcal{F} = \{f_h \colon h \in H\}$ is a system of generators of a free algebra of unbounded continuous functions on I.

Furthermore, the algebra generated by $\mathcal{F}$ is contained in $\mathcal{R}_u(I)$. Indeed, first we will prove that every algebraic combination of functions in $\mathcal{F}$ is Riemann integrable. Take $f_{h_1}, \ldots, f_{h_m} \in \mathcal{F}$, where not all functions need to be distinct. Then, for every $n \in \mathbb{N}$ and every $x \in J_n$ we have

$$f_{h_1} \cdots f_{h_m} = e^{\beta x} T_{J_n}^m(x),$$

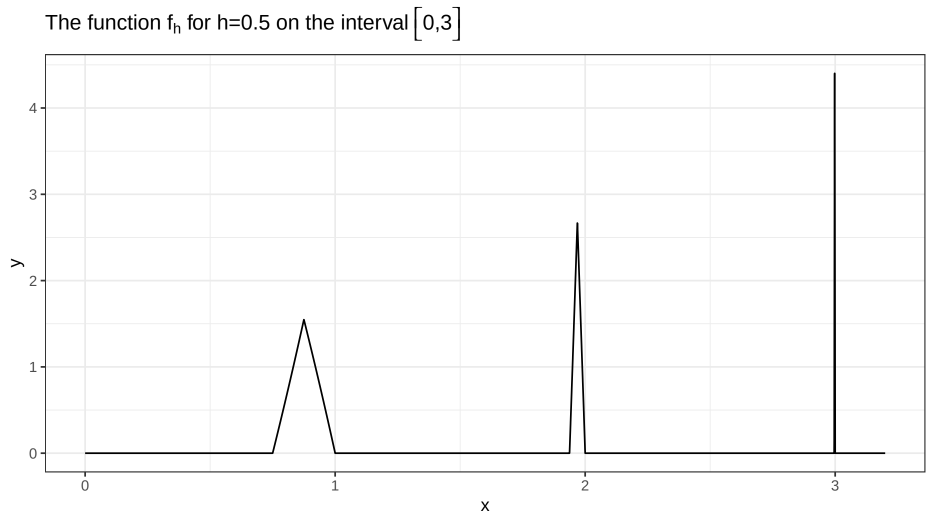

FIGURE 3. Approximate representation of the functions f_h defined in the proof of Theorem 2.11.

where $\beta = h_1 + \cdots + h_n$, and $f_{h_1} \cdots f_{h_m}(x) = 0$ for every $x \in I_n$. Notice that $\beta > 0$ because H is a positive Hamel basis. Thus,

$$\int_{J_n} f_{h_1} \cdots f_{h_m}(t)dt = \int_{J_n} e^{\beta t} T^m(t)dt \leq 2^{-2^n} e^{\beta n}$$

Hence, since $\lim_{n\to\infty} \frac{e^{\beta(n+1)}2^{-2^{(n+1)}}}{e^{\beta n}2^{-2^n}} = 0$, we have by the ratio test that $\int_0^\infty f(t)dt = \sum_{n=1}^\infty 2^{-2^n} e^{\beta n} < \infty$. Therefore, every linear combination of functions $f_{h_1} \cdots f_{h_m}$ is Riemann integrable. It remains to prove that every algebraic combination of functions in $\mathcal{F}$ is unbounded. Let P be a polynomial of positive degree without constant term in $n \in \mathbb{N}$ variables and take $f_{h_1}, \ldots, f_{h_n} \in \mathcal{F}$ distinct. Then,

$$P\left(f_{h_1}, \ldots, f_{h_n}\right)(x) = \sum_{i=1}^{k} a_i e^{\beta_i x} T_{J_m}^{m_i}(x)$$

for every $x \in J_m$ and $m \in \mathbb{N}$, and $P\left(f_{h_1}, \ldots, f_{h_n}\right) \equiv 0$ on $\bigcup_{m=1}^\infty I_m$. Since $\beta_i > 0$ for every $i \in \{1, \ldots, k\}$ and $T^{m_i}\left(m + 2^{-(2^m+1)}\right) = 1$, we have $P\left(f_{h_1}, \ldots, f_{h_n}\right)$ unbounded on $[0, \infty)$. Indeed, assume without loss of generality that $\beta_1 < \cdots < \beta_k$. Then,

$$\lim_{m\to\infty} P\left(f_{h_1}, \ldots, f_{h_n}\right)\left(m + 2^{-(2^m+1)}\right) = \mathrm{sign}(a_k)\infty,$$

where $\mathrm{sign}(a_k)$ denotes the sign of a_k. $\square$

3. Lineability and interaction among the classes of Denjoy, Khintchine, Lebesgue, and Riemann integrable functions

Theorem 3.1. *Let $I = [a, b]$. Then the family $\mathcal{D}(I) \setminus (\mathcal{L}_1(I) \cup \mathcal{R}(I))$ is $\mathfrak{c}$-lineable.*

Proof. Assume, without loss of generality, that $I = [0, 1]$. By Theorem 2.4 (or Remark 2.5) and Remark 2.9, let $S_1 = \{g_\xi : \xi < \mathfrak{c}\}$ and $S_2 = \{h_\xi : \xi < \mathfrak{c}\}$ be Hamel basis of dimension $\mathfrak{c}$ such that $\operatorname{span}\{S_1\} \subset \mathcal{R}\left(\left[0,\frac{1}{2}\right]\right) \setminus \mathcal{L}_1\left(\left[0,\frac{1}{2}\right]\right)$ and $\operatorname{span}\{S_2\} \subset \mathcal{L}_1\left(\left[\frac{1}{2},1\right]\right) \cap \mathcal{NR}\left(\left[\frac{1}{2},1\right]\right)$. For every $\xi < \mathfrak{c}$, let us define $f_\xi \colon [0,1] \to \mathbb{R}$ as

$$f_\xi(x) = \begin{cases} g_\xi(x) & \text{if } x \in \left(0,\frac{1}{2}\right), \\ h_\xi(x) & \text{if } \left(\frac{1}{2},1\right), \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have that $\mathcal{F} = \{f_\xi \colon \xi < \mathfrak{c}\}$ is a family of linearly independent functions such that $\operatorname{span}\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{D}(I) \setminus (\mathcal{L}_1(I) \cup \mathcal{R}(I))$. Indeed, it is easy to see that the functions in $\operatorname{span}\{\mathcal{F}\}\setminus\{0\}$ do not belong to $\mathcal{L}_1(I)\cup\mathcal{R}(I)$. But also $\operatorname{span}\{\mathcal{F}\} \subset \mathcal{D}(I)$ by the additivity of $\mathcal{D}(I)$ with the decomposition of the integral into subintervals and the linearity of $\mathcal{D}(I)$ (see [23, theorems 7.3 (b) and 7.4]). $\square$

Theorem 3.2. *Let $I = [a,b]$. Then the family $\mathcal{K}(I) \setminus \mathcal{D}(I)$ is $\mathfrak{c}$-lineable.*

Proof. Assume, without loss of generality, that $I = [0,1]$. Take C the standard Cantor set, $\bigcup_{n\in\mathbb{N}}(a_n,b_n) = [0,1] \setminus C$ and the functions f and F defined as follows (see Figure 4):

$$f(x) = \begin{cases} \beta_n & \text{if } x \in (a_n,c_n], \\ -\beta_n & \text{if } x \in (c_n,b_n), \end{cases}$$

where c_n is the midpoint of (a_n,b_n) and $\beta_n = \frac{1}{k(b_n-a_n)}$ with $k > 0$ such that $\frac{1}{3^k} = b_n - a_n$, and

$$F(x) = \begin{cases} \int_{a_n}^x f(t)dt & \text{if } x \in [a_n,b_n], \\ 0 & \text{if } x \in C, \end{cases}$$

(see [23, exercise 7.9] and its proof for more details on f and F). Notice that $F(a_n) = F(b_n) = 0$. Let us define the functions

$$r_\xi(x) = \begin{cases} \frac{2^n}{n} f(2^n x - 1) & \text{if } x \in \left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right] \cap M_{N_\xi}, \\ 0 & \text{otherwise;} \end{cases}$$

$$R_\xi(x) = \begin{cases} \frac{1}{n} F(2^n x - 1) & \text{if } x \in \left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right] \cap M_{N_\xi}, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that the family of functions $\mathcal{F} = \{r_\xi \colon \xi < \mathfrak{c}\}$ is linearly independent and $\operatorname{span}\{\mathcal{F}\} \setminus \{0\} \subset \mathcal{K}(I) \setminus \mathcal{D}(I)$. Notice that, since $F' = f$ a.e., we have $R'_\xi = r_\xi$ a.e. for any $\xi < \mathfrak{c}$. Hence, for any $\xi < \mathfrak{c}$, $(R_\xi)'_{ap} = r_\xi$ a.e. Also, R_ξ is continuous on $[0,1]$ for any $\xi < \mathfrak{c}$. Indeed, if $n \in N_\xi^c$, then $R_\xi \equiv 0$ on $\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right)$. If $n \in N_\xi$, then $R_\xi(x) = \frac{1}{n}F(2^n x - 1)$ is continuous on $\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right]$ because F is continuous on $[0,1]$. Observe that the points of the form $\frac{1}{2^n}$ might be problematic but this is not the case since

$$R_\xi\left(\frac{1}{2^n}\right) = \begin{cases} 0 & \text{if } n+1 \in N_\xi^c, \\ \frac{1}{n+1}F(1) = 0 & \text{if } n+1 \in N_\xi. \end{cases}$$

Finally, notice that $\lim_{x\to 0} R_\xi(x) = 0 = R_\xi(0)$. Furthemore, for every $\xi < \mathfrak{c}$, the function R_ξ is ACG. To see this, observe first that R_ξ is AC on $\{0\}$ and on the intervals $\left(\frac{1}{2^n},\frac{1}{2^{n-1}}\right]$ provided $n \in N_\xi^c$. Also, since F is AC on the intervals $[a_k,b_k]$ and $a_k \leq 2^n x - 1 \leq b_k$ if, and only if, $\frac{a_n+1}{2^n} \leq x \leq \frac{b_n+1}{2^n}$ we have that R_ξ is AC on $\left[\frac{a_k+1}{2^n}\frac{b_k+1}{2^n}\right]$ for every $k \in \mathbb{N}$ and $n \in N_\xi$. Moreover, since F is AC on C and

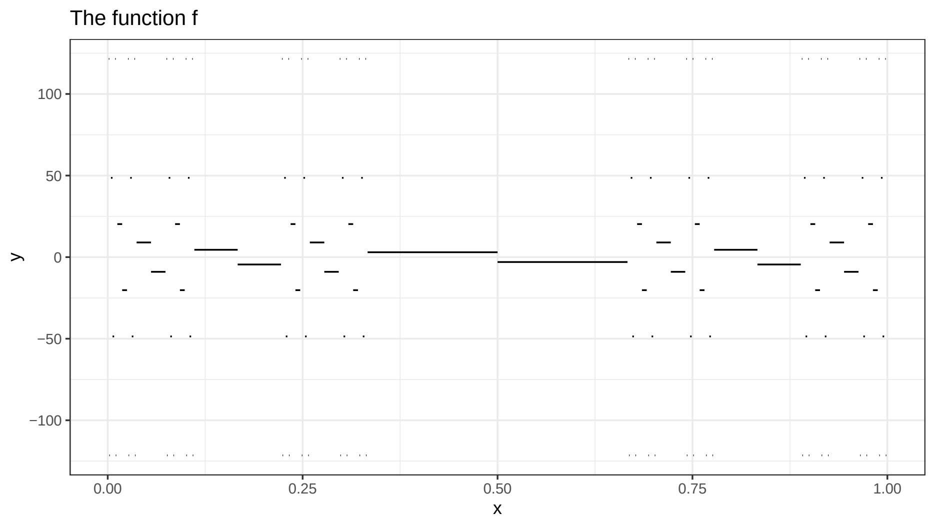

FIGURE 4. Approximate representation of the function f defined in the proof of Theorem 3.2.

$2^n x - 1 = y$ for every $y \in C$ if, and only if, $x = \frac{y+1}{2^n}$ for every $y \in C$ we have that R_ξ is AC on $\frac{1}{2^n}(C+1) := \left\{\frac{1}{2^n}(y+1) \colon y \in C\right\}$. Thus, $\mathcal{F} \subset \mathcal{K}(I)$.

To finish the proof, we will show that the functions in $\mathcal{F}$ are linearly independent and $\operatorname{span}\{\mathcal{F}\} \cap \mathcal{D}(I) = \{0\}$. Let $r = \sum_{i=1}^m \alpha_i r_{\xi_i}$ and $R = \sum_{i=1}^m \alpha_i R_{\xi_i}$, where $r_{\xi_1}, \dots, r_{\xi_m} \in \mathcal{F}$ are distinct, $\alpha_1, \dots, \alpha_n \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{N}$. It is easy to see that $R' = r$ a.e. Also, notice that by contruction of the sets $\{N_\xi \colon \xi < \mathfrak{c}\}$, there exists an interval of the form $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ such that $r = \alpha_1 r_{\xi_1}$. Thus, r cannot be the zero function which implies the linear independence. Furthermore, $r(x) = \frac{2^n}{n} f(2^n x - 1)$ on $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. Hence, since F is not ACG_* on $[0,1]$, we have that R is not ACG_* on $\left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$. Therefore, since R is ACG (ACG is closed under linear combinations) and by the last paragraph of the proof of [23, exercise 7.9], we have that $\operatorname{span}\{\mathcal{F}\} \cap \mathcal{D}(I) = \{0\}$. Thus, by the linearity of Khintchine integral (see [23, theorem 15.6]), we have the desired result. $\square$

Theorem 3.3. *Let $I = [a,b]$. Then the family of Lebesgue measurable functions from I to $\mathbb{R}$ that are not in $\mathcal{K}(I)$ is $\mathfrak{c}$-lineable.*

Proof. Assume, without loss of generality, that $I = [0,1]$. Let $\{a_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration, without repetition, of $\left[0, \frac{1}{2}\right)$. Now, for every $\xi < \mathfrak{c}$, let $\langle (a_n^\xi, b_n^\xi) \colon n \in \mathbb{N}\rangle$ be a sequence of disjoint non-degenerate intervals contained in $[a_\xi, 1]$ such that

(i) $b_1^\xi < 1$ and $b_{n+1}^\xi < b_n^\xi$,
(ii) $\{a_n^\xi\}_{n \in \mathbb{N}}$ converges to a_ξ, and
(iii) a_ξ is a point of dispersion in the sense of Lebesgue of $\bigcup_{n=1}^\infty (a_n^\xi, b_n^\xi)$.

For any $\xi < \mathfrak{c}$, we define $f_\xi \colon [0,1] \to \mathbb{R}$ as

$$f_\xi(x) = \begin{cases} \sin^2\left(\frac{x - a_n^\xi}{b_n^\xi - a_n^\xi}\pi\right) & \text{if } x \in (a_n^\xi, b_n^\xi) \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The functions in $\mathcal{F} = \{f_\xi : \xi < \mathfrak{c}\}$ are linearly independent and $\operatorname{span}\{\mathcal{F}\} \setminus \{0\}$ is a family of Lebesgue measurable functions that are not Khintchine integrable.

To see this, first we will show that they are linearly independent. Take $f_{\xi_1}, \dots, f_{\xi_m} \in \mathcal{F}$ distinct, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and $m \in \mathbb{N}$. Without loss of generality, assume that $a_{\xi_1} < \cdots < a_{\xi_m}$. If $\sum_{i=1}^m \alpha_i f_{\xi_i} = 0$, then $\alpha_i = 0$ for every $i \in \{1, \dots, m\}$. Indeed, since $a_n^{\xi_1}$ converges to a_{ξ_1}, $\{a_n^{\xi_1}\}_{n\in\mathbb{N}}$ is strictly decreasing and $a_{\xi_1} < \cdots < a_{\xi_m}$, there exists $n_0 \in \mathbb{N}$ such that $(a_n^{\xi_1}, b_n^{\xi_1}) \cap (a_{\xi_1}, a_{\xi_2})$ is nonempty for every $n \geq n_0$. Take $x_0 \in (a_{n_0}^{\xi_1}, b_{n_0}^{\xi_1}) \cap (a_{\xi_1}, a_{\xi_2})$, then $0 = \alpha_1 f_{\xi_1}(x_0)$ and where $f_{\xi_1}(x_0) \neq 0$. Thus, $\alpha_1 = 0$. Repeating the same arguments, we have that $\alpha_i = 0$ for any $i \in \{1, \dots, m\}$.

It is easy to see that the functions in $\mathcal{F}$ are Lebesgue measurable, which implies that any linear combination is also Lebesgue measurable. Notice now that by construction f_ξ is differentiable in $(a_\xi, 1]$, continous in $(a_\xi, 1]$ but not in a_ξ, and approximately differentiable in a_ξ. Thus, for every $\xi < \mathfrak{c}$, the function f_ξ is not Khintchine integrable. Indeed, assume otherwise. Following the proof of [23, exercise 15.3], the function f_ξ is continuous at a_ξ which is absurd.

It remains to show that given any linear combination f of functions in $\mathcal{F}$ which is not identically zero we have $f \notin \mathcal{K}(I)$. Let $f = \sum_{i=1}^m \alpha_i f_{\xi_i}$, where $f_{\xi_1}, \dots, f_{\xi_m} \in \mathcal{F}$ are distinct, $\alpha_1, \dots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and $m \in \mathbb{N}$. Assume, without loss of generality, that $a_{\xi_1} < \cdots < a_{\xi_m}$. By way of of contradiction, assume that $f \in \mathcal{K}(I)$, then, by [23, theorem 15.4 (a)], $f \upharpoonright [a_{\xi_1}, a_{\xi_2}] \in \mathcal{K}([a_{\xi_1}, a_{\xi_2}])$. But in this case, using a similar argument as above that comes from [23, exercise 15.3], we have $f \upharpoonright [a_{\xi_1}, a_{\xi_2}] \notin \mathcal{K}([a_{\xi_1}, a_{\xi_2}])$, which is absurd. □

4. Final remarks

In this last small section, we shall provide some comments on [21, §4], which deals with continuous unbounded functions on arbitrary non-compact metric spaces (providing an improvement of [21, Theorem 4.1]).

Theorem 4.1. *In every non-compact metric space X, the family of all continuous unbounded functions from X to $\mathbb{R}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. We will follow the argument found in [21, section 4]. Since X is not compact, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with no convergent subsequence (we can assume that all elements x_n are distinct). Thus, the set $A = \{x_n : n \in \mathbb{N}\}$ is closed and has the discrete topology. Take H a positive Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. For every $h \in H$ and every $n \in \mathbb{N}$, let $g_h(x_n) = e^{hn}$. Since A is closed and has the discrete topology, by Tietze's Extension Theorem (see [24, theorem 35.1]) there exists a continuous extension f_h to X of g_h such that $f_h \upharpoonright A = g_h$. It is easy to see that $\{f_h : h \in H\}$ is a system of generators of a free algebra of continous unbounded functions on X. □

Remark 4.2. *Let $p \in \beta\mathbb{N} \setminus \mathbb{N}$, where $\mathbb{N}$ is endowed with the discrete topology and $\beta\mathbb{N}$ is the Stone–Čech compactification by ultrafilters of $\mathbb{N}$ (i.e., any maximal family of subsets of $\mathbb{N}$ closed under taking supersets and finite intersections, see* [16]*). Consider the set $X = \beta\mathbb{N} \setminus \{p\}$. Notice that X is not compact since p is not isolated.*

Let $f : X \to \mathbb{R}$ be a continuous function and take $\overline{f} = f \upharpoonright \mathbb{N}$. Then we have two possible cases:

(i) *If $\overline{f}$ is bounded, then there exists a unique continuous and bounded extension of $\overline{f}$ to $\beta\mathbb{N}$ denoted by $\beta\overline{f}$. Since $f \restriction \mathbb{N} = \beta\overline{f} \restriction \mathbb{N}$, we have that $f = \beta\overline{f} \restriction X$. Thus, f is bounded.*

(ii) *If $\overline{f}$ is bounded, take an increasing sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\overline{f}(x_n) < \overline{f}(x_{n+1})$. Consider the basic neighborhoods $V_{\{x_q,x_{q+1},\ldots\}}$ and an element r in their intersection. Such an r exists since there are infinitely many ultrafilters that contain the set $\{\{x_q, x_{q+1}, \ldots\} : q \in \mathbb{N}\}$. In a basic neighborhood of r there are always natural numbers bigger than any real number, thus f cannot be continuous at r and we have contradiction. Hence, there are no continuous unbounded functions on X.*

To sum it up we have just proven that, in Theorem 4.1, the metric space cannot replaced by a completely regular space.

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