Lineability, algebrability, and sequences of random variables

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We show that, when omitting one condition in several well-known convergence results from probability and measure theory (such as the Dominated Convergence Theorem, Fatou's Lemma, or the Strong Law of Large Numbers), we can construct "very large" (in terms of the cardinality of their systems of generators) spaces and algebras of counterexamples. Moreover, we show that on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ the families of sequences of random variables converging in probability but (i) not converging outside a set of measure 0 or (ii) not converging in arithmetic mean are also "very large".

1 Introduction, preliminaries and background

The study of linearity within non linear settings (and, in particular, the search for linear structures of mathematical objects enjoying certain *special* property) has, for the past decade, become a sort of a trend in several different areas of Mathematics. A vast literature on this topic has been built during the last decade, from Linear Chaos to Real and Complex Analysis [4, 9, 10, 15, 20, 27, 28, 37], passing through Set Theory [21, 22, 29] and Linear and Multilinear Algebra [16], Operator Theory [13, 17], Topology and Measure Theory [15], Functional Analysis [14, 20], or Abstract Algebra [2].

Let us recall some terminology we shall need throughout this work (which can be found in, for instance, [2–7, 9, 16, 23, 36]). Assume that X is a vector space and α is a cardinal number. Then a subset $A \subset X$ is said to be:

- *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$.
- α*-lineable* if there exists a vector space M with $\dim(M) = \alpha$ and $M \setminus \{0\} \subset A$.

If, in addition, X is a topological vector space, then the subset A is said to be:

- *spaceable* in X whenever there is a closed infinite-dimensional vector subspace M of X such that $M \setminus \{0\} \subset A$.
- *dense-lineable* in X whenever there is a dense vector subspace M of X satisfying $M \setminus \{0\} \subset A$.

And, provided that X is a vector space contained in some (linear) algebra, then A is called:

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- *algebrable* if there is an algebra M so that $M\setminus\{0\}\subset A$ and M is infinitely generated, that is, the cardinality of any system of generators of M is infinite.

- *strongly α-algebrable* if there exists an α-generated *free* algebra M with $M\setminus\{0\}\subset A$. Recall that if X is contained in a commutative algebra, then a set $B\subset X$ is a generating set of some free algebra contained in A if and only if for any $N\in\mathbb{N}$, any nonzero polynomial P in N variables without constant term and any distinct $f_1,\dots,f_N\in B$, we have $P(f_1,\dots,f_N)\in A\setminus\{0\}$.

Recall that a *Riesz space,* also called a vector lattice, is a partially ordered (with, say, the order $\preceq$) vector space X where the order structure is a lattice, that is, the order $\preceq$ satisfies the following properties for every pair of vectors $x,y\in X$: there is a supremum $x\vee y\in X$; for any $z\in X$ and any scalar $\alpha\geq 0$, the fact $x\preceq y$ implies $x+z\preceq y+z$ and $\alpha x\preceq\alpha y$. Then the existence of infimum $x\wedge y\in X$ is automatically satisfied; namely, $x\wedge y=-((-x)\vee(-y))$. Also, if given $x,y\in X$ we have that neither $x\preceq y$ nor $y\preceq x$ then we shall say that x and y are non-comparable or incomparable.

A *Banach lattice* is a Riesz space X endowed with a norm $\|\cdot\|$ such that $(X,\|\cdot\|)$ is a Banach space and $|x|\preceq|y|$ implies $\|x\|\leq\|y\|$, where $|z|:=z\vee(-z)$. See, for instance, the book [35] for fundamentals of Banach lattices.

Finally, and as a variation of the notion of "latticeable" introduced in [36] by T. Oikhberg, let us present the following notion.

Definition 1.1 Suppose that X is a Riesz space and that α is a cardinal number. Then a set $A\subset X$ is said to be:

- *$\alpha-$Riesz-latticeable* if there exists a Riesz space M such that $M\setminus\{0\}\subset A$ and M is a vector space of dimension α.

- *$\alpha-$incomparably-latticeable* if there exists a lattice M such that $M\setminus\{0\}\subset A$ and M contains α incomparable vectors.

- *$\alpha-$incomparably-Riesz-latticeable* if there exists a Riesz space M such that $M\setminus\{0\}\subset A$ and M is a α-dimensional vector space with a basis of incomparable vectors.

One of the earliest results in this direction was provided by V. I. Gurariy, who showed that the set of Weierstrass' monsters (continuous nowhere differentiable functions in $\mathbb{R}$) is lineable [30]. Also, and more recently, Enflo et al. [27] proved that, for every infinite dimensional closed subspace X of $\mathcal{C}[0,1]$, the set of functions in X having infinitely many zeros in $[0,1]$ is spaceable in X. Negative results have also been obtained within this area of research. Indeed, let V stand for a subspace of $\mathcal{C}(\mathbb{R})$ such that every nonzero function in V attains its maximum at one (and only one) point, then $\dim(V)\leq 2$, that is, we cannot have lineability in this case ([12]).

The aim of this paper is twofold, firstly to continue with the ongoing research within this area of study (providing both new results and directions of research) and, secondly, to complete and improve some results on algebraic genericity within the framework of Probability Theory and Stochastic Processes established in [24] (see, also, [8] for more results in this direction of research).

Although it is a trivial notion, and in order to avoid confusion, we will say that a sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers *diverges* to $+\infty$ (in which case we will write $\lim_{n\to\infty} s_n=+\infty$) if and only if for every $L\in\mathbb{R}$ there exists some index $n_0=n_0(L)\in\mathbb{N}$ such that $s_n>L$ for all $n\geq n_0$.
As an standard terminology, given measure space $(\Omega,\mathcal{A},\mu)$, we will let $\mathcal{M}(\Omega,\mathcal{A},\mu)$ denote the family of all measurable, real-valued functions on $(\Omega,\mathcal{A},\mu)$. Also, λ stands for the Lebesgue measure and $\mathcal{B}$ denotes the Borel sets, as usual. In case $\mathbb{P}$ is a probability measure we will refer to $\mathcal{M}(\Omega,\mathcal{A},\mathbb{P})$ as the family of all (real-valued) random variables on $(\Omega,\mathcal{A},\mathbb{P})$. Whenever no confusion may arise we will simply write $\mathcal{M}$ in the sequel. Analogously, $\mathcal{M}_{0,1}$ will denote the family corresponding to the probability space $([0,1],\mathcal{B}([0,1]),\lambda)$ (we make use of standard measure theory notation).
For every measure μ we will write $[\mu]$ if the property of consideration holds for all ω outside a set $N\in\mathcal{A}$ fulfilling $\mu(N)=0$. As usual, $\mathbb{E}(X)$ will denote the expectation of the random variable X, $\mathbf{1}_A$ will denote the indicator of the set A. Moreover, $\mathfrak{c}$ shall stand, as usual, for the cardinality of the continuum and $\aleph_0$ for the cardinality of the natural numbers $\mathbb{N}$.

2 The results

We start with the following simple lemma which, however, is key for Theorem 2.2. In the lemma and subsequently we will write $[N] := \{1, \ldots, N\}$ for every $N \in \mathbb{N}$.

Lemma 2.1 *Suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space which is not isomorphic to $([N], 2^{[N]}, \mu)$ for any $N \in \mathbb{N}$. Then there exists a partition $B_1, B_2, \ldots \in \mathcal{A}$ of Ω such that $\mathbb{P}(B_n) > 0$ for every $n \in \mathbb{N}$.*

Proof. Considering that $(\Omega, \mathcal{A}, \mathbb{P})$ is not isomorphic to $([N], 2^{[N]}, \mu)$ for any probability measure on $2^{[N]}$ and any $N \in \mathbb{N}$ it suffices to consider the following three cases:
a) Suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ has no atoms. Then according to [25, 38] we can find an increasing family $(E_x)_{x\in[0,1]}$ of measurable sets such that $\mathbb{P}(E_x) = x$ holds for every $x \in [0, 1]$. Setting $B_n := E_{\frac{1}{n}} \cap (E_{\frac{1}{n+1}})^c$ for every $n \geq 2$ as well as $B_1 := (E_{\frac{1}{2}})^c$ yields a partition $B_1, B_2, \ldots \in \mathcal{A}$ with the desired property.
b) If $(\Omega, \mathcal{A}, \mathbb{P})$ has infinitely many atoms $A_2, A_3, \ldots \in \mathcal{A}$ then, without loss of generality, we may assume that these atoms are pairwise disjoint. Set $A_1 := (\bigcup_{n=2}^{\infty} A_n)^c$. If $\mathbb{P}(A_1) > 0$ then defining $B_n := A_n$ for every $n \in \mathbb{N}$ yields a partition with the desired property. If we have $\mathbb{P}(A_1) = 0$ then $(B_n)_{n\in\mathbb{N}}$, defined by $B_1 := A_1 \cup A_2, B_2 = A_3, B_3 = A_4, \ldots$ is as desired.
c) If $(\Omega, \mathcal{A}, \mathbb{P})$ has a finite number of atoms $A_1, \ldots A_N$ then, by the hypothesis, $\sum_{n=1}^{N} \mathbb{P}(A_i) < 1$. As before we may assume that $A_1, \ldots A_N$ are pairwise disjoint. Since $\Omega' := \bigcap_{n=1}^{N} A_n^c$ fulfills $\mathbb{P}(\Omega') > 0$ we can consider the non-atomic probability space $(\Omega', \mathcal{A} \cap \Omega', \mathbb{P}')$ with $\mathbb{P}'$ being defined as $\mathbb{P}'(A \cap \Omega') = \dfrac{\mathbb{P}(A \cap \Omega')}{\mathbb{P}(\Omega')}$ and apply a) to obtain a partition $C_1, C_2, \ldots$ of Ω' with $\mathbb{P}'(C_i) > 0$ for every $i \in \mathbb{N}$. Therefore, setting $B_i := A_i$ for every $i \in \{1, \ldots, N\}$ as well as $B_{j+N} := C_j$ for every $j \in \mathbb{N}$ completes the proof. □

Next, defining $\mathcal{S}_1$ as

$$\begin{aligned}\mathcal{S}_1 := \{\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}} : \exists X \in \mathcal{M} \text{ such that}\\ X_n \to X\ [\mathbb{P}] \text{ and } \mathbb{E}(|X_n|) \to +\infty \text{ as } n \to \infty\}\end{aligned}$$

the following result (which shares deep connections with [24, theorem 1] and [8, theorem 10]) holds.

For the next result let us introduce the following notation. Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a space $(\Omega_1, \mathcal{A}_1)$ and a measurable mapping (in particular, a random variable) $X : (\Omega, \mathcal{A}) \to (\Omega_1, \mathcal{A}_1)$, $\mathbb{P}^X$ stands for a probability defined on $(\Omega_1, \mathcal{A}_1)$ as

$$\mathbb{P}^X(A) := \mathbb{P}(X^{-1}(A))$$

for every $A \in \mathcal{A}_1$.

Theorem 2.2 *Suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ is as in Lemma 2.1. Then $\mathcal{S}_1$ is strongly $\mathfrak{c}-$algebrable (and $\aleph_0-$incomparably-Riesz-latticeable.)*

Proof. According to Lemma 2.1 there exists a partition $B_1, B_2, \ldots$ of Ω such that $\mathbb{P}(B_n) > 0$ for every $n \in \mathbb{N}$. Defining the random variable X by $X := \sum_{n\in\mathbb{N}} n\mathbf{1}_{B_n}$ we obviously have $\mathbb{P}^X(\mathbb{N}) = 1$ and $p_n := \mathbb{P}^X(\{n\}) > 0$ for every $n \in \mathbb{N}$. Choose a sequence $(r_n)_{n\in\mathbb{N}}$ of natural numbers verifying that $\sum_{n=1}^{\infty} p_n e^{r_n} = +\infty$ and let $\mathcal{H} \subseteq (1, 2)$ denote a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. For every pair $(\beta, n) \in (1, \infty) \times \mathbb{N}$ define the random variables X_n^β, X^β as

$$X_n^\beta(\omega) = e^{\beta r_{X(\omega)}} \mathbf{1}_{[1,n]}(X(\omega)),$$

$$X^\beta(\omega) = e^{\beta r_{X(\omega)}},$$

and set $\mathcal{X}^\beta = (X_1^\beta, X_2^\beta, \ldots)$. For every $\beta \in (1, \infty)$ we then have $\lim_{n\to\infty} X_n^\beta = X^\beta\ [\mathbb{P}]$ and

$$\lim_{n\to\infty} \mathbb{E}(|X_n^\beta|) = \lim_{n\to\infty} \mathbb{E}(X_n^\beta) = \lim_{n\to\infty} \sum_{i=1}^{n} p_i e^{\beta r_i} = +\infty,$$

i.e., $\mathcal{X}^\beta \in \mathcal{S}_1$ holds for every $\beta \in (1,\infty)$.
To prove this theorem it suffices to show that $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent and that it generates an algebra contained in $\mathcal{S}_1$, which can be achieved as follows:
(i) Suppose that $p : \mathbb{R}^s \to \mathbb{R}$ is a non-zero polynomial without constant term and that $h_1, \dots, h_s$ are different elements in $\mathcal{H}$. Then, by expanding the polynomial evaluated on the h_j's powers of X_n it is straightforward to see that there exists some $m \in \mathbb{N}$ such that $p(X_n^{h_1}, \dots, X_n^{h_s})$ can be written as

$$p(X_n^{h_1}, \dots, X_n^{h_s}) = \sum_{i=1}^{m} \alpha_i X_n^{\beta_i}$$

for every $n \in \mathbb{N}$. Thereby the coefficients $\alpha_1, \dots, \alpha_m \in \mathbb{R} \setminus \{0\}$ do not depend on n and $1 < \beta_1 < \ldots < \beta_m$ (notice that the property of $\mathcal{H}$ being a Hamel basis is essential for having $m \geq 1$). Assume now that $p(X_n^{h_1}, \dots, X_n^{h_s}) = 0$ on Ω for every $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ considering $\omega \in B_n$ we obviously have $\sum_{i=1}^m \alpha_i e^{\beta_i r_n} = 0$. On the other hand, considering $\lim_{k\to\infty} r_k = +\infty$ yields

$$\lim_{n\to\infty} \frac{\sum_{i=1}^m \alpha_i e^{\beta_i r_n}}{\alpha_m e^{\beta_m r_n}} = 1,$$

which implies that

$$\left| \sum_{i=1}^{m} \alpha_i e^{\beta_i r_n} \right| \geq \frac{1}{2} |\alpha_m| e^{\beta_m r_n} > 0 \tag{1}$$

holds for sufficiently large $n \in \mathbb{N}$. Since the latter contradicts $\alpha_m \neq 0$ and since p and $h_1, \dots, h_s \in \mathcal{H}$ were arbitrary, $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent.
(ii) Obviously $p(X_n^{h_1}, \dots, X_n^{h_s})$ converges $\mathbb{P}$-almost everywhere to $p(X^{h_1}, \dots, X^{h_s})$ for $n \to \infty$. To show that

$$\lim_{n\to\infty} \mathbb{E}\left(|p(X_n^{h_1}, \dots, X_n^{h_s})|\right) = +\infty$$

we can reuse inequality (1) to show that for sufficiently large k_0 and $n > k_0$ we have

$$\mathbb{E}\left(|p(X_n^{h_1}, \dots, X_n^{h_s})|\right) = \sum_{k=1}^{n} p_k \left| \sum_{i=1}^{m} \alpha_i e^{\beta_i r_k} \right| \geq \sum_{k=k_0}^{n} p_k \, \tfrac{1}{2} \, |\alpha_m| e^{\beta_m r_k}.$$

Since the right hand-side tends to $+\infty$ as $n \to \infty$, the proof of the first statement is complete.
In order to prove the second assertion let $(A_k)_{k\in\mathbb{N}}$ denote a countable partition of $\mathbb{N}$ into sets having, each of them, cardinality $\aleph_0$. Given A_1 we can choose a sequence $(a_{1,n})_{n\in\mathbb{N}}$ in $[0,\infty)$ in such a way that $a_{1,n} = 0$ for every $n \in A_1^c$, $a_{1,n} > 1$ for every $n \in A_1$, and $\sum_{n=1}^\infty a_{1,n} p_n = \infty$. Proceed analogously for every $k \in \mathbb{N}$ in order to define the sequence $(a_{k,n})_{n\in\mathbb{N}}$ and for all $(k,n) \in \mathbb{N}^2$ define the random variable $Y_{k,n} : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ by

$$Y_{k,n}(\omega) = a_{k,X(\omega)} \mathbf{1}_{[1,n]}(X(\omega)).$$

Then, for every $k \in \mathbb{N}$, we obviously have $\lim_{n\to\infty} Y_{k,n} = a_{k,X(\omega)}$ $[\mathbb{P}]$. Considering $\mathbb{E}(|Y_{k,n}|) = \sum_{j=1}^n a_{k,j}\, p_j$, it follows that $(Y_{k,n})_{n\in\mathbb{N}} \in \mathcal{S}_1$ for every $k \in \mathbb{N}$. Moreover, the vector space $\mathcal{M}$ generated by $(Y_{k,n})_{n\in\mathbb{N}}$ is contained in $\mathcal{S}_1$ as well. Letting $(U_n)_{n\in\mathbb{N}}, (V_n)_{n\in\mathbb{N}}$ stand for elements in $\mathcal{M}$, we can find sequences $(c_n^U)_{n\in\mathbb{N}}, (c_n^V)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that all but finitely many elements are different from 0 and $U_n = \sum_{k=1}^\infty c_k^U Y_{k,n}$, $V_n = \sum_{k=1}^\infty c_k^V Y_{k,n}$ holds for every $n \in \mathbb{N}$. The construction of $Y_{k,n}$ implies that $U = (U_n)_{n\in\mathbb{N}}$, $V = (V_n)_{n\in\mathbb{N}}$ fulfill

$$(U \vee V)_n = \sum_{k=1}^{\infty} \max\{c_k^U, c_k^V\}\, Y_{k,n} \quad \text{and} \quad (U \wedge V)_n = \sum_{k=1}^{\infty} \min\{c_k^U, c_k^V\}\, Y_{k,n},$$

implying that $U \vee V, U \wedge V \in \mathcal{M}$. Since each pair $(Y_{k_1,n})_{n\in\mathbb{N}}, (Y_{k_2,n})_{n\in\mathbb{N}}$ with $k_1 \neq k_2$ is incomparable, this completes the proof. □

Remark 2.3 *Notice that Theorem 2.2 is a stronger version of Theorem 1 in* [24] *in two respects: here algebrability instead of lineability is obtained and an algebraically independent set of cardinality* $\mathfrak{c}$ *is constructed as well. Also, notice that additional assumptions on the probability space are necessary for Theorem 1 in* [24]. *In fact, for the probability space* $(\{0,1\}, 2^{\{0,1\}}, \frac{1}{2}(\delta_0 + \delta_1))$*, e.g., the set* $\mathcal{S}_1$ *is empty.*

Slightly modifying the construction in the proof of Theorem 2.2 and setting

$$X_n^h(\omega) = e^{hrX(\omega)}\mathbf{1}_{[n,\infty)}(X(\omega))$$

directly yields the following result on the family $\mathcal{S}_2$

$$\begin{aligned}\mathcal{S}_2 := \{&\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}} : X_n \to 0\,[\mathbb{P}] \text{ and} \\ &\mathbb{E}(|X_n|) = +\infty \text{ for every } n \in \mathbb{N}\}\end{aligned}$$

which can be regarded as being symmetric to the class $\mathcal{S}_1$.

Theorem 2.4 *Suppose that* $(\Omega, \mathcal{A}, \mathbb{P})$ *is as in Lemma 2.1. Then* $\mathcal{S}_2$ *is strongly* $\mathfrak{c}-$*algebrable.*

For the next three results we will consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$. For every $\vartheta \in [0,1)$ let $R_\vartheta : [0,1] \to [0,1)$ denote the rotation by ϑ, defined by

$$R_\vartheta(x) = (x + \vartheta)(\text{mod } 1).$$

Given $s \in \mathbb{N}$ and $\vartheta = (\vartheta_1, \ldots \vartheta_s) \in [0,1)^s$ we will refer to the mapping $R_\vartheta : [0,1]^s \to [0,1)^s$, defined by

$$R_\vartheta(x_1, \ldots, x_d) = (R_{\vartheta_1}(x_1), \ldots, R_{\vartheta_s}(x_s))$$

as rotation of $[0,1)^s$. It is well known (see [31, 32]) that the orbit

$$O^+_{R_\vartheta}(0, \ldots, 0) := \left\{(R^n_{\vartheta_1}(0), \ldots, R^n_{\vartheta_s}(0)) : n \in \mathbb{N}\right\}$$

is dense in $[0,1]^s$ if $\vartheta_1, \ldots \vartheta_s$ are rationally independent, i.e., the only integers $k_1, \ldots, k_s$ for which $\sum_{i=1}^s k_i\vartheta_i$ is an integer are $k_1 = k_2 = \cdots = k_s = 0$. The just mentioned result will be key for proving that $\mathcal{S}_3$, defined by

$$\begin{aligned}\mathcal{S}_3 := \{&\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}_{0,1} : \exists X \in \mathcal{M}_{0,1} \text{ such that} \\ &X_n \to X\,[\lambda] \text{ and } \nexists \lim_{n\to\infty} \mathbb{E}(X_n)\},\end{aligned}$$

is lineable.

Theorem 2.5 $\mathcal{S}_3$ *is* $\mathfrak{c}-$*lineable (and* $\aleph_0-$*incomparably-Riesz-latticeable).*

Proof. As before let $\mathcal{H} \subseteq (0,1)$ be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. For every pair $(h,n) \in \mathcal{H} \times \mathbb{N}$ define the random variable $X_n^h \in \mathcal{M}_{0,1}$ by

$$X_n^h(\omega) = \text{sign}\left(\sin(2\pi \cdot R_h^n(0))\right) n\mathbf{1}_{(0,1/n]}(\omega)$$

and set $\mathcal{X}^h = (X_1^h, X_2^h, \ldots)$. Considering that $(X_n^h)_{n\in\mathbb{N}}$ converges pointwise to 0 and that $\mathbb{E}(X_n^h) = \text{sign}\big(\sin(2\pi\cdot R_h^n(0))\big) \in \{-1,1\}$, we get $\mathcal{X}^h \in \mathcal{S}_3$ for every $h \in \mathcal{H}$. It therefore suffices to show that $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is linearly independent and that every linear combination of elements in this family is again an element of $\mathcal{S}_3$, which can be done as follows: Suppose that $h_1, \ldots, h_s$ are pairwise different and that $\alpha_1, \ldots, \alpha_s \in \mathbb{R} \setminus \{0\}$. Since

$$\left\{(2\pi R^n_{h_1}(0), \ldots, 2\pi R^n_{h_s}(0)) : n \in \mathbb{N}\right\}$$

is dense in $[0,2\pi]^s$ we can find infinitely many $n \in \mathbb{N}$ such that

$$\sum_{i=1}^s \alpha_i X_n^{h_i}(\omega) \geq ns \cdot \min\left\{|\alpha_i| : i \in \{1, \ldots, s\}\right\} \tag{2}$$

holds for every $\omega \in (0, 1/n)$. As a direct consequence, $\sum_{i=1}^{s} \alpha_i X_n^{h_i}$ can not be 0 λ-almost everywhere, so $\mathcal{X}^{h_1}, \ldots, \mathcal{X}^{h_s}$ are linearly independent.
Additionally, we obviously have that $\sum_{i=1}^{s} \alpha_i X_n^{h_i}$ converges to 0 pointwise on $[0, 1]$, and, arguing analogously as for inequality (2), we get the existence of infinitely many $\overline{n} \in \mathbb{N}$ and infinitely many $\underline{n} \in \mathbb{N}$ such that

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{s} \alpha_i X_{\overline{n}}^{h_i}\right) &\geq s \cdot \min\left\{|\alpha_i| : i \in \{1, \ldots, s\}\right\} \\ \mathbb{E}\left(\sum_{i=1}^{s} \alpha_i X_{\underline{n}}^{h_i}\right) &\leq -s \cdot \min\left\{|\alpha_i| : i \in \{1, \ldots, s\}\right\} \end{aligned} \quad (3)$$

holds. Setting $c := s \min\left\{|\alpha_i| : i \in \{1, \ldots, s\}\right\} > 0$ it follows that

$$\liminf_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{s} \alpha_i X_n^{h_i}\right) \leq -c < c \leq \limsup_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{s} \alpha_i X_n^{h_i}\right),$$

so $\sum_{i=1}^{s} \alpha_i X_n^{h_i} \in \mathcal{S}_3$.

To prove the second assertions we proceed as follows: For every $k \in \mathbb{N}$ let $(I_{k,n})_{n\in\mathbb{N}}$ denote a partition of $(\frac{1}{k+1}, \frac{1}{k})$ into non degenerate intervals. Choose an irrational number $h \in (0, 1)$ and let us set

$$Z_{k,n}(\omega) = \frac{R_h^n(0)}{\lambda(I_{k,n})} \mathbf{1}_{I_{k,n}}(\omega)$$

for every $(k, n) \in \mathbb{N}^2$. Then for every $k \in \mathbb{N}$ we obviously have $\lim_{n\to\infty} Z_{k,n}(\omega) = 0$ for every $\omega \in [0, 1]$, hence, considering $\mathbb{E}(Y_{k,n}) = R_h^n(0)$ we have $(Z_{k,n})_{n\in\mathbb{N}} \in \mathcal{S}_3$ for every $k \in \mathbb{N}$. Moreover the vector space $\mathcal{M}$ generated by $\{(Z_{k,n})_{n\in\mathbb{N}} : k \in \mathbb{N}\}$ is contained in $\mathcal{S}_3$ too. Letting $(U_n)_{n\in\mathbb{N}}, (V_n)_{n\in\mathbb{N}}$ denote elements of $\mathcal{M}$ we can find sequences $(c_n^U)_{n\in\mathbb{N}}, (c_n^V)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that all but finitely many elements are different from 0 and $U_n = \sum_{k=1}^{\infty} c_k^U Y_{k,n}, V_n = \sum_{k=1}^{\infty} c_k^V Y_{k,n}$ holds for every $n \in \mathbb{N}$. The construction of $Y_{k,n}$ implies that $U = (U_n)_{n\in\mathbb{N}}, V = (V_n)_{n\in\mathbb{N}}$ fulfill

$$(U \vee V)_n = \sum_{k=1}^{\infty} \max\{c_k^U, c_k^V\} Z_{k,n} \quad \text{and} \quad (U \wedge V)_n = \sum_{k=1}^{\infty} \min\{c_k^U, c_k^V\} Z_{k,n},$$

implying that $U \vee V, U \wedge V \in \mathcal{M}$. Since each pair $(Z_{k_1,n})_{n\in\mathbb{N}}, (Z_{k_2,n})_{n\in\mathbb{N}}$ with $k_1 \neq k_2$ is incomparable the proof is complete. □

In a nutshell, Theorem 2.2, Theorem 2.4 and Theorem 2.5 show that, without a dominating integrable function, the family of sequences not fulfilling the Dominated Convergence Theorem (see [11]) is very large. We continue in this direction and derive additional results related to the interchangeability of the integral and the limit (also compare with Theorem 2 in [24]). Doing so we start with the family $\mathcal{S}_4$, defined by

$$\begin{aligned} \mathcal{S}_4 := \{&\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : \exists X \in \mathcal{M}_{0,1} \text{ such that} \\ &X_n \to X\,[\lambda] \text{ and } \lim_{n\to\infty} \mathbb{E}(X_n) \neq \mathbb{E}(X)\}. \end{aligned}$$

Let us recall that, given κ a finite or infinite cardinality, a subset M of a linear space X is called *positively κ-coneable* (see [1]) in X if there exists an κ-dimensional set M such that $\alpha M \subset M$ for every $\alpha > 0$.

Theorem 2.6 *$\mathcal{S}_4$ is positively $\mathfrak{c}$-coneable.*

Proof. Let $\mathcal{H} \subseteq (-1, 0)$ denote a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. For every pair $(h, n) \in \mathcal{H} \times \mathbb{N}$ define the random variable $X_n^h \in \mathcal{M}_{0,1}$ by

$$X_n^h(\omega) = n^{1+h}(1 + h)\,\omega^h \mathbf{1}_{(0,1/n]}(\omega).$$

and set $\mathcal{X}^h = (X_1^h, X_2^h, \ldots)$. Obviously $\mathbb{E}(X_n^h) = 1$ and $(X_n^h)_{n\in\mathbb{N}}$ converges pointwise to 0 for $n \to \infty$, so $\mathcal{X}^h \in \mathcal{S}_4$ for every $h \in \mathcal{H}$.

Suppose now that $n \in \mathbb{N}$, that $h_1, h_2, \ldots, h_s \in \mathcal{H}$ fulfill $h_1 < h_2 < \cdots < h_s$ and that $\alpha_1, \alpha_2, \cdots, \alpha_s \in \mathbb{R} \setminus \{0\}$. For $\omega \in (0, 1/n]$ we obviously have

$$\sum_{i=1}^{s} \alpha_i X_n^{h_i}(\omega) = \sum_{i=1}^{s} \alpha_i n^{1+h_i} (1 + h_i)\, \omega^{h_i},$$

so setting $y = 1/\omega \geq n$ as well as $\beta_i := \alpha_i n^{1+h_i}(1 + h_i)$ yields

$$\sum_{i=1}^{s} \alpha_i X_n^{h_i}(\omega) = \sum_{i=1}^{s} \beta_i\, y^{-h_i} \longrightarrow \text{sign}(\beta_1) \cdot \infty \quad \text{for } y \to \infty.$$

As direct consequence we can not have $\sum_{i=1}^{s} \alpha_i X_n^{h_i} = 0$, so the family $\{\mathcal{X}_h : h \in \mathcal{H}\}$ is linearly independent and it remains to check that each positive linear combination of elements in $\{\mathcal{X}_h : h \in \mathcal{H}\}$ is an element of $\mathcal{S}_4$ too. The latter, however, is clear since for $\alpha_1, \ldots, \alpha_s > 0$ we have that $\sum_{i=1}^{s} \alpha_i X_n^{h_i}$ converges to 0 for $n \to \infty$, so considering

$$\mathbb{E}\left(\sum_{i=1}^{s} \alpha_i X_n^{h_i}\right) = \sum_{i=1}^{s} \alpha_i > 0$$

the assertion follows. □

Remark 2.7 *The proof of Theorem 2.6 also shows that the family $\mathcal{S}_4'$, defined by*

$$\mathcal{S}_4' := \Big\{\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : \exists X \in \mathcal{M}_{0,1} \text{ such that } \mathbb{E}\left(\liminf_{n\to\infty} X_n\right) < \liminf_{n\to\infty} \mathbb{E}(X_n)\Big\},$$

is positively $\mathfrak{c}$-coneable (also see Theorem 3 in [24]*). In other words: The family of sequences $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_{0,1}$ for which the inequality in Fatou's Lemma is sharp is very large.*

We continue in the spirit of Theorem 3 in [24], leave the probabilistic setup for two results and first consider the set $\mathcal{S}_5$, defined by

$$\mathcal{S}_5 := L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda) \setminus L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda). \tag{4}$$

The following result is a generalization of [18, theorem 2.6].

Theorem 2.8 *$\mathcal{S}_5$ is strongly $\mathfrak{c}$-algebrable (and $\aleph_0-$incomparably-Riesz-latticeable).*

P r o o f. For every $\beta > 0$ define the (Borel measurable) function $f_\beta : \mathbb{R} \to \mathbb{R}$ by

$$f_\beta(x) = \sum_{n=1}^{\infty} n^\beta \mathbf{1}_{[n, n+2^{-n}]}(x).$$

It is clear that f_β is not essentially bounded but it is an element of $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ since

$$\|f_\beta\|_1 = \int_{\mathbb{R}} f_\beta \, d\lambda = \sum_{n=1}^{\infty} \frac{n^\beta}{2^n} < \infty.$$

Suppose that $\mathcal{H} \subset (0, \infty)$ is a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. We will show that $\{f_h : h \in \mathcal{H}\}$ is algebraically independent and that it generates an algebra contained in $\mathcal{S}_5$.

(i) Suppose that $p : \mathbb{R}^s \to \mathbb{R}$ is a non-zero polynomial without constant term and that $h_1, \ldots, h_s$ are different elements in $\mathcal{H}$. Then there exists some $m \in \mathbb{N}$ such that $p(f_{h_1}, \ldots, f_{h_s})$ can be expressed as

$$p(f_{h_1}, \ldots, f_{h_s}) = \sum_{i=1}^{m} \alpha_i f_{\beta_i} = \sum_{n=1}^{\infty} \left(\mathbf{1}_{[n,n+2^{-n}]} \cdot \sum_{i=1}^{m} \alpha_i n^{\beta_i} \right)$$

for every $n \in \mathbb{N}$. Thereby the coefficients fulfill $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and $0 < \beta_1 < \ldots < \beta_m$ holds. Considering $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{m} \alpha_i n^{\beta_i} = \text{sign}(\alpha_m) \cdot \infty$ it follows that in this situation we can not have $p(X_{h_1}, \ldots, X_{h_s}) = 0$, so $\{f_h : h \in \mathcal{H}\}$ is algebraically independent.
(ii) The fact that $p(X_{h_1}, \ldots, X_{h_s}) \in \mathcal{S}_5$ is straightforward.
In order to prove the second assertion let $(A_k)_{k\in\mathbb{N}}$ stand for a countable partition of $\mathbb{N}$ into sets having, each of them, cardinality $\aleph_0$ and define the function $h : \mathbb{R} \to \mathbb{R}$ as

$$h(x) = \sum_{n=1}^{\infty} n \mathbf{1}_{[n,n+\frac{1}{2^n}]}(x).$$

Furthermore, for every $k \in \mathbb{N}$ set $N_k = \bigcup_{n\in A_k} (n, n+1]$ and define the function $h_k : \mathbb{R} \to \mathbb{R}$ by $h_k(x) = h(x)\mathbf{1}_{N_k}(x)$. Letting $\mathcal{M}$ denote the vector space generated by $\{h_k : k \in \mathbb{N}\}$ it follows that $\mathcal{M}$ is contained in $\mathcal{S}_5$. Letting U, V denote elements of $\mathcal{M}$ we can find sequences $(c_n^U)_{n\in\mathbb{N}}, (c_n^V)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that all but finitely many elements are different from 0 and $U = \sum_{k=1}^{\infty} c_k^U h_k, V = \sum_{k=1}^{\infty} c_k^V h_k$ holds. The construction of h_k implies that U, V fulfill

$$U \vee V = \sum_{k=1}^{\infty} \max\{c_k^U, c_k^V\}\, h_k \quad \text{and} \quad U \wedge V = \sum_{k=1}^{\infty} \min\{c_k^U, c_k^V\}\, h_k,$$

implying that $U \vee V, U \wedge V \in \mathcal{M}$. Since each pair k_{k_1}, h_{k_2} with $k_1 \neq k_2$ is incomparable the proof is complete. □

Remark 2.9 *The proof of Theorem 2.8 also works if we replace λ by any measure μ which is absolutely continuous w.r.t. λ and whose density $k(x)$ is bounded and fulfills $\int_{[n,n+2^{-n}]} k\, d\lambda > 0$ for every $n \in \mathbb{N}$. Additionally, the proof can directly be translated to*

$$\mathcal{S}_5' := L^1(\Omega, \mathcal{A}, \mathbb{P}) \setminus L^\infty(\Omega, \mathcal{A}, \mathbb{P}) \tag{5}$$

if the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ admits a (measurable) partition $A_1, A_2, \ldots$ of Ω fulfilling $p_n := \mathbb{P}(A_n) > 0$ for every $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} n^h p_n < \infty$ for every $h > 0$.

For the next theorem we again consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$. We will let $\mathcal{C}_0$ denote the family of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ vanishing in $\pm\infty$, i.e., verifying $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. Slightly abusing notation and letting $\mathcal{C}_0$ also denote the family of all equivalence classes of elements in $\mathcal{C}_0$ (equivalence w.r.t. equality λ-almost everywhere) define the set $\mathcal{S}_6$ by

$$\mathcal{S}_6 := \mathcal{C}_0 \setminus \bigcup_{p\in[1,\infty)} L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda). \tag{6}$$

Then the following result holds:

Theorem 2.10 *$\mathcal{S}_6$ is strongly $\mathfrak{c}$-algebrable.*

Proof. For every $\beta > 1$ define

$$f_\beta(x) = \begin{cases} 1 & \text{if } x \in [-e, e], \\ \dfrac{1}{(\ln|x|)^\beta} & \text{otherwise.} \end{cases}$$

Then we clearly have that $f_\beta \in \mathcal{S}_6$ for every $\beta \in (1, \infty)$. Now, letting $\mathcal{H} \subseteq (1, \infty)$ denote a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ it suffices to show that $\{f_h : h \in \mathcal{H}\}$ is algebraically independent and that it generates an algebra

contained in $\mathcal{S}_6$.
Suppose that $p : \mathbb{R}^s \to \mathbb{R}$ is a non-zero polynomial without constant term and that $h_1, \ldots, h_n$ are different elements in $\mathcal{H}$. Then there exists some $m \in \mathbb{N}$ such that $p(f_{h_1}, \ldots, f_{h_s})$ can be expressed as

$$p(f_{h_1}, \ldots, f_{h_s}) = \sum_{i=1}^{m} \alpha_i f_{\beta_i}$$

whereby the coefficients fulfill $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and $1 < \beta_1 < \ldots < \beta_m$ holds. If $p(f_{h_1}, \ldots, f_{h_s}) = 0$ would hold then we would have

$$0 = \left| \frac{\sum_{i=1}^{m} \alpha_i f_{\beta_i}(x)}{\alpha_1 f_{\beta_1}(x)} \right|$$

which, however, contradicts the fact that the right handside converges to 1 for $x \to \infty$. Hence $p(f_{h_1}, \ldots, f_{h_s}) \neq 0$ and since $p(f_{h_1}, \ldots, f_{h_s}) \notin L^p(\mathbb{R})$ for any p (use, for instance, the Comparison Test with $|\alpha_1| f_{\beta_1}(x)$) we have that $p(f_{h_1}, \ldots, f_{h_s}) \in \mathcal{S}_6$ and the proof is complete. □

We return to the probabilistic setting and focus on convergence of the arithmetic mean of a sequence of independent, centered, integrable random variables which are not identically distributed. Define the set $\mathcal{S}_7$ by

$$\mathcal{S}_7 := \{\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}} : (X_n)_{n\in\mathbb{N}} \text{ independent}, \mathbb{E}(X_n) = 0 \\ \text{for every } n \in \mathbb{N}, \text{ and } |\overline{X}_n| \to \infty \ [\mathbb{P}] \text{ as } n \to \infty\},$$

where, as usual, $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. We will say that a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ enjoys the $\star$-property if it allows a sequence of independent random variables $(X_n^s)_{(n,s)\in\mathbb{N}^2}$ such that

$$\mathbb{P}(X_n^s = -n^s) = 1 - \frac{1}{n^{2s}},\ \mathbb{P}(X_n^s = n^{3s} - n^s) = \frac{1}{n^{2s}} \tag{7}$$

holds for all $(n, s) \in \mathbb{N}^2$. Notice that all Polish spaces with a non-atomic probability measure fulfill the $\star$-property (see, e.g., [33]).

Theorem 2.11 *Suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ fulfills the $\star$-property. Then $\mathcal{S}_7$ is lineable.*

P r o o f. In the following $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ denotes the bijection corresponding to the enumeration scheme sketched in Figure 1. Assuming that the random variables $(X_n^s)_{(n,s)\in\mathbb{N}^2}$ are independent and distributed according to equation (7) we obviously get $\mathbb{E}(X_n^s) = 0$ for all $(n, s) \in \mathbb{N}^2$.
Fix $s \in \mathbb{N}$. Setting $B_n := (X_n^s)^{-1}(\{-n^s\})$ for every $n \in \mathbb{N}$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n^c) = \sum_{n=1}^{\infty} \frac{1}{n^{2s}} < \infty$$

so the Borel-Cantelli Lemma (see [34]) implies $\mathbb{P}(\limsup_{n\to\infty} B_n^c) = 0$. Considering the measurable set

$$A^s := \left(\limsup_{n\to\infty} B_n^c\right)^c$$

it follows that $\mathbb{P}(A^s) = 1$ and for every $\omega \in A^s$ there exists some index $n_0 = n_0(\omega, s) \in \mathbb{N}$ such that $X_n^s(\omega) = -n^s$ for all $n \geq n_0$. As direct consequence, setting $Y_n^s := |\frac{1}{n}\sum_{i=1}^{n} X_i^s|$ we have $\lim_{n\to\infty} Y_n(\omega) = +\infty$ for every $\omega \in A^s$, i.e., $(X_1^s, X_2^s, \ldots) \in \mathcal{S}_7$.
The proof is complete if we show that $\{\mathcal{X}^s : s \in \mathbb{N}\}$ is linearly independent and that every linear combination of elements in the latter set is in $\mathcal{S}_7$ too, which can be done as follows. (i) Suppose that $m \in \mathbb{N}$, that

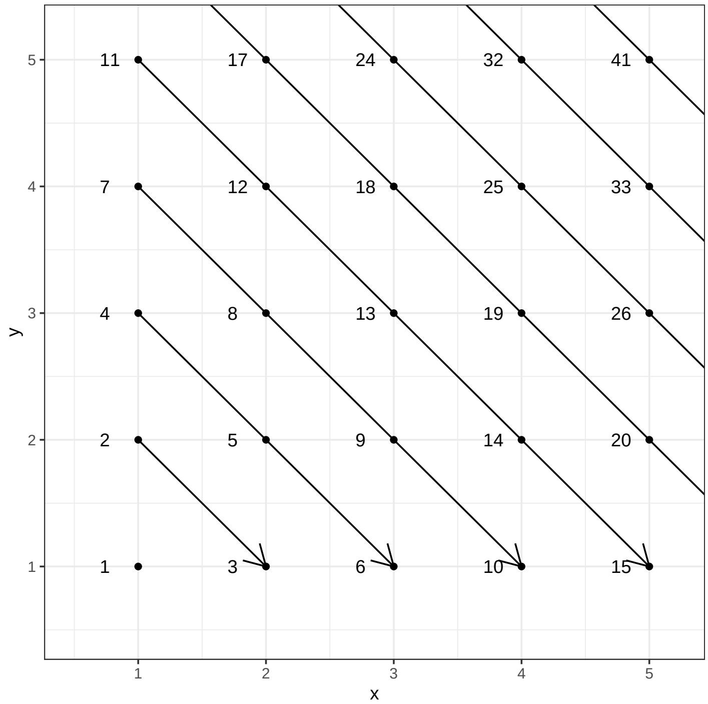

Fig. 1 Enumeration scheme describing the function φ.

$1 \leq s_1 < s_2 < \cdots < s_m$, that $\alpha_1, \ldots \alpha_m \in \mathbb{R} \setminus \{0\}$ and set $Z := \sum_{i=1}^m \alpha_i X_n^{s_i}$. Then obviously $\mathbb{E}(Z) = 0$, so considering the fact that $\mathbb{V}(X_n^{s_i}) > 0$ holds for every $i \in \{1, \ldots, m\}$ and using independence we get

$$\mathbb{V}(Z) \quad = \quad \mathbb{E}\left(Z^2\right) = \mathbb{E}\left(\sum_{i,j=1}^m \alpha_i \alpha_j X_n^{s_i} X_n^{s_j}\right) = \sum_{i=1}^m \alpha_i^2 \mathbb{V}(X_n^{s_i}) > 0,$$

implying that Z can not be identically 0 almost surely.
(ii) Let $s_1, \ldots, s_m$ and $\alpha_1, \ldots, \alpha_m$ as before, set $\mathcal{Y} = \sum_{i=1}^m \alpha_i \mathcal{X}^{s_i}$ as well as $Z_n := \frac{1}{n} \sum_{i=j}^n Y_i$ for every $n \in \mathbb{N}$. Proceeding as before yields the existence of a set $A^{s_i} \in \mathcal{A}$ with $\mathbb{P}(A^{s_i}) = 1$ for every $i \in \{1, \ldots, m\}$. Setting $A := \bigcap_{i=1}^m A^{s_i}$ we get $\mathbb{P}(A) = 1$ and for every $\omega \in \mathcal{A}$ there exists some index $n_0 = n_0(\omega) \in \mathbb{N}$ such that for every $n \geq n_0$ we have $X_n^{s_i}(\omega) = -n^{s_i}$ for every $i \in \{1, \ldots, m\}$. For such ω and $n \geq n_0$ we therefore get

$$Y_n(\omega) = \sum_{i=1}^m \alpha_i X_n^{s_i}(\omega) = -\sum_{i=1}^m \alpha_i n^{s_i},$$

implying

$$\begin{aligned} Z_n(\omega) =& \frac{1}{n} \sum_{j=1}^{n_0-1} Y_j(\omega) + \frac{1}{n} \sum_{j=n_0}^{n} Y_j(\omega) \\ =& \underbrace{\frac{1}{n} \sum_{j=1}^{n_0-1} Y_j(\omega)}_{\longrightarrow 0 \text{ for } n \to \infty} + \underbrace{\frac{n - n_0 + 1}{n} \left(-\sum_{i=1}^m \alpha_i n^{s_i} \right)}_{\longrightarrow -\text{sign}(\alpha_m) \cdot \infty}. \end{aligned}$$

We have shown $|Z_n| \longrightarrow +\infty$ $[\mathbb{P}]$, i.e., $\mathcal{Y} \in \mathcal{S}_7$. □

It is well known that almost sure convergence of a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables implies convergence in probability but not vice versa (see [34]). We now show that the family of all sequences converging in probability but in almost no point is large. Let us now consider

$$\begin{aligned}\mathcal{S}_8 := \big\{&\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : \exists X \in \mathcal{M}_{0,1} \text{ such that}\\ &X_n \to X \text{ in probability, but}\\ &\lambda(\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 0\big\},\end{aligned}$$

Recall that the following result was recently proved in [19, theorem 2.2], although (for the sake of completeness) we include here below a similar proof since it will be useful to understand the forthcoming Remarks 2.13 and 2.14, which are results of independent interest that rely on Theorem 2.12 ([19, theorem 2.2]).

Theorem 2.12 *$\mathcal{S}_8$ is strongly $\mathfrak{c}$-algebrable.*

P r o o f. For every $\beta > 1$ define

$$X_n^\beta(\omega) = e^{\beta w}\mathbf{1}_{I_n}(\omega) \tag{8}$$

with $I_n = [\frac{k}{2^l}, \frac{k+1}{2^l})$ whereby (l,k) denotes the unique pair of integers with $m \in \{1,\ldots,n\}, k \in \{0,\ldots,2^l-1\}$ fulfilling $n = 2^l + k$. Obviously we have $(X_n^\beta)_{n\in\mathbb{N}} \in \mathcal{S}_8$ for every $\beta > 1$, thus convergence to 0 in probability is clear and we have pointwise convergence only for $\omega = 1$. Let $\mathcal{H} \subseteq (1,\infty)$ denote a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. It suffices to show that $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent and that it generates an algebra contained in $\mathcal{S}_8$, which can be done as follows:
(i) Suppose that $p : \mathbb{R}^s \to \mathbb{R}$ is a non-zero polynomial without constant term and that $h_1,\ldots,h_s$ are different elements in $\mathcal{H}$. Then there exists some $m \in \mathbb{N}$ such that $p(X_n^{h_1},\ldots,X_n^{h_s})$ can be expressed as

$$p(X_n^{h_1},\ldots,X_n^{h_s}) = \sum_{i=1}^m \alpha_i X_n^{\beta_i} \tag{9}$$

for every $n \in \mathbb{N}$, whereby the coefficients fulfill $\alpha_1,\ldots,\alpha_m \in \mathbb{R}\setminus\{0\}$ and $1 < \beta_1 < \ldots < \beta_m$ (and neither the coefficients nor $\beta_1,\ldots,\beta_m$ depend on n). It is well known that the family of exponential functions is linearly independent on each interval. While the standard proof considers the Wronskian, in our setting we can alternatively simply calculate higher order derivatives and use the fact that $\sum_{i=1}^m \alpha_i\beta_i^k e^{\beta_i\omega}$ converges to $\operatorname{sign}(\alpha_m)\cdot\infty$ for $k\to\infty$. Altogether we conclude that $p(X_n^{h_1},\ldots,X_n^{h_s})$ can not be identical to 0, i.e. $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent.
(ii) For every $n \in \mathbb{N}$ set

$$Y_n(\omega) := p(X_n^{h_1},\ldots,X_n^{h_s})(\omega) = \mathbf{1}_{I_n}(\omega)\sum_{i=1}^m \alpha_i e^{\beta_i\omega}.$$

Obviously $(Y_n)_{n\in\mathbb{N}}$ converges to 0 in probability. Since $Y_n(\omega)$ can only converge to 0 for $\omega = 1$ and for each $\omega \in [0,1)$ fulfilling $\sum_{i=1}^m \alpha_i e^{\beta_i\omega} = 0$, the set $\{\omega \in [0,1] : \lim_{n\to\infty} X_n(\omega) = 0\}$ is finite, hence of Lebesgue measure 0. □

Remark 2.13 *Slightly modifying the proof of Theorem 2.12 and considering*

$$X_n^\beta(\omega) = e^{n\beta w}\mathbf{1}_{I_n}(\omega)$$

it is straightforward to show that $\mathcal{S}_8'$, defined by

$$\begin{aligned}\mathcal{S}_8' := \big\{&\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : X_n \to 0 \textit{ in probability,}\\ &\limsup_{n\to\infty}|X_n| = +\infty\,[\mathbb{P}], \quad \liminf_{n\to\infty}|X_n| = 0\,[\mathbb{P}]\big\},\end{aligned}$$

is strongly $\mathfrak{c}$-algebrable.

Remark 2.14 *As we did in, for example, Theorem 2.8 it can be shown, in a similar fashion, that $\mathcal{S}_8$ is $\aleph_0-$incomparably-Riesz-latticeable as well.*

We stick to convergence on probability and show that the family of all sequences converging in probability but not being Cesàro[1] convergent is large too. More precisely, defining

$$\mathcal{S}_9 := \big\{\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : \exists X \in \mathcal{M}_{0,1} \text{ such that } X_n \to X \text{ in probability, but } \lambda(\{\omega \in [0,1] : \lim_{n\to\infty} \overline{X}_n(\omega) = X(\omega)\}) = 0\big\}$$

the following result holds:

Theorem 2.15 *$\mathcal{S}_9$ is strongly $\mathfrak{c}$-algebrable.*

P r o o f. We proceed similar as in the proof of Theorem 2.12 and set

$$X_n^\beta(\omega) = e^{n\beta(w+1)}\mathbf{1}_{I_n}(\omega)$$

for every $\beta > 1$ and $\omega \in [0,1]$. If $(X_n^\beta(\omega))_{n\in\mathbb{N}}$ would be Cesáro convergent then we would also have $\lim_{n\to\infty} \frac{X_n^\beta(\omega)}{n} = 0$. It is, however, not difficult to show that the latter can not hold: Fix $\omega \in [0,1)$ and let $(b_1, b_2, \ldots) \in \{0,1\}^{\mathbb{N}}$ denote the binary expansion of ω (to assure uniqueness of the representation we only consider representations with infinitely many zeros), i.e. $\omega = \sum_{i=1}^\infty \frac{b_i}{2^i}$. Furthermore, for every $l \in \mathbb{N}$ set

$$\omega_l := \sum_{i=1}^{l} \frac{b_i}{2^i} =: \frac{k_l}{2^l}.$$

Considering $n_l = 2^l + k_l$ we have $\omega_l \in I_{n_l}$ for every $l \in \mathbb{N}$ and it follows that $\frac{1}{n_l}X_{n_l}^\beta(\omega) = \frac{1}{n_l}e^{n_l\beta(\omega+1)}$. Convergence to $+\infty$ of the right hand side yields a contradiction. Hence we have shown that $(X_n^\beta)_{n\in\mathbb{N}} \in \mathcal{S}_9$ for every $\beta > 1$. Letting $\mathcal{H} \subseteq (1,\infty)$ again denote a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$ the proof is complete if we can show that $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent and generates an algebra contained in $\mathcal{S}_9$. Letting $p : \mathbb{R}^s \to \mathbb{R}$ denote a non-zero polynomial without constant term and $h_1, \ldots, h_n$ pairwise different elements in $\mathcal{H}$ we get the presentation (9) and it follows analogously that $\{\mathcal{X}^h : h \in \mathcal{H}\}$ is algebraically independent.
To show $p(X_n^{h_1}, \ldots, X_n^{h_s}) \in \mathcal{S}_9$ we set $Y_n = p(X_n^{h_1}, \ldots, X_n^{h_s})$, proceed as in the first part of the proof and use that fact that for every $\omega \in [0,1)$ and sufficiently large $l \in \mathbb{N}$ we have $|Y_{n_l}(\omega)| \geq |\alpha_m| e^{n_l\beta_m(\omega+1)}\frac{1}{2}$, implying that $\frac{Y_{n_l}(\omega)}{n_l}$ does not converge to 0 for $l \to \infty$. □

Remark 2.16 *The proof of Theorem 2.15 directly yields the following byproduct: The family $\mathcal{S}_9'$, defined by*

$$\mathcal{S}_9' := \big\{\mathcal{X} = (X_n)_{n\in\mathbb{N}} \in \mathcal{M}_{0,1}^{\mathbb{N}} : \exists X \in \mathcal{M}_{0,1} \textit{ such that } X_n \to X \textit{ in probability but} \|X_n - X\|_1 \not\longrightarrow 0 \textit{ as } n \to \infty\big\}$$

is strongly $\mathfrak{c}$-algebrable.

Remark 2.17 *As we did in, for example, Theorem 2.8 it can be shown, in a similar fashion, that $\mathcal{S}_9$ is $\aleph_0-$incomparably-Riesz-latticeable as well.*

We complete the paper with one more result outside the probabilistic setting. Suppose that $(\Omega, \mathcal{A}, \mu)$ is an infinite measure space. We will say that a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions that converges to a measurable function f μ-almost everywhere does not fulfill Egorov's property if it does not converge to f uniformly outside a set of finite measure. Recall that Egorov's theorem says that on finite measure spaces μ-almost everywhere convergence implies uniform convergence outside arbitrarily small sets (see [26]). Define the set $\mathcal{S}_{10}$ by

$$\mathcal{S}_{10} := \{\mathcal{F} = (f_n)_{n\in\mathbb{N}} \in \mathcal{M}(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)^{\mathbb{N}} : \exists\, f \text{ with } f_n \to f\ [\mu] \text{ but } (f_n)_{n\in\mathbb{N}} \text{ does not enjoy Egorov's property}\}.$$

Then $\mathcal{S}_{10}$ has the following property:

[1] Let us recall that the Cesàro sum is defined as the limit, as n tends to infinity, of the sequence of arithmetic means of the first n random variables of a given sequence, that is, $\overline{X_n} \to \infty$ $(n \to \infty)$.

Theorem 2.18 *$\mathcal{S}_{10}$ is $\mathfrak{c}$-strongly algebrable.*

Proof. For every $\beta \in (0,\infty)$ define the measurable function $f_n^\beta : \mathbb{R} \to \mathbb{R}$ by $f_n^\beta(\omega) := e^{\beta\omega}\mathbf{1}_{[n,\infty)}(\omega)$. Then obviously $(f_n^\beta)_{n\in\mathbb{N}} \in \mathcal{S}_{10}$ for every $\beta \in (0,\infty)$. Let $\mathcal{H} \subseteq (1,2)$ denote a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. We are going to show that $\{\mathcal{F}^h : h \in \mathcal{H}\}$ is algebraically independent and generates an algebra contained in $\mathcal{S}_{10}$. If $p : \mathbb{R}^s \to \mathbb{R}$ denotes a non-zero polynomial without constant term and $h_1, \ldots, h_s$ pairwise different elements in $\mathcal{H}$ then $p(f_n^{h_1}, \ldots, f_n^{h_s})$ can be expressed as

$$p(f_n^{h_1}, \ldots, f_n^{h_s}) = \sum_{i=1}^{m} \alpha_i f_n^{\beta_i} =: g_n$$

for every $n \in \mathbb{N}$, whereby $\alpha_1, \ldots, \alpha_m \in \mathbb{R} \setminus \{0\}$ and $1 < \beta_1 < \ldots < \beta_m$ do not depend on n. Consider $\lim_{\omega\to\infty} g_n(\omega) = \operatorname{sign}(\alpha_m) \cdot \infty$, so g_n can not be identical to 0, i.e., $\{\mathcal{F}^h : h \in \mathcal{H}\}$ is algebraically independent. To show that the algebra generated by the latter set is contained in $\mathcal{S}_{10}$ a simple growth argument (already used in the paper) can be applied. □

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