On quantile based co-risk measures and their estimation

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Abstract

Conditional Value-at-Risk (CoVaR) is defined as the Value-at-Risk of a certain risk given that the related risk equals a given threshold ($\text{CoVaR}^{=}$) or is smaller/larger than a given threshold ($\text{CoVaR}^{<}$/$\text{CoVaR}^{\geq}$). We extend the notion of Conditional Value-at-Risk to quantile based co-risk measures that are weighted mixtures of CoVaR at different levels and hence involve the stochastic dependence that occurs among the risks and that is captured by copulas. We show that every quantile based co-risk measure is a quantile based risk measure and hence fulfills all related properties. We further discuss continuity results of quantile based co-risk measures from which consistent estimators for $\text{CoVaR}^{<}$ and $\text{CoVaR}^{\geq}$ based risk measures immediately follow when plugging in empirical copulas. Although estimating co-risk measures based on $\text{CoVaR}^{=}$ is a non-trivial endeavour since conditioning on events with zero probability is necessary we show that working with so-called empirical checkerboard copulas allows to construct strongly consistent estimators for $\text{CoVaR}^{=}$ and related co-risk measures under very mild regularity conditions. A small simulation study illustrates the performance of the obtained estimators for special classes of copulas.

1 Introduction

Conditional risk measures (*co-risk measures* for short) are systemic risk measures which are used to quantify a risk that may be related to another risk. This may involve quantifying the impact of a risk on another risk (for instance, the impact of an asset on another asset, the impact of a financial institution on another financial institution, etc.) as well as quantifying the systemic contribution of a risk to a set of risks (for instance, the impact of an asset on the whole portfolio, the impact of a financial institution on the corresponding financial sector, etc.).

The most prominent co-risk measure is the Co-Value-at-Risk (CoVaR) which equals the common Value-at-Risk (VaR) of the conditional distribution of a certain risk given that the related risk equals a given threshold ($\text{CoVaR}^{=}$), is smaller than a given threshold ($\text{CoVaR}^{<}$), or is larger than a given threshold ($\text{CoVaR}^{\geq}$). The former Co-Value-at-Risk was introduced in [3], the latter so called 'modified' Co-Value-at-Risks were discussed in [19, 29]; see also [4, 21, 22, 34]. The stochastic dependence that occurs among the risks is captured in terms of bivariate copulas. In contrast to the former version, the modified Co-Value-at-Risks exhibit the advantage that they are 'dependence consistent' (see, e.g., [4, 29, 34]), i.e. they preserve the concordance order of copulas.

In the present paper we apply L/P (loss/profit) notation and focus on the class of quantile based co-risk measures, which are risk measures that are weighted mixtures of the Co-Value-at-Risk at different levels and hence involve the stochastic dependence among the risks. This class

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includes the Co-Value-at-Risk itself, but also the very prominent Co-Expected-Shortfall (CoES; see, e.g., [29, 34]) and the Marginal Expected-Shortfall (MES; see, e.g., [2, 29]). Quantile based co-risk measures are hence defined as conditional analogues of quantile based risk measures (which are related to distortion risk measures); for more information on this approach we refer to [1, 16, 20, 36, 41, 42, 44] and references therein. It turns out that every quantile based co-risk measures is a quantile based risk measure whose distortion function depends on the involved copula. Thus, quantile based co-risk measures fulfill all the properties that apply to quantile based risk measures including positive homogeneity, translativity, comonoton additivity and monotonicity with respect to the stochastic order, and properties such as subadditivity or convexity depend on the specific shape of the related distortion function.

We further study continuity properties of quantile based co-risk measures taking into account the underlying dependence structure between the risks captured by copulas. It turns out that, depending on the type of co-risk measure, different notions of copula convergence need to be considered; this includes pointwise/uniform convergence for co-risk measures based on $\mathrm{CoVaR}^{<}$ and $\mathrm{CoVaR}^{\geq}$ (Theorem 5.1), and the recently introduced weak conditional convergence (see [25]) for co-risk measures based on $\mathrm{CoVaR}^{=}$ (Theorem 5.2). Applying the obtained continuity results strongly consistent estimators for co-risk measures based on $\mathrm{CoVaR}^{<}$ and $\mathrm{CoVaR}^{\geq}$ can be achieved by simply plugging in the empirical copula and the empirical marginal distribution function.

The situation turns out to be more challenging when co-risk measures based on $\mathrm{CoVaR}^{=}$ are to be estimated. In the literature several approaches have been discussed for the estimation of Co-Value-at-Risk $\mathrm{CoVaR}^{=}$ (and related co-risk measures) including quantile regression and GARCH estimation methods, but also using rather restrictive assumptions on the underlying bivariate distribution; see, e.g., [5, 8, 19]. In the current paper we present a quite general estimation procedure using checkerboard aggregations of the empirical copula. In Theorem 6.1 we show that under some very mild regularity conditions on the copula or the marginal distribution function, the estimator based on checkerboard approximations is strongly consistent.
As an additional feature of our approach we finally present a more sophisticated estimation approach that involves information about the underlying dependence structure using results from [23] and, in an Extreme Value copula setting, we compare the performance of the estimators incorporating or ignoring information about the dependence structure.

The rest of this contribution is organized as follows: Section 2 gathers preliminaries and notations that will be used throughout the paper. In Section 3 we start with the unconditional setting and define quantile based risk measures that are weighted mixtures of the Value-at-Risk at different levels and quantify single risks. In Section 4 we then proceed with the conditional setting and introduce the notion of quantile based co-risk measures that involve the stochastic dependence among the risks. Section 5 derives the afore-mentioned continuity results and in Section 6 we discuss consequences of these results for the estimation of quantile based co-risk measures.

2 Notation and preliminaries

In the sequel, let $\mathbb{I} := [0,1]$ and let $\mathcal{C}$ denote the family of all bivariate copulas. For each copula C the corresponding probability measure (also known as doubly stochastic measure or copula measure) will be denoted by μ_C, i.e. $\mu_C([0,u] \times [0,v]) = C(u,v)$ for all $u, v \in \mathbb{I}$. For more background on copulas we refer to [11, 32]. For every metric space (S,d) the Borel σ-field on S will be denoted by $\mathcal{B}(S)$.

In what follows Markov kernels will play a prominent role. A *Markov kernel* from $\mathbb{R}$ to $\mathbb{R}$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to \mathbb{I}$ such that for every fixed $E \in \mathcal{B}(\mathbb{R})$ the mapping $x \mapsto K(x, E)$ is (Borel-)measurable and for every fixed $x \in \mathbb{R}$ the mapping $E \mapsto K(x, E)$ is a probability measure. Given two real-valued random variables X, Y on a probability space $(\Omega, \mathcal{A}, P)$ we say that a Markov kernel K is a *regular conditional distribution* of Y given $X = x$ if

$$K(x, E) = E(\mathbb{1}_E \circ Y \,|\, X = x) =: P^Y(E \,|\, X = x) \tag{2.1}$$

holds P^X-almost surely for every $E \in \mathcal{B}(\mathbb{R})$. It is well-known (see, e.g., [24, 26]) that for X, Y as above, a regular conditional distribution of Y given X always exists and is unique for P^X-a.e. $x \in \mathbb{R}$, i.e. any two versions may differ only on a P^X null set.
If (X, Y) has distribution function H (in which case we will also write $(X, Y) \sim H$ and let μ_H denote the corresponding probability measure on $\mathcal{B}(\mathbb{R}^2)$) we will let K_H denote (a version of) the regular conditional distribution of Y given X and simply refer to it as *Markov kernel of H*. If C is a copula then we will consider the Markov kernel of C automatically as mapping $K_C : \mathbb{I} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$. Defining the x-section of a set $G \in \mathcal{B}(\mathbb{R}^2)$ as $G_x := \{y \in \mathbb{R} : (x, y) \in G\}$ the so-called disintegration theorem (see [24, 26]) yields

$$\int_{\mathbb{R}} K_H(x, G_x) \,\mathrm{d}P^X(x) = \mu_H(G)$$

As a direct consequence, for every $C \in \mathcal{C}$ we get

$$\int_{\mathbb{I}} K_C(u, E) \,\mathrm{d}P^X(u) = \int_{\mathbb{I}} K_C(u, E) \,\mathrm{d}\lambda(u) = \lambda(E)$$

for every $E \in \mathcal{B}(\mathbb{I})$, whereby λ denotes the Lebesgue measure on $\mathbb{R}$. For more background on conditional expectation and general disintegration we refer to [24, 26].

We denote by M the comonotonicity copula, by Π the independence copula and by W the countermonotonicity copula (see, e.g., [11, 32]). As copulas are Lipschitz continuous, Rademacher's theorem (see, e.g., [35]) implies that every copula is differentiable λ-a.e. It hence follows that $\partial_1 C(u, v) := \frac{\partial}{\partial u} C(u, v)$ exists a.e. and, for every $v \in \mathbb{I}$, the identity $\partial_1 C(u, v) = K_C(u, [0, v])$ holds for a.e. $u \in \mathbb{I}$; see, e.g., [11, Theorem 3.4.4].

3 Quantile based risk measures

We start with the unconditional setting and provide a brief overview of quantile based risk measures with which single risks can be quantified.

A function $D : \mathbb{I} \to \mathbb{I}$ is said to be a *distortion function* if it is increasing, right-continuous and fulfills $D(0) = 0$ and $\sup_{u \in (0,1)} D(u) = 1$ (and hence $D(1) = 1$). Thus, a distortion function corresponds to the distribution function of a probability measure μ on $\mathcal{B}(\mathbb{I})$ fulfilling $\mu((0, 1)) = 1$.

Example 3.1. The terms attached to the following examples are the names of the risk measures resulting from the respective distortion functions via Equation (3.1) below.

1. **Expectation:** The function $D_{\mathrm{E}} : \mathbb{I} \to \mathbb{I}$ given by $D_{\mathrm{E}}(t) := t$ is a distortion function.

2. **Value-at-Risk:** For $\alpha \in (0, 1)$ the function $D_{\mathrm{VaR}_\alpha} : \mathbb{I} \to \mathbb{I}$ given by $D_{\mathrm{VaR}_\alpha}(t) := \mathbb{1}_{[\alpha,1]}(t)$ is a distortion function.

3. **Expected-Shortfall:** For $\alpha \in (0,1)$ the function $D_{\mathrm{ES}_\alpha} : \mathrm{I} \to \mathrm{I}$ given by

$$D_{\mathrm{ES}_\alpha}(t) := \frac{t-\alpha}{1-\alpha}\, \mathbb{1}_{[\alpha,1]}(t)$$

is a distortion function.

4. **Range-Value-at-Risk:** For $\alpha, \beta \in [0,1]$ with $\alpha < \beta$, the function $D_{\mathrm{RVaR}_{\alpha,\beta}} : \mathrm{I} \to \mathrm{I}$ given by

$$D_{\mathrm{RVaR}_{\alpha,\beta}}(t) := \frac{t-\alpha}{\beta-\alpha}\, \mathbb{1}_{[\alpha,\beta]}(t) + \mathbb{1}_{(\beta,1]}(t)$$

is a distortion function. In particular, for $\alpha, \beta \in (0,1)$ with $\alpha < \beta$

$$\begin{aligned} D_{\mathrm{RVaR}_{0,1}} &= D_{\mathrm{E}} \\ D_{\mathrm{RVaR}_{\alpha,1}} &= D_{\mathrm{ES}_\alpha} \\ D_{\mathrm{RVaR}_{\beta,\beta}} := \lim_{\alpha \to \beta} D_{\mathrm{RVaR}_{\alpha,\beta}} &= D_{\mathrm{VaR}_\beta} \end{aligned}$$

For further examples of distortion functions including Expected-Shortfall of higher orders, Proportional Hazard transform or Wang's distortion function, we refer to [16, 39, 40].

Throughout this paper, we consider pairs (D, Q) consisting of a distortion function D and the probability measure Q corresponding to D, and we use identical sub- or superscripts for both, D and Q, in the case of a particular choice of D or Q. Note that, due to the properties of D we have $Q[(0,1)] = 1$.

We denote by $\mathcal{L}^0$ the vector lattice of all random variables, by $\mathcal{L}^1$ the vector lattice of all integrable random variables, and by $\mathcal{L}^\infty$ the vector lattice of all almost surely bounded random variables. Then we have $\mathcal{L}^\infty \subseteq \mathcal{L}^1 \subseteq \mathcal{L}^0$. For a random variable Y, we further denote by F_Y its univariate distribution function.

Define

$$\mathcal{L}_Q \;:=\; \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} |F_Y^{\leftarrow}(u)|\, \mathrm{d}Q(u) < \infty \right\}$$

where $F_Y^{\leftarrow} : \mathrm{I} \to \mathbb{R}$ denotes the (lower) quantile function of F_Y given, for every $u \in (0,1]$, by $F_Y^{\leftarrow}(u) := \inf\{y \in \mathbb{R} : F_Y(y) \geq u\}$ with $F_Y^{\leftarrow}(0) := \inf\{y \in \mathbb{R} : F_Y(y) > 0\}$; see [11]. Note that $\mathcal{L}_Q$ may fail to be a vector space. Then we have $\mathcal{L}^\infty \subseteq \mathcal{L}_Q$ and the map $\varrho_Q : \mathcal{L}_Q \to \mathbb{R}$ given by

$$\varrho_Q(Y) := \int_{\mathrm{I}} F_Y^{\leftarrow}(u)\, \mathrm{d}Q(u) \tag{3.1}$$

is said to be a *quantile based risk measure.* The notion of a quantile based risk measure naturally generalizes that of a spectral risk measure that has been introduced in [1] in P/L (profit/loss) setting.

Example 3.2. We present some classical risk measures that belong to the class of quantile based risk measures:

1. **Expectation:** The distortion function D_{E} satisfies $D_{\mathrm{E}} \circ F_Y = F_Y$ and hence

$$\varrho_{Q_{\mathrm{E}}}(Y) = \mathrm{E}(Y)$$

for every $Y \in \mathcal{L}_{Q_{\mathrm{E}}} = \mathcal{L}^1$.

2. **Value-at-Risk**: For $\alpha \in (0,1)$, the probability measure Q_{VaR_α} corresponding to D_{VaR_α} is the Dirac measure at α. This yields

$$\mathrm{VaR}_\alpha(Y) := \varrho_{Q_{\mathrm{VaR}_\alpha}}(Y) = F_Y^{\leftarrow}(\alpha)$$

for every $Y \in \mathcal{L}_{Q_{\mathrm{VaR}_\alpha}} = \mathcal{L}^0$ (such that $\mathcal{L}_{Q_{\mathrm{VaR}_\alpha}}$ does not depend on α). The quantile risk measure VaR_α is called *Value-at-Risk* at (confidence) level α.

3. **Expected-Shortfall:** For $\alpha \in (0,1)$, the probability measure Q_{ES_α} corresponding to D_{ES_α} satisfies $Q_{\mathrm{ES}_\alpha} = \int \frac{1}{1-\alpha} \mathbb{1}_{[\alpha,1]}(u)\,\mathrm{d}\lambda(u)$. This yields

$$\mathrm{ES}_\alpha(Y) := \varrho_{Q_{\mathrm{ES}_\alpha}}(Y) = \frac{1}{1-\alpha} \int_{[\alpha,1]} F_Y^{\leftarrow}(u)\,\mathrm{d}\lambda(u)$$

for every $Y \in \mathcal{L}_{Q_{\mathrm{ES}_\alpha}} = \mathcal{L}^{1,+} := \{Y \in \mathcal{L}^0 : \mathrm{E}(Y^+) < \infty\}$ (such that $\mathcal{L}_{Q_{\mathrm{ES}_\alpha}}$ does not depend on α). The quantile risk measure ES_α is called *Expected-Shortfall* at (confidence) level α.

4. **Range-Value-at-Risk:** For $\alpha, \beta \in [0,1]$ with $\alpha < \beta$, the probability measure $Q_{\mathrm{RVaR}_{\alpha,\beta}}$ corresponding to $D_{\mathrm{RVaR}_{\alpha,\beta}}$ satisfies $Q_{\mathrm{RVaR}_{\alpha,\beta}} = \int \frac{1}{\beta-\alpha} \mathbb{1}_{[\alpha,\beta]}(u)\,\mathrm{d}\lambda(u)$. This yields

$$\mathrm{RVaR}_{\alpha,\beta}(Y) := \varrho_{Q_{\mathrm{RVaR}_{\alpha,\beta}}}(Y) = \frac{1}{\beta-\alpha} \int_{[\alpha,\beta]} F_Y^{\leftarrow}(u)\,\mathrm{d}\lambda(u)$$

for every $Y \in \mathcal{L}_{Q_{\mathrm{RVaR}_{\alpha,\beta}}}$. If $\alpha, \beta \in (0,1)$ then $\mathcal{L}_{Q_{\mathrm{RVaR}_{\alpha,\beta}}} = \mathcal{L}^0$ (such that $\mathcal{L}_{Q_{\mathrm{RVaR}_{\alpha,\beta}}}$ does not depend on α and β). The quantile risk measure $\mathrm{RVaR}_{\alpha,\beta}$ is called *Range-Value-at-Risk* at level (α,β); see [9, 12]. In particular, for $\alpha, \beta \in (0,1)$ with $\alpha < \beta$

$$\begin{aligned} \mathrm{RVaR}_{0,1} &= \mathrm{E} \\ \mathrm{RVaR}_{\alpha,1} &= \mathrm{ES}_\alpha \\ \mathrm{RVaR}_{\beta,\beta} := \lim_{\alpha\to\beta} \mathrm{RVaR}_{\alpha,\beta} &= \mathrm{VaR}_\beta \end{aligned}$$

Every quantile based risk measure is positively homogeneous, translative, comonoton additive and monotone with respect to the stochastic order; see, e.g., [16, Lemma 3.1]. Moreover, a quantile based risk measure is subadditive (and hence convex), if and only if, D is convex; see, e.g., [16, Theorem 5.4].

The composition $D \circ F$ of a distortion function D and a univariate distribution function F is again a distribution function, and we denote by $\mu_{D\circ F}$ the probability measure corresponding to $D \circ F$. We note in passing that quantile based risk measures may also be represented in terms of the composition $D \circ F$, as is well-known in the literature: For every distortion function D and every distribution function F the identity

$$\varrho_Q(Y) = \int_{\mathbb{R}} y\,\mathrm{d}\mu_{D\circ F_Y}(y) = \int_{(0,\infty)} 1 - (D\circ F)(y)\,\mathrm{d}\lambda(y) + \int_{(-\infty,0)} (D\circ F)(y)\,\mathrm{d}\lambda(y)$$

holds for every $Y \in \mathcal{L}_Q$; see, e.g., [14, 16, 43].

4 Quantile based co-risk measures

We proceed with the conditional setting and introduce the notion of quantile based co-risk measures which are risk measures that involve the stochastic dependence among risks.

In what follows, we consider two random variables X and Y on $(\Omega, \mathcal{A}, P)$ with continuous distribution functions F_X and F_Y, respectively.

1. For every $\gamma \in (0,1)$ we denote by $F_{Y|X<F_X^{\leftarrow}(\gamma)}$ the distribution function corresponding to the probability measure $P^Y(\cdot \,|\, \{X < F_X^{\leftarrow}(\gamma)\})$,
2. for every $\gamma \in (0,1)$ we denote by $F_{Y|X\geq F_X^{\leftarrow}(\gamma)}$ the distribution function corresponding to the probability measure $P^Y(\cdot \,|\, \{X \geq F_X^{\leftarrow}(\gamma)\})$, and
3. for every $\gamma \in (0,1)$ we denote by $F_{Y|X=F_X^{\leftarrow}(\gamma)}$ the distribution function corresponding to the probability measure $P^Y(\cdot \,|\, X = F_X^{\leftarrow}(\gamma))$ (notice that as soon as a Markov kernel of Y given X is chosen $F_{Y|X=F_X^{\leftarrow}(\gamma)}$ is defined for every $\gamma \in (0,1)$).

We assume that the random variables X and Y are connected via the unique copula C (see, e.g., [11]), i.e. the identity $H(x,y) = C(F_X(x), F_Y(y))$ holds for all $(x,y) \in \mathbb{R}^2$. In this case, since $(F_Y \circ F_Y^{\leftarrow})(u) = u$ for every $u \in \mathrm{I}$

1. the identity
$$F_{Y|X<F_X^{\leftarrow}(\gamma)}(y) = \frac{C\big(\gamma, F_Y(y)\big)}{\gamma}$$
holds for every $y \in \mathbb{R}$ and every $\gamma \in (0,1)$,
2. the identity
$$F_{Y|X\geq F_X^{\leftarrow}(\gamma)}(y) = \frac{F_Y(y) - C\big(\gamma, F_Y(y)\big)}{1-\gamma}$$
holds for every $y \in \mathbb{R}$ and every $\gamma \in (0,1)$, and
3. the identity
$$F_{Y|X=F_X^{\leftarrow}(\gamma)}(y) = K_C\big(\gamma, [0, F_Y(y)]\big) \tag{4.1}$$
holds for every $y \in \mathbb{R}$ and a.e. $\gamma \in (0,1)$. Indeed, since $\lambda^{F_X^{\leftarrow}} = P^X$ and $(F_X \circ F_X^{\leftarrow})(u) = u$ for every $u \in \mathrm{I}$, we obtain
$$\begin{aligned}\int_{[0,u]} F_{Y|X=F_X^{\leftarrow}(\gamma)}(y)\; \mathrm{d}\lambda(\gamma) &= \int_{(F_X^{\leftarrow})^{-1}((-\infty, F_X^{\leftarrow}(u)])} F_{Y|X=F_X^{\leftarrow}(\gamma)}(y)\; \mathrm{d}\lambda(\gamma) \\ &= \int_{(-\infty, F_X^{\leftarrow}(u)]} F_{Y|X=x}(y)\; \mathrm{d}\lambda^{F_X^{\leftarrow}}(x) \\ &= \int_{(-\infty, F_X^{\leftarrow}(u)]} F_{Y|X=x}(y)\; \mathrm{d}P^X(x) \\ &= H\big(F_X^{\leftarrow}(u), y\big) \\ &= C\big((F_X \circ F_X^{\leftarrow})(u), F_Y(y)\big) \\ &= C(u, F_Y(y)) \\ &= \int_{[0,u]} K_C\big(\gamma, [0, F_Y(y)]\big)\; \mathrm{d}\lambda(\gamma)\end{aligned}$$
for every $u \in \mathrm{I}$. The assertion hence follows from the Radon-Nikodym theorem. The previous identity was also discussed in [31].

Note that, in Equation (4.1), we can choose any Markov kernel that is a regular conditional distribution of Y given X (which may be the "canonical version" constructed in [4, 22]). Thus, the Markov kernel on the right hand side of Equation (4.1) is defined for every $\gamma \in (0,1)$ but is unique only for a.e. $\gamma \in (0,1)$. Therefore, although the results related to Markov kernels are valid only a.s., we may define quantities related to the Markov kernel for arbitrary $\gamma \in (0,1)$.

For $C \in \mathcal{C}$ and $\gamma \in (0,1)$, we now define the maps $\delta^{<}_{\gamma,C}, \delta^{\geq}_{\gamma,C}, \delta^{=}_{\gamma,C} : \mathrm{I} \to \mathrm{I}$ by letting

$$\delta^{<}_{\gamma,C}(t) := \frac{C(\gamma,t)}{\gamma} \qquad \delta^{\geq}_{\gamma,C}(t) := \frac{t - C(\gamma,t)}{1-\gamma} \qquad \delta^{=}_{\gamma,C}(t) := K_C(\gamma,[0,t])$$

Then, for every $* \in \{<, \geq, =\}$, $\delta^{*}_{\gamma,C}$ is a distribution functions on I satisfying

$$F_{Y|X * F_X^{\leftarrow}(\gamma)} = \delta^{*}_{\gamma,C} \circ F_Y$$

which in turn implies (compare [29, Theorem 3.1])

$$F^{\leftarrow}_{Y|X * F_X^{\leftarrow}(\gamma)} = F_Y^{\leftarrow} \circ (\delta^{*}_{\gamma,C})^{\leftarrow}$$

Note that the two above equations for $* \in \{=\}$ are valid only a.s. Due to Lipschitz continuity of copulas, the maps $\delta^{<}_{\gamma,C}$ and $\delta^{\geq}_{\gamma,C}$ are Lipschitz continuous. In contrast to that, $\delta^{=}_{\gamma,C}$ may fail to be continuous.

Remark 4.1. Note that the maps $\delta^{<}_{\gamma,C}$ and $\delta^{\geq}_{\gamma,C}$ are related to each other via $\delta^{\geq}_{\gamma,C} = \delta^{<}_{1-\gamma,\nu_1(C)}$ where $\nu_1(C)$ denotes the reflection of C in the first coordinate given by $(\nu_1(C))(u,v) := v - C(1-u,v)$ for all $(u,v) \in \mathrm{I}^2$ and hence

$$F_{Y|X \geq F_X^{\leftarrow}(\gamma)} = \delta^{<}_{1-\gamma,\nu_1(C)} \circ F_Y$$

Note that ν_1 is an involution satisfying $\nu_1(M) = W$ and $\nu_1(\Pi) = \Pi$; we refer to [10, 15, 17] for more information on copula reflections.
The first approach applying conditional distribution functions $F_{Y|X<F_X^{\leftarrow}(\gamma)}$ is usually considered when X and Y are interpreted as financial positions describing the wealth of an institution or the profit of an asset portfolio; see, e.g., [4]. The second approach using conditional distribution functions $F_{Y|X \geq F_X^{\leftarrow}(\gamma)}$ is common for instance when considering X and Y as *random loss* variables; see, e.g., [29].
Thus, in either case we can restrict our consideration to the functions $\delta^{<}$ and $\delta^{=}$.

It turns out that the maps $\delta^{<}_{\gamma,C}$ and $\delta^{=}_{\gamma,C}$ themselves are distortion functions and may be used to construct distortion functions for co-risk measures. The next result is straightforward:

Corollary 4.2. *For every $\gamma \in (0,1)$ and every copula C the maps $\delta^{<}_{\gamma,C}$ and $\delta^{=}_{\gamma,C}$ are distortion functions.*

We now apply $\delta^{<}_{\gamma,C}$ and $\delta^{=}_{\gamma,C}$ to the comonotonicity copula M, the independence copula Π and the countermonotonicity copula W:

C	$\delta^{<}_{\gamma,C}$		$\delta^{=}_{\gamma,C}$	
M	$D_{\mathrm{RVaR}_{0,\gamma}}$		$D_{\mathrm{RVaR}_{\gamma,\gamma}}$	$= D_{\mathrm{VaR}_\gamma}$
Π	$D_{\mathrm{RVaR}_{0,1}}$	$= D_{\mathrm{E}}$	$D_{\mathrm{RVaR}_{0,1}}$	$= D_{\mathrm{E}}$
W	$D_{\mathrm{RVaR}_{1-\gamma,1}}$	$= D_{\mathrm{ES}_{1-\gamma}}$	$D_{\mathrm{RVaR}_{1-\gamma,1-\gamma}}$	$= D_{\mathrm{VaR}_{1-\gamma}}$

Remark 4.3. Corollary 4.2 reflects the idea of considering $\delta^{<}_{\gamma,C}$ (and possibly also $\delta^{=}_{\gamma,C}$) as a distortion function whose corresponding quantile based risk measure is based on contagion from an external scenario whereas the dependence is modeled with a copula having horizontal concave sections; see, e.g., [7, 44]. As can be seen from the previous table, the Expected-Shortfall possesses such an interpretation.

The quantile functions of $\delta^{<}_{\gamma,C}$ and $\delta^{=}_{\gamma,C}$ satisfy:

C	$(\delta^{<}_{\gamma,C})^{\leftarrow}(u)$	$(\delta^{=}_{\gamma,C})^{\leftarrow}(u)$
M	γu	γ
Π	u	u
W	$\gamma u + 1 - \gamma$	$1 - \gamma$

From [11, Theorem 1.7.3] we may conclude that the map $C \mapsto \delta^{<}_{\gamma,C}$ is order preserving and that the map $C \mapsto (\delta^{<}_{\gamma,C})^{\leftarrow}$ is order reversing with respect to PLOD order on copulas:

Corollary 4.4. *For $\gamma \in (0,1)$ the inequalities $\delta^{<}_{\gamma,W} \leq \delta^{<}_{\gamma,C_1} \leq \delta^{<}_{\gamma,C_2} \leq \delta^{<}_{\gamma,M}$ and*

$$(\delta^{<}_{\gamma,M})^{\leftarrow} \leq (\delta^{<}_{\gamma,C_2})^{\leftarrow} \leq (\delta^{<}_{\gamma,C_1})^{\leftarrow} \leq (\delta^{<}_{\gamma,W})^{\leftarrow}$$

hold for all $C_1 \leq C_2$.

The following result extends Corollary 4.2:

Corollary 4.5. *For every $\gamma \in (0,1)$, every copula C and every distortion function D the maps $D^{<}_{\gamma,C} := D \circ \delta^{<}_{\gamma,C}$ and $D^{=}_{\gamma,C} := D \circ \delta^{=}_{\gamma,C}$ are distortion functions.*

Example 4.6. The terms attached to the following examples are the names of the co-risk measures resulting from the respective distortion functions.

1. **Marginal Expected-Shortfall** or **Co-Expectation:** The distortion functions $(D_{\mathrm{E}})^{<}_{\gamma,C}$ and $(D_{\mathrm{E}})^{=}_{\gamma,C}$ satisfy

$$\begin{aligned} (D_{\mathrm{E}})^{<}_{\gamma,C} &= \delta^{<}_{\gamma,C} \\ (D_{\mathrm{E}})^{=}_{\gamma,C} &= \delta^{=}_{\gamma,C} \end{aligned}$$

2. **Co-Value-at-Risk:** For $\alpha \in (0,1)$, the distortion functions $(D_{\mathrm{VaR}_\alpha})^{<}_{\gamma,C}$ and $(D_{\mathrm{VaR}_\alpha})^{=}_{\gamma,C}$ satisfy

$$\begin{aligned} (D_{\mathrm{VaR}_\alpha})^{<}_{\gamma,C} &= \mathbb{1}_{[(\delta^{<}_{\gamma,C})^{\leftarrow}(\alpha),1]} = D_{\mathrm{VaR}_{(\delta^{<}_{\gamma,C})^{\leftarrow}(\alpha)}} \\ (D_{\mathrm{VaR}_\alpha})^{=}_{\gamma,C} &= \mathbb{1}_{[(\delta^{=}_{\gamma,C})^{\leftarrow}(\alpha),1]} = D_{\mathrm{VaR}_{(\delta^{=}_{\gamma,C})^{\leftarrow}(\alpha)}} \end{aligned}$$

3. **Co-Expected-Shortfall:** For $\alpha \in (0,1)$, the distortion functions $(D_{\mathrm{ES}_\alpha})^{<}_{\gamma,C}$ and $(D_{\mathrm{ES}_\alpha})^{=}_{\gamma,C}$ satisfy

$$\begin{aligned} (D_{\mathrm{ES}_\alpha})^{<}_{\gamma,C}(t) &= \frac{C(\gamma,t) - \alpha\gamma}{\gamma - \alpha\gamma} \, \mathbb{1}_{[\alpha\gamma,1]}(C(\gamma,t)) \\ (D_{\mathrm{ES}_\alpha})^{=}_{\gamma,C}(t) &= \frac{K_C(\gamma,[0,t]) - \alpha}{1-\alpha} \, \mathbb{1}_{[\alpha,1]}(K_C(\gamma,[0,t])) \end{aligned}$$

for every $t \in \mathrm{I}$.

4. **Co-Range-Value-at-Risk:** For $\alpha, \beta \in [0,1]$ with $\alpha < \beta$, the distortion functions $(D_{\mathrm{RVaR}_{\alpha,\beta}})^{<}_{\gamma,C}$ and $(D_{\mathrm{RVaR}_{\alpha,\beta}})^{=}_{\gamma,C}$ satisfy

$$\begin{aligned}(D_{\mathrm{RVaR}_{\alpha,\beta}})^{<}_{\gamma,C}(t) &= \frac{C(\gamma,t)-\alpha\gamma}{\beta\gamma-\alpha\gamma}\,\mathbb{1}_{[\alpha\gamma,\beta\gamma]}(C(\gamma,t)) + \mathbb{1}_{(\beta\gamma,\gamma]}(C(\gamma,t)) \\ (D_{\mathrm{RVaR}_{\alpha,\beta}})^{=}_{\gamma,C}(t) &= \frac{K_C(\gamma,[0,t])-\alpha}{\beta-\alpha}\,\mathbb{1}_{[\alpha,\beta]}(K_C(\gamma,[0,t])) + \mathbb{1}_{(\beta,1]}(K_C(\gamma,[0,t]))\end{aligned}$$

for every $t \in \mathrm{I}$.

Representations for the above distortion functions in the case of comonotonic, independent and countermonotonic random variables, respectively, may be obtained from the following table:

C	$(D_{\mathrm{RVaR}_{\alpha,\beta}})^{<}_{\gamma,C}$	$(D_{\mathrm{RVaR}_{\alpha,\beta}})^{=}_{\gamma,C}$
M	$D_{\mathrm{RVaR}_{\alpha\gamma,\beta\gamma}}$	$D_{\mathrm{RVaR}_{\gamma,\gamma}}$
Π	$D_{\mathrm{RVaR}_{\alpha,\beta}}$	$D_{\mathrm{RVaR}_{\alpha,\beta}}$
W	$D_{\mathrm{RVaR}_{\alpha\gamma+1-\gamma,\beta\gamma+1-\gamma}}$	$D_{\mathrm{RVaR}_{1-\gamma,1-\gamma}}$

Now, for $\gamma \in (0,1)$ and $C \in \mathcal{C}$, define

$$\begin{aligned}\mathcal{L}^{<}_{\gamma,C,Q} &:= \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} |F^{\leftarrow}_{Y|X<F^{\leftarrow}_X(\gamma)}(u)|\,\mathrm{d}Q(u) < \infty \right\} \\ \mathcal{L}^{\geq}_{\gamma,C,Q} &:= \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} |F^{\leftarrow}_{Y|X\geq F^{\leftarrow}_X(\gamma)}(u)|\,\mathrm{d}Q(u) < \infty \right\} \\ \mathcal{L}^{=}_{\gamma,C,Q} &:= \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} |F^{\leftarrow}_{Y|X=F^{\leftarrow}_X(\gamma)}(u)|\,\mathrm{d}Q(u) < \infty \right\}\end{aligned}$$

Then we have $\mathcal{L}^\infty \subseteq \mathcal{L}^{<}_{\gamma,C,Q}, \mathcal{L}^{\geq}_{\gamma,C,Q}, \mathcal{L}^{=}_{\gamma,C,Q}$ and the maps $\varrho^{<}_{\gamma,C,Q} : \mathcal{L}^{<}_{\gamma,C,Q} \to \mathbb{R}$, $\varrho^{\geq}_{\gamma,C,Q} : \mathcal{L}^{<}_{\gamma,C,Q} \to \mathbb{R}$ and $\varrho^{=}_{\gamma,C,Q} : \mathcal{L}^{=}_{\gamma,C,Q} \to \mathbb{R}$ given by

$$\begin{aligned}\varrho^{<}_{\gamma,C,Q}(Y) &:= \int_{\mathrm{I}} F^{\leftarrow}_{Y|X<F^{\leftarrow}_X(\gamma)}(u)\,\mathrm{d}Q(u) \\ \varrho^{\geq}_{\gamma,C,Q}(Y) &:= \int_{\mathrm{I}} F^{\leftarrow}_{Y|X\geq F^{\leftarrow}_X(\gamma)}(u)\,\mathrm{d}Q(u) \\ \varrho^{=}_{\gamma,C,Q}(Y) &:= \int_{\mathrm{I}} F^{\leftarrow}_{Y|X=F^{\leftarrow}_X(\gamma)}(u)\,\mathrm{d}Q(u)\end{aligned}$$

are said to be a *quantile based co-risk measure.*

Example 4.7. Let $* \in \{<, \geq, =\}$.

1. **Marginal Expected-Shortfall** or **Co-Expectation:** The quantile co-risk measure $\varrho^{*}_{\gamma,C,Q_{\mathrm{E}}}$ satisfies
$$\mathrm{MES}^{*}_{\gamma,C}(Y|X) := \varrho^{*}_{\gamma,C,Q_{\mathrm{E}}}(Y) = E(Y|X * F^{\leftarrow}_X(\gamma))$$
for every $Y \in \mathcal{L}^{*}_{\gamma,C,Q_{\mathrm{E}}}$ and is called *Marginal Expected-Shortfall* or *Co-Expectation*; see, e.g., [2, 29].

2. **Co-Value-at-Risk:** For $\alpha \in (0,1)$, the quantile co-risk measure $\varrho^*_{\gamma,C,Q_{\mathrm{VaR}_\alpha}}$ satisfies

$$\mathrm{CoVaR}^*_{\alpha,\gamma,C}(Y|X) := \varrho^*_{\gamma,C,Q_{\mathrm{VaR}_\alpha}}(Y) = F^{\leftarrow}_{Y|X*F^{\leftarrow}_X(\gamma)}(\alpha)$$

   for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{VaR}_\alpha}}$ and is called *Co-Value-at-Risk* at level α; see, e.g., [3, 19, 29].

3. **Co-Expected-Shortfall:** For $\alpha \in (0,1)$, the quantile co-risk measure $\varrho^*_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$ satisfies

$$\mathrm{CoES}^*_{\alpha,\gamma,C}(Y|X) := \varrho^*_{\gamma,C,Q_{\mathrm{ES}_\alpha}}(Y) = \frac{1}{1-\alpha}\int_{[\alpha,1]} F^{\leftarrow}_{Y|X*F^{\leftarrow}_X(\gamma)}(u)\,\mathrm{d}\lambda(u)$$

   for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$ and is called *Co-Expected-Shortfall* at level α; see, e.g., [29].

4. **Co-Range-Value-at-Risk:** For $\alpha,\beta \in [0,1]$ with $\alpha < \beta$, the quantile co-risk measure $\varrho^*_{\gamma,C,Q_{\mathrm{RVaR}_{\alpha,\beta}}}$ satisfies

$$\mathrm{CoRVaR}^*_{\alpha,\beta,\gamma,C}(Y|X) := \varrho^*_{\gamma,C,Q_{\mathrm{RVaR}_{\alpha,\beta}}}(Y) = \frac{1}{\beta-\alpha}\int_{[\alpha,\beta]} F^{\leftarrow}_{Y|X*F^{\leftarrow}_X(\gamma)}(u)\,\mathrm{d}\lambda(u)$$

   for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{RVaR}_{\alpha,\beta}}}$ and is called *Co-Range-Value-at-Risk* at level (α,β). In particular, for $\alpha,\beta \in (0,1)$ with $\alpha < \beta$

$$\begin{aligned} \mathrm{CoRVaR}_{0,1} &= \mathrm{MES} \\ \mathrm{CoRVaR}_{\alpha,1} &= \mathrm{CoES}_\alpha \\ \mathrm{CoRVaR}_{\beta,\beta} := \lim_{\alpha\to\beta} \mathrm{CoRVaR}_{\alpha,\beta} &= \mathrm{CoVaR}_\beta \end{aligned}$$

Since $\varrho^*_{\gamma,C,Q}$ satisfies

$$\varrho^*_{\gamma,C,Q}(Y) = \int_{\mathrm{I}} \mathrm{CoVaR}^*_{u,\gamma,C}(Y|X)\,\mathrm{d}Q(u)$$

($* \in \{<,\geq,=\}$), every quantile based co-risk measure is a CoVaR-based risk measure. It even turns out that every quantile based co-risk measure is a quantile based risk measure for which the distortion function depends on $\gamma \in (0,1)$ and $C \in \mathcal{C}$:

Theorem 4.8. *Let $\gamma \in (0,1)$, $C \in \mathcal{C}$ and consider $* \in \{<,\geq,=\}$. Then*

$$\mathcal{L}^*_{\gamma,C,Q} = \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} \left| \left(F^{\leftarrow}_Y \circ (\delta^*_{\gamma,C})^{\leftarrow}\right)(u)\right| \,\mathrm{d}Q(u) < \infty \right\} = \mathcal{L}^*_{Q_{\gamma,C}}$$

and the identity

$$\varrho^*_{\gamma,C,Q}(Y) = \int_{\mathrm{I}} \mathrm{VaR}_{(\delta^*_{\gamma,C})^{\leftarrow}(u)}(Y)\,\mathrm{d}Q(u) = \int_{\mathrm{I}} F^{\leftarrow}_Y(u)\,\mathrm{d}Q^*_{\gamma,C}(u) = \varrho^*_{Q_{\gamma,C}}(Y)$$

holds for every $Y \in \mathcal{L}^*_{\gamma,C,Q}$.

Proof. First note that $Q^<_{\gamma,C} = Q^{(\delta^<_{\gamma,C})^{\leftarrow}}$ which is immediate from Corollary 4.5. Then the identities

$$\int_{\mathrm{I}} \left(F^{\leftarrow}_{Y|X<F^{\leftarrow}_X(\gamma)}(u)\right)^+ \mathrm{d}Q(u) = \int_{\mathrm{I}} \left(\left(F^{\leftarrow}_Y \circ (\delta^<_{\gamma,C})^{\leftarrow}\right)(u)\right)^+ \mathrm{d}Q(u)$$

$$
\begin{aligned}
&= \int_{(0,1]} \big((F_Y^{\leftarrow} \circ (\delta_{\gamma,C}^{<})^{\leftarrow})(u) \big)^{+} \, \mathrm{d}Q(u) \\
&= \int_{((\delta_{\gamma,C}^{<})^{\leftarrow}(0),(\delta_{\gamma,C}^{<})^{\leftarrow}(1)]} (F_Y^{\leftarrow}(u))^{+} \, \mathrm{d}Q^{(\delta_{\gamma,C}^{<})^{\leftarrow}}(u) \\
&= \int_{((\delta_{\gamma,C}^{<})^{\leftarrow}(0),(\delta_{\gamma,C}^{<})^{\leftarrow}(1)]} (F_Y^{\leftarrow}(u))^{+} \, \mathrm{d}Q_{\gamma,C}^{<}(u) \\
&= \int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^{+} \, \mathrm{d}Q_{\gamma,C}^{<}(u)
\end{aligned}
$$

and

$$
\int_{\mathrm{I}} \big(F_{Y|X<F_X^{\leftarrow}(\gamma)}^{\leftarrow}(u) \big)^{-} \, \mathrm{d}Q(u) = \int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^{-} \, \mathrm{d}Q_{\gamma,C}^{<}(u)
$$

hold for every distortion function D and every $Y \in \mathcal{L}^0$. Thus,

$$
\mathcal{L}_{\gamma,C,Q}^{<} = \mathcal{L}_{Q_{\gamma,C}}^{<} = \left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathrm{I}} \big| \big(F_Y^{\leftarrow} \circ (\delta_{\gamma,C}^{<})^{\leftarrow} \big)(u) \big| \, \mathrm{d}Q(u) \right\}
$$

and

$$
\varrho_{\gamma,C,Q}^{<} = \varrho_{Q_{\gamma,C}}^{<} = \int_{\mathrm{I}} \big(F_Y^{\leftarrow} \circ (\delta_{\gamma,C}^{<})^{\leftarrow} \big)(u) \, \mathrm{d}Q(u)
$$

This proves the assertion for $* \in \{<\}$. The same reasoning applies to $* \in \{\geq\}$ and $* \in \{=\}$ where the latter follows from the fact that $\delta_{\gamma,C}^{=}$ is right-continuous at 0 and left-continuous at 1. □

Remark 4.9. Since quantile based co-risk measures are quantile based risk measures they fulfill all the properties that apply to quantile based risk measures including positive homogeneity, translativity, comonoton additivity and monotonicity with respect to the stochastic order; see, e.g., [16, Lemma 3.1]. Properties such as subadditivity or convexity depend on the shape of $D_{\gamma,C}^{*}$. For instance, by [16, Theorem 5.4], $\varrho_{\gamma,C,Q}^{*}$ is subadditive (and hence convex), if and only if, $D_{\gamma,C}^{*}$ is convex.

As a first consequence of Theorem 4.8 we obtain the following corollary which is immediate from the identity $\delta_{\gamma,C}^{\geq} = \delta_{1-\gamma,\nu_1(C)}^{<}$ discussed in Remark 4.1:

Corollary 4.10. *The identity*

$$
\varrho_{\gamma,C,Q}^{\geq}(Y) \;=\; \varrho_{1-\gamma,\nu_1(C),Q}^{<}(Y)
$$

holds for every $Y \in \mathcal{L}_{\gamma,C,Q}^{\geq} = \mathcal{L}_{1-\gamma,\nu_1(C),Q}^{<}$.

Thus, in what follows we may restrict our consideration to the quantile co-risk measures $\varrho_{\gamma,C,Q}^{<}$ and $\varrho_{\gamma,C,Q}^{=}$. Nevertheless, we add results for $\varrho_{\gamma,C,Q}^{\geq}$ where appropriate.

For $C \in \{M, \Pi, W\}$ we immediately obtain the following identities:

C	$\varrho_{\gamma,C,Q}^{<}(Y)$	$\varrho_{\gamma,C,Q}^{=}(Y)$
M	$\int_{\mathrm{I}} \mathrm{VaR}_{\gamma u}(Y) \, \mathrm{d}Q(u)$	$\mathrm{VaR}_{\gamma}(Y)$
Π	$\int_{\mathrm{I}} \mathrm{VaR}_{u}(Y) \, \mathrm{d}Q(u)$	$\int_{\mathrm{I}} \mathrm{VaR}_{u}(Y) \, \mathrm{d}Q(u)$
W	$\int_{\mathrm{I}} \mathrm{VaR}_{\gamma u+1-\gamma}(Y) \, \mathrm{d}Q(u)$	$\mathrm{VaR}_{1-\gamma}(Y)$

Remarkably, no matter how Q is chosen, $\varrho^{=}_{\gamma,M,Q}$ (which refers to the case when the random variables are comonotonic) equals the Value-at-Risk at level γ, and $\varrho^{=}_{\gamma,W,Q}$ (which refers to the case when the random variables are countermonotonic) equals the Value-at-Risk at level $1-\gamma$. The copulas M and W belong to the class of all completely dependent copulas: A copula C is said to be *completely dependent* if there exists some λ-preserving transformation $h : \mathbb{I} \to \mathbb{I}$ (i.e., a transformation fulfilling $\lambda(h^{-1}(F)) = \lambda(F)$ for every $F \in \mathcal{B}(\mathbb{I})$) such that $K(\gamma, E) := \mathbb{1}_E(h(\gamma))$ is a Markov kernel of C. For more properties of complete dependence we refer to [27] as well as to [13] and the references therein.

Corollary 4.11. *For every completely dependent copula C with λ-preserving transformation h the identity*

$$\varrho^{=}_{\gamma,C,Q} = \mathrm{VaR}_{h(\gamma)}$$

holds for every $\gamma \in (0,1)$ such that $h(\gamma) \in (0,1)$, and the map $\varrho^{=}_{\gamma,C,Q}$ does not depend on the choice of Q. In particular, $\varrho^{=}_{\gamma,M,Q} = \mathrm{VaR}_\gamma$ and $\varrho^{=}_{\gamma,W,Q} = \mathrm{VaR}_{1-\gamma}$.

Theorem 4.8 in combination with Corollary 4.4 and the fact that the pseudo inverse is increasing implies that the map $C \mapsto \varrho^{<}_{Q_{\gamma,C}}(Y)$ is order preserving with respect to the PLOD order on copulas; Theorem 4.12 hence slightly extends [29, Theorem 3.4] and [34, Theorem 12]:

Theorem 4.12. *Consider $\gamma \in (0,1)$.*

(1) The inequality $\varrho^{<}_{Q_{\gamma,C_2}}(Y) \leq \varrho^{<}_{Q_{\gamma,C_1}}(Y)$ holds for all $C_1 \leq C_2$ and every $Y \in \mathcal{L}^{<}_{\gamma,C_1,Q} \cap \mathcal{L}^{<}_{\gamma,C_2,Q}$. In particular, the inequality

$$\varrho^{<}_{Q_{\gamma,M}}(Y) \leq \varrho^{<}_{Q_{\gamma,C}}(Y) \leq \varrho^{<}_{Q_{\gamma,W}}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{<}_{\gamma,C,Q}$.

(2) The inequality $\varrho^{\geq}_{Q_{\gamma,C_1}}(Y) \leq \varrho^{\geq}_{Q_{\gamma,C_2}}(Y)$ holds for all $C_1 \leq C_2$ and every $Y \in \mathcal{L}^{\geq}_{\gamma,C_1,Q} \cap \mathcal{L}^{\geq}_{\gamma,C_2,Q}$. In particular, the inequality

$$\varrho^{\geq}_{Q_{\gamma,W}}(Y) \leq \varrho^{\geq}_{Q_{\gamma,C}}(Y) \leq \varrho^{\geq}_{Q_{\gamma,M}}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{\geq}_{\gamma,C,Q}$.

Additional results including order properties of Y can be derived from [29].

The next result is a consequence of Corollary 4.4 and yields relations for the domain of a co-risk measure:

Corollary 4.13. *Consider $\gamma \in (0,1)$.*

(1) The inclusion

$$\left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathbb{I}} (F_Y^{\leftarrow}(u))^+ \, \mathrm{d}Q^{<}_{\gamma,W}(u) + \int_{\mathbb{I}} (F_Y^{\leftarrow}(u))^- \, \mathrm{d}Q^{<}_{\gamma,M}(u) < \infty \right\} \subseteq \mathcal{L}^{<}_{\gamma,C,Q}$$

holds for every $C \in \mathcal{C}$.

(2) The inclusion

$$\left\{ Y \in \mathcal{L}^0 \;\middle|\; \int_{\mathbb{I}} (F_Y^{\leftarrow}(u))^+ \, \mathrm{d}Q^{\geq}_{\gamma,M}(u) + \int_{\mathbb{I}} (F_Y^{\leftarrow}(u))^- \, \mathrm{d}Q^{\geq}_{\gamma,W}(u) < \infty \right\} \subseteq \mathcal{L}^{\geq}_{\gamma,C,Q}$$

holds for every $C \in \mathcal{C}$.

Proof. We first prove (1). To this end, consider $Y \in \mathcal{L}^0$. Corollary 4.4 together with the fact that the pseudo inverse is increasing then yields

$$\int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^+ \, \mathrm{d}Q^{<}_{\gamma,C}(u) \leq \int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^+ \, \mathrm{d}Q^{<}_{\gamma,W}(u)$$

and

$$\int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^- \, \mathrm{d}Q^{<}_{\gamma,C}(u) \leq \int_{\mathrm{I}} (F_Y^{\leftarrow}(u))^- \, \mathrm{d}Q^{<}_{\gamma,M}(u)$$

which proves (1). Analogously, we obtain (2). □

We now resume Example 4.7:

Example 4.14. Let $* \in \{<, \geq, =\}$.

1. **Marginal Expected-Shortfall** or **Co-Expectation:** The Marginal Expected-Shortfall satisfies

$$\mathrm{MES}^*_{\gamma,C}(Y|X) = \int_{\mathrm{I}} \mathrm{VaR}_{(\delta^*_{\gamma,C})^{\leftarrow}(u)}(Y) \, \mathrm{d}\lambda(u)$$

for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{E}}}$. Moreover,

(a) the inequality

$$\mathrm{RVaR}_{0,\gamma}(Y) \leq \mathrm{MES}^{<}_{\gamma,C}(Y|X) \leq \mathrm{ES}_{1-\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{<}_{\gamma,C,Q_{\mathrm{E}}}$, and the inclusion $\mathcal{L}^1 \subseteq \mathcal{L}^{<}_{\gamma,C,Q_{\mathrm{E}}}$ holds for every $C \in \mathcal{C}$.

(b) the inequality

$$\mathrm{RVaR}_{0,1-\gamma}(Y) \leq \mathrm{MES}^{\geq}_{\gamma,C}(Y|X) \leq \mathrm{ES}_{\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{\geq}_{\gamma,C,Q_{\mathrm{E}}}$, and the inclusion $\mathcal{L}^1 \subseteq \mathcal{L}^{\geq}_{\gamma,C,Q_{\mathrm{E}}}$ holds for every $C \in \mathcal{C}$.

2. **Co-Value-at-Risk:** For $\alpha \in (0,1)$, the Co-Value-at-Risk satisfies (see [29, Theorem 3.1])

$$\mathrm{CoVaR}^*_{\alpha,\gamma,C}(Y|X) = \mathrm{VaR}_{(\delta^*_{\gamma,C})^{\leftarrow}(\alpha)}(Y)$$

for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{VaR}_\alpha}} = \mathcal{L}^0$. In particular, every Co-Value-at-Risk at level α is a Value-at-Risk at level $(\delta^*_{\gamma,C})^{\leftarrow}(\alpha)$. Moreover,

(a) the inequality

$$\mathrm{VaR}_{\alpha\gamma}(Y) \leq \mathrm{CoVaR}^{<}_{\alpha,\gamma,C}(Y|X) \leq \mathrm{VaR}_{\alpha\gamma+1-\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \mathcal{L}^0$.

(b) the inequality

$$\mathrm{VaR}_{\alpha(1-\gamma)}(Y) \leq \mathrm{CoVaR}^{\geq}_{\alpha,\gamma,C}(Y|X) \leq \mathrm{VaR}_{\alpha(1-\gamma)+\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \mathcal{L}^0$.

3. **Co-Expected-Shortfall:** For $\alpha \in (0,1)$, the Co-Expected-Shortfall satisfies

$$\mathrm{CoES}^*_{\alpha,\gamma,C}(Y|X) = \frac{1}{1-\alpha} \int_{[\alpha,1]} \mathrm{VaR}_{(\delta^*_{\gamma,C})^{\leftarrow}(u)}(Y) \, \mathrm{d}\lambda(u)$$

for every $Y \in \mathcal{L}^*_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$. Moreover,

(a) the inequality

$$\mathrm{RVaR}_{\alpha\gamma,\gamma}(Y) \leq \mathrm{CoES}^{<}_{\alpha,\gamma,C}(Y|X) \leq \mathrm{ES}_{\alpha\gamma+1-\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{<}_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$, and the inclusion $\mathcal{L}^{1,+} \subseteq \mathcal{L}^{<}_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$ holds for every $C \in \mathcal{C}$.

(b) the inequality

$$\mathrm{RVaR}_{\alpha(1-\gamma),1-\gamma}(Y) \leq \mathrm{CoES}^{\geq}_{\alpha,\gamma,C}(Y|X) \leq \mathrm{ES}_{\alpha(1-\gamma)+\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \bigcap_{C\in\mathcal{C}} \mathcal{L}^{\geq}_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$, and the inclusion $\mathcal{L}^{1,+} \subseteq \mathcal{L}^{\geq}_{\gamma,C,Q_{\mathrm{ES}_\alpha}}$ holds for every $C \in \mathcal{C}$.

4. **Co-Range-Value-at-Risk:** For $\alpha, \beta \in (0,1)$ with $\alpha < \beta$, the Co-Range-Value-at-Risk satisfies

$$\mathrm{CoRVaR}^{*}_{\alpha,\beta,\gamma,C}(Y|X) \;=\; \frac{1}{\beta-\alpha}\int_{[\alpha,\beta]} \mathrm{VaR}_{(\delta^{*}_{\gamma,C})^{\leftarrow}(u)}(Y)\,\mathrm{d}\lambda(u)$$

for every $Y \in \mathcal{L}^{*}_{\gamma,C,Q_{\mathrm{RVaR}_{\alpha,\beta}}} = \mathcal{L}^{0}$. Moreover,

(a) the inequality

$$\mathrm{RVaR}_{\alpha\gamma,\beta\gamma}(Y) \leq \mathrm{CoRVaR}^{<}_{\alpha,\beta,\gamma,C}(Y|X) \leq \mathrm{RVaR}_{\alpha\gamma+1-\gamma,\beta\gamma+1-\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \mathcal{L}^{0}$.

(b) the inequality

$$\mathrm{RVaR}_{\alpha(1-\gamma),\beta(1-\gamma)}(Y) \leq \mathrm{CoRVaR}^{\geq}_{\alpha,\beta,\gamma,C}(Y|X) \leq \mathrm{RVaR}_{\alpha(1-\gamma)+\gamma,\beta(1-\gamma)+\gamma}(Y)$$

holds for every $C \in \mathcal{C}$ and every $Y \in \mathcal{L}^{0}$.

Remark 4.15. Although every quantile co-risk measure is a quantile risk measure (Theorem 4.8), it needs not to be a quantile risk measure of the same type. For instance, even though every Co-Value-at-Risk at level α is a Value-at-Risk at level $(\delta^{*}_{\gamma,C})^{\leftarrow}(\alpha)$, a Co-Expected-Shortfall may fail to be of Expected-Shortfall type.

Remark 4.16. A summary of $\mathrm{CoVaR}^{=}_{\alpha,\gamma,C}$ and $\mathrm{CoVaR}^{<}_{\alpha,\gamma,C}$ expressions for various families of copulas including among others Fréchet copulas, EFGM copulas, Marshall-Olkin copulas, Archimedean copulas and Extreme Value copulas may be found in [4, 22].

5 Continuity results

In this section we study continuity properties of quantile based co-risk measures taking into account the underlying dependence structure. It turns out that, depending on the type of co-risk measure, different notions of copula convergence need to be considered including pointwise/uniform convergence (for co-risk measures based on $\mathrm{CoVaR}^{<}$ and $\mathrm{CoVaR}^{\geq}$) and weak convergence of the corresponding Markov kernels known as weak conditional convergence (for co-risk measures based on $\mathrm{CoVaR}^{=}$).

We start with two results that are key to this section:

Theorem 5.1. *Consider a sequence of copulas $(C_n)_{n\in\mathbb{N}}$ converging pointwise to C and a sequence of continuous univariate distribution functions $(F_{Y,n})_{n\in\mathbb{N}}$ converging weakly to F_Y. Then, for every $\gamma \in (0,1)$, the identities*

$$\begin{aligned}\lim_{n\to\infty}(\delta^{<}_{\gamma,C_n} \circ F_{Y,n})(y) &= (\delta^{<}_{\gamma,C} \circ F_Y)(y)\\ \lim_{n\to\infty}(\delta^{\geq}_{\gamma,C_n} \circ F_{Y,n})(y) &= (\delta^{\geq}_{\gamma,C} \circ F_Y)(y)\end{aligned}$$

hold for every $y \in \mathbb{R}$.

Proof. As F_Y is continuous, weak convergence of $(F_{Y,n})_{n\in\mathbb{N}}$ to F_Y coincides with pointwise convergence. Moreover, as pointwise convergence $(C_n)_{n\in\mathbb{N}}$ to C equals uniform and hence continuous convergence, we have that the sequence $(\delta^{<}_{\gamma,C_n} \circ F_{Y,n})_{n\in\mathbb{N}}$ converges pointwise to $\delta^{<}_{\gamma,C} \circ F_Y$. The same reasoning yields the second identity. □

Viewing bivariate copulas in terms of their conditional distributions and considering weak convergence leads to the notion of weak conditional convergence introduced in [25]: Suppose that $C, C_1, C_2, \ldots$ are copulas and let $K_C, K_{C_1}, K_{C_2}, \ldots$ be (versions of) the corresponding Markov kernels. We will say that $(C_n)_{n\in\mathbb{N}}$ converges *weakly conditional* to C if and only if for λ-almost every $u \in \mathbb{I}$ we have that the sequence $(K_{C_n}(u,\cdot))_{n\in\mathbb{N}}$ of probability measures on $\mathcal{B}(\mathbb{I})$ converges weakly to the probability measure $K_C(u,\cdot)$. In the latter case we will write $C_n \xrightarrow{\text{wcc}} C$ (where 'wcc' stands for 'weak conditional convergence'). Following the construction in [22] it is straightforward to verify that weak conditional convergence coincides with almost sure convergence of the partial derivatives on a dense set.
As is well known, many standard parametric classes $\{C_\theta : \theta \in \Theta\}$ of copulas depend on the parameter θ weakly conditional in the sense that if $\theta_n \to \theta$ then $C_n \xrightarrow{\text{wcc}} C$: (among many others)

- the family of EFGM copulas: see [11, Section 6.3].
- the family of Gaussian copulas: see [11, Section 6.7].
- the family of t-copulas: see [11, Section 6.7].
- the family of Marshall-Olkin copulas: see [11, Section 6.4].

fulfill this property (see [25]). In [25], the authors have also proved that within the class of Archimedean copulas and the class of Extreme Value copulas standard pointwise convergence and weak conditional convergence are equivalent. The equivalence between the two notions of convergence reveals its potential when studying risk measures based on CoVaR$^=$ (as we will see in Section 6).

Theorem 5.2. *Consider a sequence of copulas $(C_n)_{n\in\mathbb{N}}$ converging weakly conditional to C and a sequence of continuous univariate distribution functions $(F_{Y,n})_{n\in\mathbb{N}}$ converging weakly to F_Y. Then there exists a set $\Lambda \in \mathcal{B}((0,1))$ with $\lambda(\Lambda) = 1$ such that for every $\gamma \in \Lambda$*

$$\lim_{n\to\infty}(\delta^{=}_{\gamma,C_n} \circ F_{Y,n})(y) = (\delta^{=}_{\gamma,C} \circ F_Y)(y) \tag{5.1}$$

for every $y \in \mathbb{R}$ with $K_C(\gamma, \{F_Y(y)\}) = 0$.
Additionally, if either (a) F_Y is strictly increasing or (b) the map $v \mapsto K_C(\gamma,[0,v])$ is continuous, then Equation (5.1) holds for all but at most countable infinitely many $y \in \mathbb{R}$ (interpreting the expressions as conditional distribution functions on $\mathbb{R}$ we hence have weak convergence).

Proof. Letting $\Lambda \in \mathcal{B}((0,1))$ denote the set of all $x \in (0,1)$ such that $K_{C_n}(x,\cdot)$ converges weakly to $K_C(x,\cdot)$ we have $\lambda(\Lambda) = 1$ by assumption. Suppose now that $\gamma \in \Lambda$ and that $K_C(\gamma, \{F_Y(y)\}) = 0$ holds. Considering

$$\begin{aligned} &\left|K_{C_n}(\gamma,[0,F_{Y,n}(y)]) - K_C(\gamma,[0,F_Y(y)])\right| \\ &\quad \leq \underbrace{\left|K_{C_n}(\gamma,[0,F_{Y,n}(y)]) - K_{C_n}(\gamma,[0,F_Y(y)])\right|}_{=:I_n} + \underbrace{\left|K_{C_n}(\gamma,[0,F_Y(y)]) - K_C(\gamma,[0,F_Y(y)])\right|}_{=:J_n} \end{aligned}$$

we obviously have $\lim_{n\to\infty} J_n = 0$ since $F_Y(y)$ is a continuity point of $v \mapsto K_C(\gamma,[0,v])$. For I_n we can proceed as follows: For every $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$K_C(\gamma, [F_Y(y) - \delta, F_Y(y) + \delta]) < \varepsilon$$

Since $(F_{Y,n})_{n\in\mathbb{N}}$ converges weakly (in this case pointwise) to F_Y there exists some $n_0 \in \mathbb{N}$ such that $F_{Y,n}(y) \in [F_Y(y) - \delta,\ F_Y(y) + \delta]$ holds for every $n \geq n_0$. Having this, using Portmanteau theorem it follows that

$$\limsup_{n\to\infty} I_n \leq \limsup_{n\to\infty} K_{C_n}(\gamma, [F_Y(y) - \delta,\ F_Y(y) + \delta]) \leq K_C(\gamma, [F_Y(y) - \delta,\ F_Y(y) + \delta]) < \varepsilon$$

from which Equation (5.1) follows since $\varepsilon > 0$ was arbitrary. Assertions (1) and (2) on weak convergence are straightforward consequences of Equation (5.1) and the fact that if either F_Y is strictly increasing or the map $v \mapsto K_C(\gamma,[0,v])$ is continuous, then $K_C(\gamma,\{F_Y(y)\}) = 0$ holds for λ-almost all $y \in \mathbb{R}$. □

Remark 5.3. Notice that the set Λ in Theorem 5.2 depends on the versions of Markov kernels constituting the sequence. The same applies to all following results that are related to convergence of $(\delta^{=}_{\gamma,C_n})_{n\in\mathbb{N}}$.

As a direct consequence of Theorems 5.1, 5.2 and [37, Lemma 21.2], we obtain continuity of Co-Value-at-Risk under some very mild regularity conditions on the limiting distribution function F_Y or the limiting copula C:

Corollary 5.4. *Consider a sequence of copulas $(C_n)_{n\in\mathbb{N}}$, a copula C and a sequence of continuous univariate distribution functions $(F_{Y,n})_{n\in\mathbb{N}}$ converging weakly to F_Y.*

(1) If $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C, then for every $\gamma \in (0,1)$

$$\lim_{n\to\infty} \mathrm{CoVaR}^{<}_{\alpha,\gamma,C_n}(Y_n|X) = \mathrm{CoVaR}^{<}_{\alpha,\gamma,C}(Y|X)$$

for every continuity point $\alpha \in (0,1)$ of $(\delta^{<}_{\gamma,C} \circ F_Y)^{\leftarrow} = F_Y^{\leftarrow} \circ (\delta^{<}_{\gamma,C})^{\leftarrow}$.
Additionally, if F_Y and the map $v \mapsto C(\gamma, v)$, $\gamma \in (0,1)$, are strictly increasing then the above identity holds for every $\alpha \in (0,1)$.

(2) If $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C, then for every $\gamma \in (0,1)$

$$\lim_{n\to\infty} \mathrm{CoVaR}^{\geq}_{\alpha,\gamma,C_n}(Y_n|X) = \mathrm{CoVaR}^{\geq}_{\alpha,\gamma,C}(Y|X)$$

for every continuity point $\alpha \in (0,1)$ of $(\delta^{\geq}_{\gamma,C} \circ F_Y)^{\leftarrow} = F_Y^{\leftarrow} \circ (\delta^{\geq}_{\gamma,C})^{\leftarrow}$.
Additionally, if F_Y and the map $v \mapsto v - C(\gamma, v)$, $\gamma \in (0,1)$, are strictly increasing then the identity holds for every $\alpha \in (0,1)$.

(3) *Assume that either (a) F_Y is strictly increasing or (b) the map $v \mapsto K_C(\gamma,[0,v])$ is continuous. If $(C_n)_{n\in\mathbb{N}}$ converges weakly conditional to C, then there exists a set $\Lambda \in \mathcal{B}((0,1))$ with $\lambda(\Lambda)=1$ such that for every $\gamma \in \Lambda$*

$$\lim_{n\to\infty} \mathrm{CoVaR}^{=}_{\alpha,\gamma,C_n}(Y_n|X) = \mathrm{CoVaR}^{=}_{\alpha,\gamma,C}(Y|X)$$

for every continuity point $\alpha \in (0,1)$ of $(\delta^{=}_{\gamma,C} \circ F_Y)^{\leftarrow} = F_Y^{\leftarrow} \circ (\delta^{=}_{\gamma,C})^{\leftarrow}$.
Additionally, if F_Y and the map $v \mapsto K_C(\gamma,[0,v])$, $\gamma \in \Lambda$, are strictly increasing then the identity holds for every $\alpha \in (0,1)$.

Another direct consequence of Theorems 5.1 and 5.2 can be obtained for co-risk measures $\varrho^{*}_{\gamma,C,Q}$, $* \in \{<,\geq,=\}$, for which Q is absolutely continuous; the next result focusses on Co-Range-Value-at-Risk and hence Marginal Expected-Shortfall and Co-Expected-Shortfall:

Corollary 5.5. *Consider a sequence of copulas $(C_n)_{n\in\mathbb{N}}$, a copula C and a sequence of continuous univariate distribution functions $(F_{Y,n})_{n\in\mathbb{N}}$ converging weakly to F_Y. Further assume that Q is absolutely continuous with respect to λ.*

(1) *If $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C and $\int_{\mathbb{I}} \sup_{n\in\mathbb{N}} \left|\mathrm{VaR}_{(\delta^{<}_{\gamma,C_n})^{\leftarrow}(u)}(Y_n)\right| \,\mathrm{d}Q(u) < \infty$, then*

$$\lim_{n\to\infty} \rho^{<}_{\gamma,C_n,Q}(Y_n) = \rho^{<}_{\gamma,C,Q}(Y)$$

for every $\gamma \in (0,1)$.

(2) *If $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C and $\int_{\mathbb{I}} \sup_{n\in\mathbb{N}} \left|\mathrm{VaR}_{(\delta^{\geq}_{\gamma,C_n})^{\leftarrow}(u)}(Y_n)\right| \,\mathrm{d}Q(u) < \infty$, then*

$$\lim_{n\to\infty} \rho^{\geq}_{\gamma,C_n,Q}(Y_n) = \rho^{\geq}_{\gamma,C,Q}(Y)$$

for every $\gamma \in (0,1)$.

(3) *Assume that either (a) F_Y is strictly increasing or (b) the map $v \mapsto K_C(\gamma,[0,v])$ is continuous. If $(C_n)_{n\in\mathbb{N}}$ converges weakly conditional to C and $\int_{\mathbb{I}} \sup_{n\in\mathbb{N}} \left|\mathrm{VaR}_{(\delta^{=}_{\gamma,C_n})^{\leftarrow}(u)}(Y)\right| \,\mathrm{d}Q(u) < \infty$, then there exists a set $\Lambda \in \mathcal{B}((0,1))$ with $\lambda(\Lambda)=1$ such that*

$$\lim_{n\to\infty} \rho^{=}_{\gamma,C_n,Q}(Y_n) = \rho^{=}_{\gamma,C,Q}(Y)$$

for every $\gamma \in \Lambda$.

Proof. We first prove (1). By Theorems 5.1 and 5.2 we have

$$\lim_{n\to\infty} \delta^{*}_{\gamma,C_n} \circ F_{Y,n} = \delta^{*}_{\gamma,C} \circ F_Y$$

pointwise and hence weakly for $* \in \{<,\geq\}$, and weakly for $* \in \{=\}$. For $* \in \{<,\geq,=\}$, it then follows from [37, Lemma 21.2] that

$$\lim_{n\to\infty} \left(\delta^{*}_{\gamma,C_n} \circ F_{Y,n}\right)^{\leftarrow} = \left(\delta^{*}_{\gamma,C} \circ F_Y\right)^{\leftarrow}$$

weakly. Because $(\delta^{*}_{\gamma,C} \circ F_Y)^{\leftarrow}$ has at most countably many discontinuity points and Q is absolutely continuous with respect to λ, the previous identity holds Q-a.s. The assertion hence follows from dominated convergence theorem. □

The convergence results stated in Corollaries 5.4 and 5.5 for co-risk measures based on $\mathrm{CoVaR}^{=}$ are valid whenever the limiting distribution function F_Y is strictly increasing, a condition which might seem acceptable in most applications. Also the conditions to the copula seem not to be too restrictive:

Remark 5.6. Notice that, if the copula C is absolutely continuous such that its Lebesgue density c satisfies $c(u, v) > 0$ for all $(u, v) \in (0, 1)^2$, then

1. the map $v \mapsto C(\gamma, v)$ is strictly increasing for every $\gamma \in (0, 1)$;
2. the map $v \mapsto v - C(\gamma, v)$ is strictly increasing for every $\gamma \in (0, 1)$;
3. the map $v \mapsto K_C(\gamma, [0, v])$ is absolutely continuous and strictly increasing for every $\gamma \in (0, 1)$.

In this case, all the necessary and additional conditions to the limiting copula used in Corollaries 5.4 and 5.5 are fulfilled.

6 Consequences for the estimation

In this section we focus on consequences of the results presented in Section 5 to the estimation of co-risk measures. Since consistent estimators for co-risk measures based on $\mathrm{CoVaR}^{<}$ and $\mathrm{CoVaR}^{\geq}$ can be obtained by simply plugging in the empirical copula and the empirical distribution function (compare Corollaries 5.4 and 5.5), we restrict our consideration to the more challenging situation when co-risk measures based on $\mathrm{CoVaR}^{=}$ are to be estimated.

Suppose that $(X_1, Y_1), (X_2, Y_2), \ldots$ is a random sample from $(X, Y) \sim H$; recall that H is continuous and that there exists some unique copula C satisfying $H(x, y) = C(F_X(x), F_Y(y))$ for all $(x, y) \in \mathbb{R}^2$. In the following, we denote by $\widehat{F}_{Y,n}$ the empirical distribution function given $Y_1, Y_2, \ldots, Y_n$, and by $\widehat{E}_n$ the empirical copula (by which we mean the unique copula determined by bilinear interpolation of the empirical subcopula, i.e., a checkerboard copula with a quite specific structure) given $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. Then $\widehat{E}_n$ fulfills

$$d_\infty(\widehat{E}_n, C) = O\left(\sqrt{\frac{\ln(\ln(n))}{n}}\right) \tag{6.1}$$

for $n \to \infty$ with probability one; see, e.g., [23].

Plugging in the empirical distribution function and the empirical copula into the co-quantile risk measures ϱ^*, $* \in \{<, \geq\}$, by Corollaries 5.4 and 5.5, we obtain consistency of these estimators. In particular, we obtain consistent estimators for CoVaR^*, MES^*, ES^* and RVaR^* in the case $* \in \{<, \geq\}$.

For the reasons discussed in [23] it is not possible to estimate risk measures based on $\mathrm{CoVaR}^{=}$ by simply plugging in the empirical copula. One possibility to overcome this problem is to use (general) checkerboard aggregations instead of empirical copulas. Following [23], we let $\mathcal{CB}_N(C)$ be the N-checkerboard approximation of $C \in \mathcal{C}$ and we will write $N(n) := \lfloor n^s \rfloor$ for some fixed $s \in (0, \frac{1}{2})$. The next result shows that it is possible to estimate $\delta^{=}_{\gamma,C} \circ F_Y$ in almost full generality.

Theorem 6.1. *Suppose that* $(X_1, Y_1), (X_2, Y_2), \ldots$ *is a random sample from* $(X, Y) \sim H$ *and assume that either (a)* F_Y *is strictly increasing or (b) the map* $v \mapsto K_C(\gamma, [0, v])$ *is continuous. Then there exists a set* $\Lambda \in \mathcal{B}((0, 1))$ *with* $\lambda(\Lambda) = 1$ *such that for every* $\gamma \in \Lambda$

$$\lim_{n\to\infty} \big(\delta^{=}_{\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)} \circ \widehat{F}_{Y,n}\big)(y) = \big(\delta^{=}_{\gamma,C} \circ F_Y\big)(y) \tag{6.2}$$

weakly.

Proof. According to Theorem 5.2 it suffices to prove that $(\mathcal{CB}_{N(n)}(\widehat{E}_n))_{n\in\mathbb{N}}$ converges weakly conditional to C with probability one. Since the empirical copula $\widehat{E}_n$ fulfills (6.1) we may assume from now on that $d_\infty(\widehat{E}_n, C) = O(\sqrt{\ln(\ln(n))/n})$ holds. Consider

$$\begin{aligned} &\big|K_{\mathcal{CB}_{N(n)}(\widehat{E}_n)}(\gamma, [0, y]) - K_C(\gamma, [0, y])\big| \\ &\quad\leq \underbrace{\big|K_{\mathcal{CB}_{N(n)}(\widehat{E}_n)}(\gamma, [0, y]) - K_{\mathcal{CB}_{N(n)}(C)}(\gamma, [0, y])\big|}_{=:I_n} + \underbrace{\big|K_{\mathcal{CB}_{N(n)}(C)}(\gamma, [0, y]) - K_C(\gamma, [0, y])\big|}_{=:J_n}. \end{aligned}$$

Then according to [28] there exists a set $\Gamma \in \mathcal{B}([0, 1]^2)$ with $\lambda_2(\Gamma) = 1$ such that $\lim_{n\to\infty} J_n = 0$ holds for all $(x, y) \in \Gamma$. For I_n we can proceed as follows: Let Λ denote the set of all $\gamma \in (0, 1)$ such that $\lambda(\Gamma_\gamma) = 1$ and suppose that $\gamma \in \Lambda$ and $(\gamma, y) \in \Gamma$ holds. Further, define the squares R_{ij} for $i, j \in \{1, \ldots, N(n)\}$ by

$$R_{ij} := \left[\frac{i-1}{N(n)}, \frac{i}{N(n)}\right] \times \left[\frac{j-1}{N(n)}, \frac{j}{N(n)}\right]$$

Considering (see [23]) that for every $\gamma \in \big[\frac{i-1}{N(n)}, \frac{i}{N(n)}\big)$ and every copula $A \in \mathcal{C}$ we have

$$K_{\mathcal{CB}_{N(n)}(A)}\big(\gamma, [0, \tfrac{j_0}{N(n)}]\big) = N(n) \sum_{j=1}^{j_0} \mu_A(R_{i,j})$$

for every $j_0 \in \{1, 2, \ldots, N(n)\}$ it follows immediately that $I_n \leq N(n) \cdot 2 \cdot d_\infty(\widehat{E}_n, C)$ and therefore $\limsup_{n\to\infty} I_n = 0$. This completes the proof since $\lambda(\Gamma_\gamma) = 1$ and $\gamma \in \Gamma$ was arbitrary. □

Combining Theorem 6.1 and Corollaries 5.4 and 5.5 leads to consistent estimators for the co-risk measures $\text{CoVaR}^=$, $\text{MES}^=$, $\text{ES}^=$ and $\text{RVaR}^=$.

Corollary 6.2. *Suppose that* $(X_1, Y_1), (X_2, Y_2), \ldots$ *is a random sample from* $(X, Y) \sim H$ *and assume that either (a)* F_Y *is strictly increasing or (b) the map* $v \mapsto K_C(\gamma, [0, v])$ *is continuous.*

1. *Then there exists a set* $\Lambda \in \mathcal{B}((0, 1))$ *with* $\lambda(\Lambda) = 1$ *such that*

   $$\lim_{n\to\infty} \text{CoVaR}^{=}_{\alpha,\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)}(Y_n|X) = \text{CoVaR}^{=}_{\alpha,\gamma,C}(Y|X)$$

   for every $\gamma \in \Lambda$ *and every continuity point* $\alpha \in (0, 1)$ *of* $(\delta^{=}_{\gamma,C} \circ F_Y)^{\leftarrow} = F_Y^{\leftarrow} \circ (\delta^{=}_{\gamma,C})^{\leftarrow}$.
   In addition, if F_Y *and the map* $v \mapsto K_C(\gamma, [0, v])$, $\gamma \in \Lambda$, *are strictly increasing then the identity holds for every* $\alpha \in (0, 1)$.

2. *Further assume that* Q *is absolutely continuous with respect to* λ. *Then there exists a set* $\Lambda \in \mathcal{B}((0, 1))$ *with* $\lambda(\Lambda) = 1$ *such that*

   $$\lim_{n\to\infty} \rho^{=}_{\gamma,C_n,Q}(Y_n) = \rho^{=}_{\gamma,C,Q}(Y)$$

   for every $\gamma \in \Lambda$.

Remark 6.3. At this point note that Corollary 5.5 and Corollary 6.2 can be extended in a straightforward way to functionals of the form

$$Y \mapsto \int_{\mathrm{I}} F^{\leftarrow}_{Y|X * F^{\leftarrow}_X(\gamma)}(u) \, \mathrm{d}\mu(u) \qquad\qquad * \in \{<, \geq, =\}$$

where μ is a signed measure satisfying $\mu = a_1 \, Q_1 - a_2 \, Q_2$ with $a_1, a_2 \in [0, \infty)$ and probability measures Q_1 and Q_2 corresponding to distortion functions as defined in Section 3. Such functionals belong to the class of distortion risk metrics discussed in [38].

We now illustrate the performance of our estimators $\mathrm{CoVaR}^{=}_{\alpha,\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)}(Y_n|X)$ for Co-Value-at-Risk and $\mathrm{CoES}^{=}_{\alpha,\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)}(Y_n|X)$ for Co-Expected Shortfall discussed in Corollary 6.2 in a Marshall-Olkin copula setting; recall that the Marshall-Olkin family of copulas $(M_{\theta_1,\theta_2})_{\theta_1,\theta_2 \in \mathrm{I}}$ is defined by

$$M_{\theta_1,\theta_2}(u, v) := \begin{cases} u^{1-\theta_1} \, v & u^{\theta_1} \geq v^{\theta_2} \\ u \, v^{1-\theta_2} & u^{\theta_1} < v^{\theta_2} \end{cases}$$

(see [11, Section 6.4]).

Example 6.4. Consider a random variable Y following a Beta-distribution Beta$(2, 5)$ and a Marshall-Olkin copula $M_{0.2,0.7}$. Figure 1 depicts a boxplot summarizing the 1000 obtained estimates for $\mathrm{CoVaR}^{=}_{\alpha,\gamma,M_{0.2,0.7}}(Y|X)$ at level $\gamma = 0.05$. Instead of comparing Co-Value-at-Risk values for specific choices of α, we use the L_1-distance between the functions $\alpha \mapsto \mathrm{CoVaR}^{=}_{\alpha,\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)}(Y_n|X)$ and $\alpha \mapsto \mathrm{CoVaR}^{=}_{\alpha,\gamma,M_{0.2,0.7}}(Y|X)$ to illustrate the estimators' performance; at this point note that we choose a Beta$(2, 5)$-distributed random variable to avoid problems concerning the existence of the L_1-distance. Figure 2 depicts a boxplot summarizing the 1000 obtained estimates for

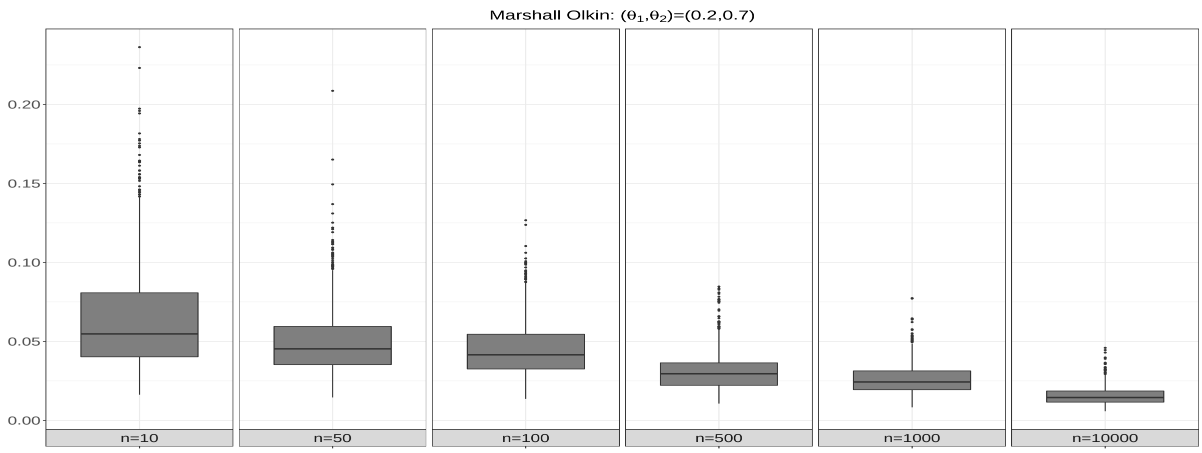

Figure 1: Boxplot summarizing the L_1-distance of the 1000 obtained estimates for $\mathrm{CoVaR}^{=}_{\alpha,\gamma,M_{0.2,0.7}}(Y|X)$ at level $\gamma = 0.05$.

$\mathrm{CoES}^{=}_{\alpha,\gamma,M_{0.2,0.7}}(Y|X)$ at levels $\alpha = 0.05$ and $\gamma = 0.05$.

Even though the co-risk measures $\mathrm{CoVaR}^{=}$, $\mathrm{MES}^{=}$, $\mathrm{ES}^{=}$ and $\mathrm{RVaR}^{=}$ can be consistently estimated by means of checkerboard aggregations of the empirical copula as shown above, in what follows we discuss a more sophisticated approach also incorporating information about the dependence structure between X and Y. We then compare the performance of the estimators incorporating or ignoring information about the dependence structure.

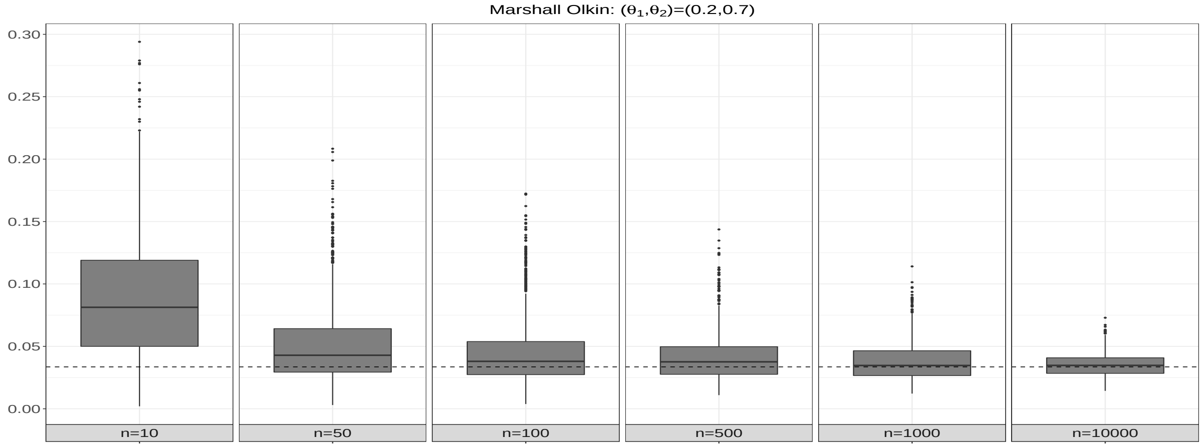

Figure 2: Boxplot summarizing the 1000 obtained estimates for $\mathrm{CoES}^{=}_{\alpha,\gamma,M_{0.2,0.7}}(Y|X)$ at levels $\alpha = 0.05$ and $\gamma = 0.05$.

Extreme Value copulas A copula $C \in \mathcal{C}$ is called an *Extreme Value copula* if there exists a convex function $A : \mathbb{I} \to \mathbb{I}$ satisfying $A(0) = A(1) = 1$ and $\max(1-t, t) \leq A(t) \leq 1$ for all $t \in \mathbb{I}$ such that for all $u, v \in (0,1)$ the copula C can be expressed in terms of A as

$$C(u,v) = C_A(u,v) := (uv)^{A\left(\frac{\ln(u)}{\ln(uv)}\right)}$$

(see [11, 33]). The function A is called *Pickands dependence function*.

Now, suppose that C_A is an Extreme Value copula with Pickands dependence function A and suppose that $(X_1,Y_1),(X_2,Y_2),\dots$ is a random sample from $(X,Y) \sim H$ with underlying copula C_A. Following [25] and letting $\hat{A}_n$ denote the CFG estimator according to [6, 18] it can be shown that, for suitable weight functions, $(\hat{A}_n)_{n\in\mathbb{N}}$ is uniformly, strongly consistent (see [6, Proposition 4.1]). Although the estimator $\hat{A}_n$ may fail to be convex in general, following an idea from [18] it can be used to construct a convex estimator $\hat{A}_n^*$ given by

$$\hat{A}_n^* := \text{greatest convex minorant of } \max\{\min\{\hat{A}_n, 1\}, \mathrm{id}, 1 - \mathrm{id}\}$$

where id denotes the identity function on $\mathbb{I}$. $\hat{A}_n^*$ is a Pickands dependence function (see [18, Section 3.3]) and the estimator $\hat{A}_n^*$ is uniformly, strongly consistent (which follows from [30]). Hence [25, Theorem 5.1] directly yields weak conditional convergence of the sequence of corresponding Extreme Value copulas $(C_{\hat{A}_n^*})_{n\in\mathbb{N}}$ to C_A. Therefore, as a consequence of Theorem 5.2, for estimating co-risk measures based on $\mathrm{CoVaR}^{=}$ in an Extreme Value setting, it is possible to replace the estimator based on checkerboard aggregations by an estimator based on $\hat{A}_n^*$.

The benefit of including information about the dependence structure is illustrated in a Galambos copula setting; the Galambos family of copula $(C_{A_\theta})_{\theta\in(0,\infty)}$ is defined by means of its Pickands function

$$A_\theta(t) := 1 - \left(t^{-\theta} + (1-t)^{-\theta}\right)^{-1/\theta}$$

Example 6.5. Consider a random variable Y following a Beta-distribution Beta$(2,5)$ and a Galambos copula C_{A_3} with parameter $\theta = 3$. Figure 3 depicts a boxplot summarizing the 1000 obtained estimates (left panel: estimator based on $\hat{A}_n^*$, right panel: estimator based on checkerboard aggregations) for $\mathrm{CoVaR}^{=}_{\alpha,\gamma,C_{A_3}}(Y|X)$ at level $\gamma = 0.05$; instead of comparing Co-Value-at-Risk values for specific choices of α, we use the L_1-distance between the functions

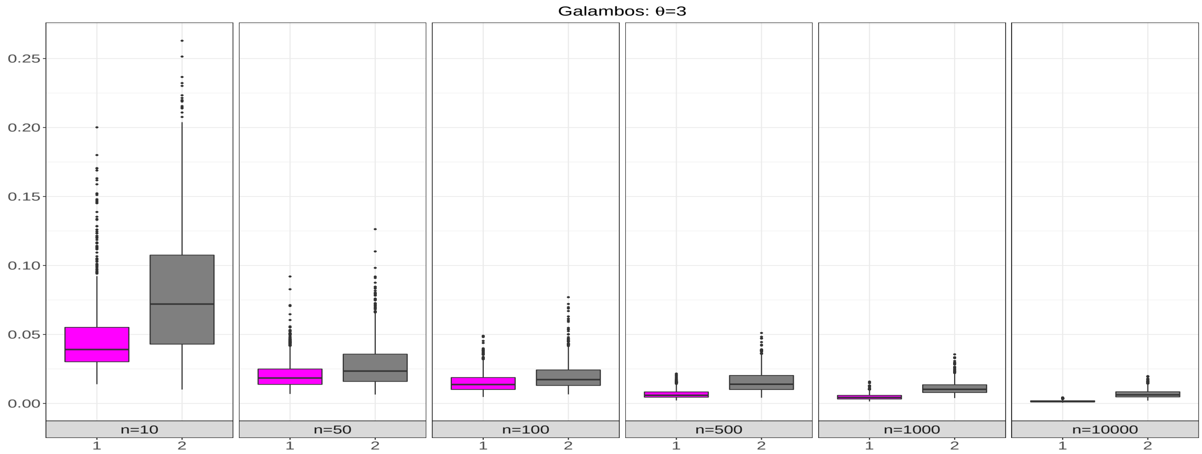

Figure 3: Boxplot summarizing the L_1-distance of the 1000 obtained estimates (pink: estimator based on $\hat{A}_n^*$, gray: estimator based on checkerboard approximations) for $\mathrm{CoVaR}^{=}_{\alpha,\gamma,C_{A_3}}(Y|X)$ at level $\gamma = 0.05$.

$\alpha \mapsto \mathrm{CoVaR}^{=}_{\alpha,\gamma,\mathcal{CB}_{N(n)}(\widehat{E}_n)}(Y_n|X)$ and $\alpha \mapsto \mathrm{CoVaR}^{=}_{\alpha,\gamma,C_{A_3}}(Y|X)$ to illustrate the estimators' performance. Figure 4 depicts a boxplot summarizing the 1000 obtained estimates (left panel: estimator based on $\hat{A}_n^*$, right panel: estimator based on checkerboard approximations) for $\mathrm{CoES}^{=}_{\alpha,\gamma,C_{A_3}}(Y|X)$ at levels $\alpha = 0.05$ and $\gamma = 0.05$.

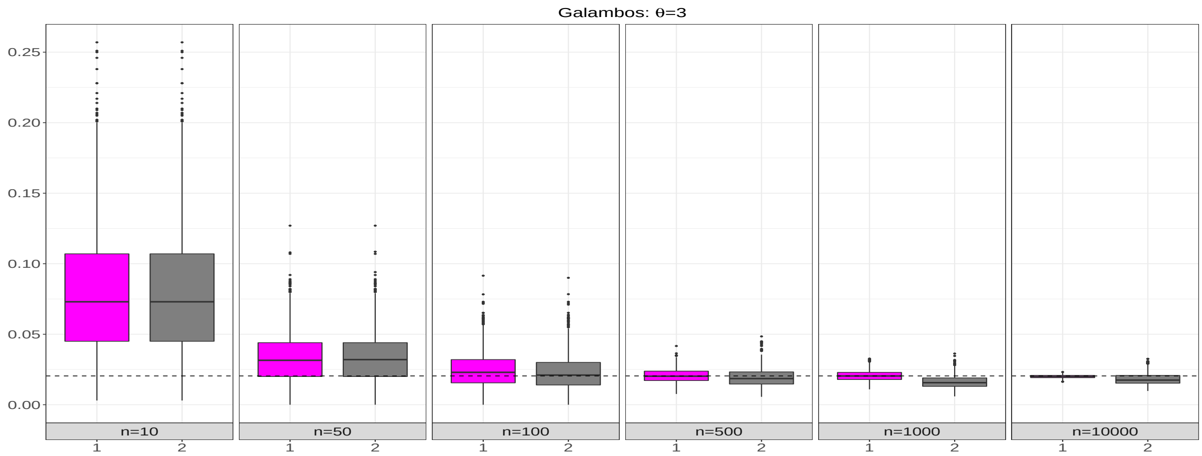

Figure 4: Boxplot summarizing the 1000 obtained estimates (pink: estimator based on $\hat{A}_n^*$, gray: estimator based on checkerboard approximations) for $\mathrm{CoES}^{=}_{\alpha,\gamma,C_{A_3}}(Y|X)$ at levels $\alpha = 0.05$ and $\gamma = 0.05$.

Not surprisingly, Figures 3 and 4 show that the plug-in estimator based on $\hat{A}_n^*$ (using the Extreme Value information) outperforms the empirical checkerboard estimator (ignoring the Extreme Value information).

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References

[1] Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *J. Bank. Financ. 26*(7), 1505–1518.

[2] Acharya, V. V., L. H. Pedersen, T. Philippon, and M. P. Richardson (2017). Measuring systemic risk. *Rev. Financ. Stud. 30*(1), 2–47.

[3] Adrian, T. and M. K. Brunnermeier (2016). CoVaR. *Am. Econ. Rev. 106*(7), 1705–1741.

[4] Bernardi, M., F. Durante, and P. Jaworski (2017). CoVaR of families of copulas. *Statist. Probab. Lett. 120*, 8–17.

[5] Bianchi, M. L. and A. M. Sorrentino (2020). Measuring CoVaR: An empirical comparison. *Comput. Econ. 55*, 511–528.

[6] Capéraà, P., A.-L. Fougères, and C. Genest (1997). A nonparametric estimation procedure for bivariate extreme value copulas. *Biometrika 84*(3), 567–577.

[7] Cherubini, U. and S. Mulinacci (2014). Contagion–based distortion risk measures. *Appl. Math. Lett. 27*, 85–89.

[8] Choi, J.-E. and D. W. Shin (2019). Three regime bivariate normal distribution: a new estimation method for co-value-at-risk, CoVaR. *Europ. J. Financ. 25*(18), 1817–1833.

[9] Cont, R., R. Deguest, and G. Scandolo (2010). Robustness and sensitivity analysis of risk measurement procedures. *Quant. Finance 10*(6), 593–606.

[10] Durante, F. and S. Fuchs (2019). Reflection invariant copulas. *Fuzzy Sets and Systems 354*, 63–73.

[11] Durante, F. and C. Sempi (2016). *Principles of Copula Theory.* CRC Press, Boca Raton FL.

[12] Embrechts, P., H. Liu, and R. Wang (2018). Quantile-based risk sharing. *Oper. Res. 66*(4), 936–949.

[13] Fernández Sánchez, J. and W. Trutschnig (2015). Conditioning-based metrics on the space of multivariate copulas and their interrelation with uniform and levelwise convergence and Iterated Function Systems. *J. Theoret. Probab. 28*, 1311–1336.

[14] Föllmer, H. and A. Schied (2011). *Stochastic Finance.* Third edition. De Gruyter, Berlin.

[15] Fuchs, S. (2014). Multivariate copulas: Transformations, symmetry, order and measures of concordance. *Kybernetika 50*(5), 725–743.

[16] Fuchs, S., R. Schlotter, and K. D. Schmidt (2017). A review and some complements on quantile risk measures and their domain. *Risks 5*(4), no. 59, 16 pp.

[17] Fuchs, S. and K. D. Schmidt (2014). Bivariate copulas: Transformations, asymmetry and measures of concordance. *Kybernetika 50*(1), 109–125.

[18] Genest, C. and J. Segers (2009). Rank-based inference for bivariate extreme-value copulas. *Ann. Statist. 37*(5B), 2990–3022.

[19] Girardi, G. and A. Ergün (2013). Systemic risk measurement: Multivariate GARCH estimation of CoVaR. *J. Bank. Financ. 37*(8), 3169–3180.

[20] Gzyl, H. and S. Mayoral (2008). On a relationship between distorted and spectral risk measures. *Rev. Econom. Financ. 15*, 8–21.

[21] Hakwa, B., M. Jäger-Ambrożewicz, and B. Rüdiger (2015). Analysing systemic risk contribution using a closed formula for conditional value at risk through copula. *Commun. Stoch. Anal. 9*(1), 131–158.

[22] Jaworski, P. (2017). On conditional value at risk (CoVaR) for tail–dependent copulas. *Depend. Model. 5*, 1–19.

[23] Junker, R., F. Griessenberger, and W. Trutschnig (2021). Estimating scale-invariant directed dependence of bivariate distributions. *Comput. Statist. Data Anal. 153*, Article ID 107058, 22 pages.

[24] Kallenberg, O. (2002). *Foundations of Modern Probability.* Second edition. Springer, New York.

[25] Kasper, T., S. Fuchs, and W. Trutschnig (2020). On weak conditional convergence of bivariate Archimedean and extreme value copulas, and consequences to nonparametric estimation. pp. Available at `https://arxiv.org/abs/2006.07131`.

[26] Klenke, A. (2008). *Wahrscheinlichkeitstheorie.* Second edition. Springer, Heidelberg.

[27] Lancaster, H. O. (1963). Correlation and complete dependence of random variables. *Ann. Math. Statist. 34*(4), 1315–1321.

[28] Li, X., P. Mikusiński, and M. D. Taylor (1998). Strong approximation of copulas. *J. Math. Anal. Appl. 255*, 608–623.

[29] Mainik, G. and E. Schaanning (2014). On dependence consistency of CoVaR and some other systemic risk measures. *Stat. Risk Model. 31*(1), 49–77.

[30] Marshall, A. W. (1970). Discussion of Barlow and van Zwet's papers. In M. L. Puri (Ed.), *Nonparametric Techniques in Statistical Inference*, pp. 175–176. Cambridge University Press, London.

[31] Mroz, T., W. Trutschnig, and J. Fernández Sánchez. Distributions with fixed marginals maximizing the mass of the endograph of a function. pp. Available at `https://arxiv.org/abs/1602.05807.`

[32] Nelsen, R. B. (2006). *An Introduction to Copulas.* Second edition. Springer, New York.

[33] Pickands, J. (1981). Multivariate extreme value distributions. *Bull. Inst. Internat. Statist. 49*(2), 859–878.

[34] Sordo, M. A., A. J. Bello, and A. Suárez-Llorens (2018). Stochastic orders and co-risk measures under positive dependence. *Insurance Math. Econom. 78*, 105–113.

[35] Tao, T. (2011). *An Introdcution to Measure Theory.* American Mathematical Society, Providence RI.

[36] Tsukahara, H. (2009). One-parameter families of distortion risk measures. *Math. Finance 19*(4), 691–705.

[37] van der Vaart, A. W. (1998). *Asymptotic Statistics.* Cambridge University Press.

[38] Wang, Q., R. Wang, and Y. Wei (2020). Distortion riskmetrics on general spaces. *Astin Bull. 50*(3), 827–851.

[39] Wang, S. (1996). Premium calculation by transforming the layer premium density. *Astin Bull. 26*(1), 71–92.

[40] Wang, S. (2002). A universal framework for pricing financial and insurance risks. *Astin Bull. 32*(2), 213–234.

[41] Wang, S. and J. Dhaene (1998). Comonotonicity, correlation order and premium principles. *Insurance Math. Econom. 22*(3), 235–242.

[42] Wirch, J. L. and M. R. Hardy (2002). Distortion risk measures − Coherence and stochastic dominance. *Proceedings of the Sixth Conference on Insurance Mathematics and Economics, Lisbon.*

[43] Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica 55*(1), 95–115.

[44] Yin, C. and D. Zhu (2018). New class of distortion risk measures and their tail asymptotics with examples on VaR. *J. Financ. Risk Mgmt. 7*, 12–38.