# Markov product invariance in classes of bivariate copulas characterized by univariate functions

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**Abstract**

We extend and sharpen some results in the literature concerning the notion of Markov product idempotence in some well-known classes of copulas. Focusing on families of copulas which are characterized by univariate functions we show that in the class of extreme-value copulas, in the class of diagonal copulas and in some special class of copulas represented by measure-preserving transformations only the usual suspects (if contained in the class) are idempotent, namely the product copula $\Pi$ and minimum copula $M$. Additionally, we prove a conjecture going back to Albanese and Sempi in 2016 saying that the only idempotent Archimedean copula is the product copula $\Pi$.

*Keywords:* Copula, Markov product, star product, invariance

## 1. Introduction

The so-called star product of copulas was introduced by Darsow et al. in 1992 (see [2]) and has since then be studied in various papers. Given bivariate copulas $A, B$ the star product $A * B$ is defined by

$$(A * B)(x, y) = \int_{[0,1]} \partial_2 A(x, u) \partial_1 B(u, y) d\lambda(u), \tag{1}$$

where $\partial_i$ denotes the partial derivative with respect to the $i$-th coordinate. In 1996 Olsen et al. (see [16]) showed that the space $(\mathcal{C}, *)$ of all bivariate copulas with the star product as binary operation and the space $(\mathcal{M}, \circ)$ of all Markov operators with the composition as binary operation are isomorphic and that every copula $A$ can be expressed in the form $A = B^t * C$ where $B, C$ are so-called completely dependent (or, equivalently, left invertible) copulas and $B^t$ denotes the transpose of $B$.

The star product is closely related to various operations on the family of bivariate copulas $\mathcal{C}$, in particular to shuffles of copulas. In fact, if $h : [0, 1] \to [0, 1]$ preserves the Lebesgue measure $\lambda$ (i.e., the push forward $\lambda^h$ of $\lambda$ via $h$ coincides with $\lambda$) then according to [21, Lemma 2] the $h$-shuffle $\mathcal{S}_h(A)$ of the copula $A$ can be expressed as $\mathcal{S}_h(A) = C_h^t * A$. Moreover, the star product is smoothing in various ways - if, for instance, $A$ is absolutely continuous and $B$ is an arbitrary copula then $A * B$ and $B * A$ are absolutely continuous too (see [22]).

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Using the one-to-one correspondence between copulas and Markov kernels as studied in [19] it is straightforward to show (see [23]) that the star product of copulas is not a new operation but an operation that has been well known for decades. In fact, the Markov kernel of $A * B$ coincides with the standard composition $K_A \circ K_B$ of the the Markov kernels of $A$ and $B$ as used in the context of Markov processes in discrete time. In the sequel we will therefore refer to the star product also as Markov product of copulas.

A bivariate copula $A$ is called idempotent if, and only if $A * A = A$ holds. In [3] a full characterization of idempotent copulas was established by Darsow and Olsen. Interpreting copulas and doubly stochastic measures as natural generalization of doubly stochastic matrices and having in mind that every $d$-dimensional doubly stochastic matrix $M$ can be expressed as $M = WQW^{-1}$ whereby $W$ is a permutation matrix and $Q$ is a block diagonal matrix (see [5]) it is surprising that the family $\mathcal{C}_{ip}$ of all idempotent bivariate copulas is quite diverse and complex. This fact was illustrated in [23] where the authors (working with Iterated Function Systems with Probabilities) showed that for every $s \in (1, 2)$ there exists some idempotent copula $A_s$ with the property that the Hausdorff dimension of the support of the corresponding doubly stochastic measure $\mu_{A_s}$ has Hausdorff dimension $s$.

In standard classes of copulas, however, idempotence seems to be a very rare property. Translating the results for Markov chains to copulas, in [13] Lagerås showed that within the class of Archimedean copulas with $d \geq 3$-monotone generators the product copula $\Pi$ is the only idempotent element. Albanese and Sempi (see [1]) studied idempotence in various other well-known classes, conjectured that $\Pi$ is the only idempotent element in the class of ALL Archimedean copula and wrote 'It is our feeling that the result still holds; however, the result would need an entirely different proof, which we have not been able to find.' In the current paper we prove their conjecture using measure-theoretic techniques and, additionally, show that $M$ and $\Pi$ are the only idempotent extreme-value copulas and that $M$ is the only idempotent diagonal copula.

The rest of this paper is organized as follows: Section 2 gathers some preliminaries and notations. Section 3 focuses on extreme-value, Section 4 on diagonal copulas. The afore-mentioned conjecture on Archimedean copulas is established in Section 5.

## 2. Notation and preliminaries

In the sequel $\mathcal{C}$ will denote the family of all two-dimensional *copulas*, i.e., the family of all bivariate distribution functions (restricted to $[0, 1]^2$) whose univariate marginals correspond the uniform distribution on $[0, 1]$. $M$ will denote the minimum copula, $\Pi$ the product copula. For properties of copulas see [4, 15]. For every $C \in \mathcal{C}$, $\mu_C$ will denote the corresponding *doubly stochastic measure*, $\mathcal{P}_\mathcal{C}$ will denote the class of all doubly stochastic measures on $[0, 1]^2$. The family of all diagonals of copulas, i.e., the family of all non-decreasing functions $\delta : [0, 1] \to [0, 1]$ fulfilling that (i) $\delta(0) = 0, \delta(1) = 1$, that (ii) $\delta$ is Lipschitz continuous with Lipschitz constant $L = 2$ and that $\delta(t) \leq t$ for every $t \in [0, 1]$ will be denoted by $\mathcal{D}$. The Lebesgue measure on $[0, 1]$ and $[0, 1]^2$ will be denoted by $\lambda$ and $\lambda_2$ respectively.

For every metric space $(\Omega, d)$ the Borel $\sigma$-field on $\Omega$ will be denoted by $\mathcal{B}(\Omega)$. For every probability measure $\mu$ on $\mathcal{B}(\Omega)$ the support of $\mu$, i.e. the complement of the union of all open sets $U$ fulfilling

$\mu(U) = 0$, will be denoted by $Supp(\mu)$. It is well-known that the support of a measure coincides with the set of all points $x \in \Omega$ fulfilling that for every $r > 0$ we have $\mu(B(x,r)) > 0$ where $B(x,r)$ denotes the open ball of radius $r$ centered at $x$ (see [18]).

Suppose that $(\Omega_1, d_1)$ and $(\Omega_2, d_2)$ are metric spaces. A *Markov kernel* from $\Omega_1$ to $\mathcal{B}(\Omega_2)$ is a mapping $K\colon \Omega_1 \times \mathcal{B}(\Omega_2) \to [0,1]$ such that $x \mapsto K(x,B)$ is measurable function for every fixed $B \in \mathcal{B}(\Omega_2)$ and $B \mapsto K(x,B)$ is a probability measure for every fixed $x \in \Omega_1$. If we only have $K(x,\Omega_2) \in [0,1]$ for every $x \in \Omega_1$ then $K(\cdot,\cdot)$ will be called *substochastic* kernel. Given real-valued random variables $X, Y$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a Markov kernel $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called a *regular conditional distribution* of $Y$ given $X$ if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ Y | X)(\omega) \tag{2}$$

holds $\mathbb{P}$-a.e. It is well know that for each pair $(X,Y)$ of real-valued random variables a regular conditional distribution $K(\cdot,\cdot)$ of $Y$ given $X$ exists, that $K(\cdot,\cdot)$ is unique $\mathbb{P}^X$-a.e. (i.e. unique for $\mathbb{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot,\cdot)$ only depend on $\mathcal{P}^{(X,Y)}$. Hence, given $C \in \mathcal{C}$ we will let $K_C(\cdot,\cdot)$ denote (a version of) the regular conditional distribution of $Y$ given $X$ and refer to $K_C(\cdot,\cdot)$ simply as regular conditional distribution of $C$ or as Markov kernel of $C$. Note that for every $C \in \mathcal{C}$, its regular conditional distribution $K_C(\cdot,\cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have $(G_x = \{y \in [0,1] : (x,y) \in G\})$

$$\int_{[0,1]} K_C(x, G_x)\, d\lambda(x) = \mu_C(G), \tag{3}$$

so in particular

$$\int_{[0,1]} K_C(x, F)\, d\lambda(x) = \lambda(F) \tag{4}$$

holds for all $E, F \in \mathcal{B}([0,1])$. A copula $C$ is called *completely dependent* if there exists a $\lambda$-preserving transformation $h : [0,1] \to [0,1]$ (i.e., a transformation fulfilling $\lambda^h = \lambda$) such that $K(x,F) = \mathbf{1}_F(h(x))$ is a Markov kernel of $C$. In the sequel $\mathcal{C}_d$ will denote the family of all completely dependent copulas, $C_h$ will denote the completely dependent copula induced by $h$, and $\mathcal{T}$ will denote the family of all $\lambda$-preserving transformations on $[0,1]$. For equivalent formulations of complete dependence we refer to [19] and the references therein. For more details and properties of conditional expectation, regular conditional distribution and disintegration see [8, 12].

As direct application of the results in [14] the Markov kernel $K_C(\cdot,\cdot)$ of an arbitrary copula $C \in \mathcal{C}$ can be decomposed into the sum of three substochastic kernels $K_C^{abs}$, $K_C^{sing}$, $K_C^{dis}$ from $[0,1]$ to $\mathcal{B}([0,1])$, i.e.

$$K_C(x,F) = K_C^{abs}(x,F) + K_C^{sing}(x,F) + K_C^{dis}(x,F) \tag{5}$$

for every $x \in [0,1]$ and $F \in \mathcal{B}([0,1])$. Thereby, the measure $K_C^{abs}(x,\cdot)$ is absolutely continuous with respect to $\lambda$, the measure $K_C^{sing}(x,\cdot)$ is singular with respect to $\lambda$ and has no point masses, and $K_C^{dis}(x,\cdot)$ is discrete for every $x \in [0,1]$. Letting $k_C$ denote the Radon-Nikodym derivative of $\mu_C$ with respect to $\lambda_2$ (almost everywhere) uniqueness of the kernel $K_C$ implies that the measure $K_C^{abs}(x,\cdot)$ and $F \mapsto \int_F k_C(x,y)\, d\lambda(y)$ coincide for almost all $x \in [0,1]$. In the sequel we will refer to the induced measure

$\mu_C^{abs}, \mu_C^{sing}, \mu_C^{dis}$, given by

$$\mu_C^{abs}(E \times F) = \int_E K_C^{abs}(x,F)\, d\lambda(x), \qquad \mu_C^{sing}(E \times F) = \int_E K_C^{sing}(x,F)\, d\lambda(x)$$
$$\mu_C^{dis}(E \times F) = \int_E K_C^{dis}(x,F)\, d\lambda(x) \tag{6}$$

and extended to $\mathcal{B}([0,1]^2)$ in the standard way simply as *absolutely continuous, singular and discrete component* of $\mu_C$. Notice that for $C_h \in \mathcal{C}_d$ only $K_C^{dis}$ is non-degenerated and we have $\mu_{C_h} = \mu_{C_h}^{dis}$ and that the standard definition of (purely) singular copulas as stated in [4, 15] translates to $\text{sing}(C) := \mu_C^{sing}([0,1]^2) + \mu_C^{dis}([0,1]^2) = 1$.

As already mentioned in the introduction, given $A, B \in \mathcal{C}$ the Markov product (a.k.a. star product) $A * B \in \mathcal{C}$ is defined by (see [2, 16])

$$(A * B)(x,y) = \int_{[0,1]} \partial_2 A(x,u)\partial_1 B(u,y) d\lambda(u). \tag{7}$$

A copula $C \in \mathcal{C}$ is called idempotent if $C * C = C$ holds, the family of all idempotent copulas will be denoted by $\mathcal{C}_{ip}$. $C \in \mathcal{C}$ is called *symmetric* if $C^t = C$, i.e. if $C(y,x) = C(x,y)$ holds for all $x, y \in [0,1]$. According to [3, 20] each idempotent copula is symmetric. The following result, stating that the Markov kernel of $A * B$ is just the standard composition of the Markov kernels of $A$ and $B$ (justifying the name Markov product) will be used throughout the paper:

**Lemma 2.1** ([23])**.** *Let $A$ and $B$ be copulas with corresponding Markov kernels $K_A$ and $K_B$. Then the Markov kernel $K_A \circ K_B$, defined by*

$$K_A \circ K_B(x,F) = \int_{[0,1]} K_B(y,F) K_A(x,dy), \tag{8}$$

*is a regular conditional distribution of $A * B$.*

The following lemma will be useful in the sequel.

**Lemma 2.2.** *Suppose that $C \in \mathcal{C}$ is idempotent. Then $\mu_C(E \times E) > 0$ holds for every $E \in \mathcal{B}([0,1])$ fulfilling $\lambda(E) > 0$.*

*Proof.* Suppose that $C$ is idempotent, fix $E \in \mathcal{B}([0,1])$ with $\lambda(E) > 0$, and define $\Lambda, \Lambda_E \in \mathcal{B}([0,1])$ by

$$\Lambda = \{x \in [0,1] : K_C(x,\cdot) = K_C \circ K_C(x,\cdot)\}, \ \Lambda_E = \{x \in \Lambda : K_C(x,E) > 0\}.$$

Then $\lambda(\Lambda) = 1$ and using symmetry of $C$ and disintegration we get

$$\mu_C(E \times \Lambda_E) \;=\; \mu_C(\Lambda_E \times E) = \int_{\Lambda_E} K_C(x,E) d\lambda(x) = \int_{[0,1]} K_C(x,E) d\lambda(x) = \lambda(E),$$

implying that $\Gamma_E := \{x \in E : K_C(x, \Lambda_E) = 1\}$ fulfills $\lambda(\Gamma_E) = \lambda(E) > 0$. For every such $x \in \Lambda \cap \Gamma_E$ it follows that

$$K_C(x,E) \;=\; K_C \circ K_C(x,E) = \int_{[0,1]} K_C(z,E) K_C(x,dz) = \int_{\Lambda_E} K_C(z,E) K_C(x,dz) > 0$$

since $K_C(x,\Lambda_E) = 1$ and $K_C(z,E) > 0$ for every $z \in \Lambda_E$. Again applying disintegration yields

$$\mu_C(E \times E) = \int_E K_C(x,E) d\lambda(x) = \int_{\Lambda \cap \Gamma_E} K_C(x,E) d\lambda(x) > 0$$

which completes the proof. □

**Lemma 2.3** ([23]). *Suppose that $A, B \in \mathcal{C}$. If the density $k_B$ of $\mu_B^{abs}$ is strictly positive on $[0,1]^2$ and $\mathrm{sing}(A) > 0$ holds then we have $\mathrm{sing}(A * B) < \mathrm{sing}(A)$.*

Before focussing on extreme-value copulas we sharpen a result by Albanese and Sempi (see [1], Theorem 6.1) saying that the transpose $C_h^t$ of a completely dependent copula $C_h$ is idempotent if, and only if $h$ fulfills $h \circ h(x) = h(x)$ for $\lambda$-almost every $x \in [0,1]$.

**Theorem 2.4.** *$C_h$ is idempotent if, and only if $C_h^t$ is idempotent. The only completely dependent idempotent copula is $M$.*

*Proof.* The first assertion is trivial since every idempotent copula is symmetric (see [3, 20]). To prove the second one assume that $C_h \in \mathcal{C}_d \cap \mathcal{C}_{ip}$. Then using eq.(8) we get $h \circ h(x) = h(x)$ for $\lambda$-almost every $x \in [0,1]$. Hence, defining $E \in \mathcal{B}([0,1])$ by

$$E := \{x \in [0,1] : h(x) \neq x\}$$

we get $h^{-1}(E) = \{x \in [0,1] : h \circ h(x) \neq h(x)\}$ and $\lambda(h^{-1}(E)) = 0$. Considering that $h$ is $\lambda$-preserving $\lambda(E) = \lambda^h(E) = 0$ follows. Since the latter implies $C_h = M$ the proof is complete. □

## 3. Extreme-value copulas

Recall that a copula $C$ is called *extreme-value copula* (EVC) if there exists a copula $B \in \mathcal{C}$ such that

$$C(x,y) = \lim_{n \to \infty} B^n(x^{\frac{1}{n}}, y^{\frac{1}{n}})$$

holds for all $x, y \in [0,1]$. Pickands [17] showed that every EVC can be expressed in terms of a so-called *Pickands dependence function*, i.e., a convex function $A : [0,1] \to [1/2, 1]$ fulfilling $\max\{1-t, t\} \leq A(t)$ for all $t \in [0,1]$, such that

$$C(x,y) = (xy)^{A\left(\frac{\ln(x)}{\ln(xy)}\right)}$$

holds for all $(x,y) \in (0,1)^2$. It is straightforward to verify that an EVC $C$ is symmetric if, and only if, $A$ is symmetric w.r.t. $t = \frac{1}{2}$. Following [24] for every $t \in (0,1)$ define the strictly increasing homeomorphism $f_t$ of $[0,1]$ by $f_t(x) = x^{\frac{1}{t}-1}$, for $t \in \{0,1\}$ define $f_t$ by $f_0(x) = 0$ and $f_1(x) = 1$ for every $x \in [0,1]$, respectively. Given a Pickands dependence function $A$ and defining $L \in [0, \frac{1}{2}]$ and $R \in [\frac{1}{2}, 1]$ by

$$L = \max\{x \in [0,1] : A(x) = 1 - x\}, \quad R = \min\{x \in [0,1] : A(x) = x\}$$

the proof of Corollary 5 in [24] implies the following slightly stronger assertion:

**Lemma 3.1.** *Suppose that $C \neq M$ is a symmetric EVC with a Pickands dependence function $A$. Then $L < \frac{1}{2}, R = 1 - L$ and the density $k_C$ of the absolutely continuous component $\mu_C^{abs}$ of $\mu_C$ fulfills $k_C(x,y) > 0$ for $\lambda_2$-almost every $(x,y) \in \Gamma_L$ with*

$$\Gamma_L = \left\{(x,y) \in (0,1)^2 : f_L(x) \leq y \leq f_{1-L}(x)\right\}. \tag{9}$$

*Furthermore $\Gamma_L$ is the support of $\mu_C$ and the support of $\mu_C^{abs}$.*

**Theorem 3.2.** *Let $C \neq M$ be an EVC. Then $C$ is idempotent if, and only if, $C = \Pi$.*

*Proof.* Let $C \neq M$ be an idempotent EVC with Pickands dependence function $A$. Then $L \in [0, \frac{1}{2})$ and, considering that idempotence implies symmetry, we get $R = 1 - L \in (\frac{1}{2}, 1]$. Furthermore according to Lemma 3.1 the density $k_C$ of the absolutely continuous component $\mu_C^{abs}$ of $\mu_C$ is greater than 0 for $\lambda_2$-almost all $(x, y) \in \Gamma_L$. Without loss of generality we may therefore assume that $k_C(x, y) > 0$ for all $(x, y) \in \Gamma_L$.

Suppose now that $L \in (0, \frac{1}{2})$ holds. Defining $\mu^*$ by

$$\mu^*(E \times F) := \int_E \int_{[0,1]} K_C^{abs}(y, F) K_C^{abs}(x, dy) d\lambda(x)$$

for all $E, F \in \mathcal{B}([0, 1])$ and extending it in the standard way to $\mathcal{B}([0, 1]^2)$ it follows that $\mu^*$ is a measure fulfilling $\mu^*(G) \leq \mu_C(G) = \mu_{C*C}(G)$ for all $G \in \mathcal{B}([0, 1]^2)$. Moreover, considering

$$\begin{aligned} \mu^*(E \times F) &= \int_E \int_{[0,1]} K_C^{abs}(y, F) K_C^{abs}(x, dy) d\lambda(x) \\ &= \int_E \int_{[0,1]} \left( \int_F k_C(y, z) d\lambda(z) \right) k_C(x, y) d\lambda(y) d\lambda(x) \\ &= \int_E \int_F \underbrace{\int_{[0,1]} k_C(x, y) k_C(y, z) d\lambda(y)}_{=:H(x,z)} d\lambda(z) \lambda(x), \end{aligned}$$

it follows that $\mu^*$ is absolutely continuous w.r.t. $\lambda_2$ and the density is given by the function $H$. Using idempotence of $C$ it follows that the density $k_C$ of $C$ fulfills $k_C(x, z) \geq H(x, z)$ for $\lambda_2$-almost all $(x, z) \in [0, 1]^2$. Suppose now that $x \in (0, 1)$ and that $z \in (f_L \circ f_L(x), f_{1-L} \circ f_{1-L}(x))$. We will show that $H(x, z) > 0$ and distinguish two cases: (i) If $z \leq x$ then setting $y = f_L(x)$ yields $f_L(y) = f_L \circ f_L(x) =: f_L^2(x) < z$ as well as $f_{1-L}(y) = f_{1-L} \circ f_L(x) = x \geq z$. Considering that $f_L$ and $f_{1-L}$ are strictly increasing homeomorphisms of $[0, 1]$ there exists some $\delta > 0$ such that for every $y_1 \in [y, y + \delta] \subseteq (0, 1)$ we have $z \in (f_L(y_1), f_{1-L}(y_1))$, which implies

$$H(x, z) = \int_{[0,1]} k_C(x, y) k_C(y, z) d\lambda(y) \geq \int_{[f_L(x), f_L(x)+\delta]} \underbrace{k_C(x, y) k_C(y, z)}_{>0} d\lambda(y) > 0$$

(ii) If $z > x$ setting $y = f_{1-L}(x)$ we get $f_L(y) = x < z$ as well as $f_{1-L} \circ f_{1-L}(x) =: f_{1-L}^2(x) > z$. Again using the fact that $f_L$ and $f_{1-L}$ are strictly increasing homeomorphisms of $[0, 1]$ we can find some $\delta > 0$ such that for every $y_1 \in [y - \delta, y + \delta] \subseteq (0, 1)$ we have $z \in (f_L(y_1), f_{1-L}(y_1))$ from which, using the same argument as in (i) we get $H(x, z) > 0$.

Since obviously $f_L^2 < f_L$ and $f_{1-L}^2 < f_{1-L}$ on $(0, 1)$ it follows that the support of $\mu_C^{abs}$ can not coincide with the set $\Gamma_L$, a contradiction to Lemma 3.1. It therefore suffices to consider $L = 0$. In this case Lemma 3.1 implies that $k_C(x, y) > 0$ on $(0, 1)^2$. Moreover, if $C$ had a singular or discrete component we had $\text{sing}(C) > 0$ from which according to Lemma 7 in [20] we would get $\text{sing}(C) = \text{sing}(C * C) < \text{sing}(C)$, a contradiction to $C$ being idempotent. Altogether it follows that $C$ is an absolutely continuous EVC whose density is strictly positive on $(0, 1)^2$, so applying Lemma 6 in [20] shows $C = \Pi$ and the proof is complete. ☐

## 4. Diagonal copulas

For every diagonal $\delta \in \mathcal{D}$ define the so-called *diagonal copula* (see [15]) $E_\delta$ by

$$E_\delta(x,y) = \min\left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\}. \tag{10}$$

(We will use the symbol $E_\delta$ instead of $K_\delta$ since the letter $K$ will denote kernels throughout the whole paper.) It is well known that $E_\delta$ is singular and that for every symmetric copula $A$ with diagonal $\delta$ we have $A \leq E_\delta$ (see, e.g., [15]).

Defining $L, U : [0,1] \to [0,1]$ by

$$L(x) := \min\big\{z \in [0,1] : g(z) \geq \delta(x)\big\},\ U(x) := \min\big\{z \in [0,1] : \delta(z) \geq g(x)\big\}.$$

whereby $g(x) = 2x - \delta(x)$ for every $x \in [0,1]$, according to [7] the functions $L, U$ have the following properties:

1. $L(x) \leq x$ for all $x \in [0,1]$. Furthermore $L$ is non-decreasing and lower semicontinuous (hence left-continuous).
2. $U(x) \geq x$ for all $x \in [0,1]$. Furthermore $U$ is non-decreasing and upper semicontinuous (hence right-continuous).
3. $\delta = \delta \circ U \circ L$ and $g = g \circ L \circ U$.
4. $L(x) \leq y$ if and only if $U(x) \geq y$.

Additionally it is straightforward to verify that $U$ is a distribution function fulfilling $U(0) = 0, U(1) = 1$, and that $L$ is the quasi-inverse (or quantile function) of $U$, i.e.

$$L(y) = \min\{x \in [0,1] : U(x) \geq y\}$$

holds for every $y \in (0,1)$. As a direct consequence

$$U \circ L(x) \geq x, \quad L \circ U(x) \leq x \tag{11}$$

holds for every $x \in [0,1]$. Letting $w_\delta : [0,1] \to [0,2]$ denote a Borel measurable function fulfilling $\delta'(x) = w_\delta(x)$ for $\lambda$-a.e. $x \in [0,1]$, according to [7] $K_{E_\delta}(\cdot,\cdot)$, defined by

$$K_{E_\delta}(x,F) = \frac{w_\delta(x)}{2}\mathbf{1}_F(L(x)) + \left(1 - \frac{w_\delta(x)}{2}\right)\mathbf{1}_F(U(x)) \tag{12}$$

is a Markov kernel of $E_\delta$. Letting $\Gamma(L)$ and $\Gamma(U)$ denote the graphs of $L$ and $U$, respectively, we therefore have $\mu_{E_\delta}(\Gamma(L) \cup \Gamma(U)) = 1$.

**Theorem 4.1.** *Let $\delta : [0,1] \to [0,1]$ be a diagonal. Then $E_\delta$ is idempotent if and only if $\delta = id_{[0,1]}$.*

*Proof.* Suppose that $\delta \in \mathcal{D}$ and that $E_\delta$ is idempotent. First notice that according to equation (8) the Markov kernel $K_{E_\delta} \circ K_{E_\delta}$ is given by

$$\begin{aligned}
K_{E_\delta} \circ K_{E_\delta}(x,F) &= \int_{[0,1]} K_{E_\delta}(z,F) K_{E_\delta}(x,dz) \\
&= \tfrac{w_\delta(x)}{2} K_{E_\delta}(L(x),F) + \left(1 - \tfrac{w_\delta(x)}{2}\right) K_{E_\delta}(U(x),F) \\
&= \tfrac{w_\delta(x)}{2}\left(\tfrac{w_\delta(L(x))}{2}\mathbf{1}_F(L \circ L(x)) + \left(1 - \tfrac{w_\delta(L(x))}{2}\right)\mathbf{1}_F(U \circ L(x))\right) \\
&\quad + \left(1 - \tfrac{w_\delta(x)}{2}\right)\left(\tfrac{w_\delta(U(x))}{2}\mathbf{1}_F(L \circ U(x)) + \left(1 - \tfrac{w_\delta(U(x))}{2}\right)\mathbf{1}_F(U \circ U(x))\right),
\end{aligned}$$

which implies that $\mu_{E_\delta * E_\delta}$ concentrates its mass on the graph of at most four functions, $L \circ L, U \circ L, L \circ U$, and $U \circ U$. According to inequality (11) $L \circ L(x) \leq L \circ U(x) \leq x$ as well as $U \circ U(x) \geq U \circ L(x) \geq x$ holds for every $x \in [0,1]$. Defining the set $\Lambda \in \mathcal{B}([0,1])$ by

$$\Lambda = \{x \in (0,1) : x \text{ Lebesgue point of } w_\delta \text{ and } K_{E_\delta} \circ K_{E_\delta}(x,\cdot) = K_{E_\delta}(x,\cdot)\}$$

then $\lambda(\Lambda) = 1$ follows immediately (see [18, Theorem 7.10]).

We will show now $\delta(x) = x$ holds for every $x \in \Lambda$ and proceed as follows: Suppose that $x \in \Lambda$ and that $\delta(x) < x$. **(a)** If $w_\delta(x) > 0$ then, using $L(x) < x < U(x)$, it follows that $L \circ L(x) = L(x)$ or $L \circ U(x) = L(x)$ holds (otherwise we would have $K_{E_\delta}(x, L(x)) = 0$, hence $w_\delta(x) = 0$). (i) If $L \circ L(x) = L(x)$ then $L$ is constant on the non-degenerated interval $[L(x), x]$, which implies

$$0 \;=\; \mu_{E_\delta}([L(x), x] \times \{L(x)\}) = \int_{[L(x),x]} \frac{w_\delta(t)}{2} d\lambda(t).$$

The latter, however, contradicts the fact that $x$ is a Lebesgue point of $w_\delta$ since in this case we would have

$$\lim_{r \to 0+} \frac{1}{r} \int_{[x-r,x]} w_\delta \, d\lambda = w_\delta(x) > 0.$$

(ii) If $L \circ U(x) = L(x)$ then $L$ is constant on the non-degenerated interval $[x, U(x)]$ and we can proceed analogously to obtain a contradiction. Altogether we have shown that $\delta(x) < x$ implies $w_\delta(x) = 0$.

**(b)** If $1 - \frac{w_\delta(x)}{2} > 0$ then, using $L(x) < x < U(x)$, it follows that $U \circ U(x) = U(x)$ or $U \circ L(x) = U(x)$ holds (otherwise we would have $K_{E_\delta}(x, U(x)) = 0$, hence $1 - \frac{w_\delta(x)}{2} = 0$). Since $x$ is a Lebesgue point of $w_\delta$ if and only if it is a Lebesgue point of $1 - \frac{w_\delta(x)}{2}$ we can proceed as in (a) and consider two options: (i) If $U \circ L(x) = U(x)$ then $U$ is constant on the non-degenerated interval $[L(x), x]$, which implies

$$0 \;=\; \mu_{E_\delta}([L(x), x] \times \{U(x)\}) = \int_{[L(x),x]} 1 - \tfrac{w_\delta(t)}{2} d\lambda(t).$$

The latter, however, contradicts the fact that $x$ is a Lebesgue point of $1 - \frac{w_\delta(x)}{2}$ since in this case we would have

$$\lim_{r \to 0+} \frac{1}{r} \int_{[x-r,x]} 1 - \tfrac{w_\delta}{2} \, d\lambda = 1 - \tfrac{w_\delta(x)}{2} > 0.$$

(ii) If $U \circ U(x) = x$ we proceed analogously to obtain a contradiction. Altogether we have shown that $\delta(x) < x$ implies $1 - \frac{w_\delta(x)}{2} = 0$.

Combining (a) and (b) yields $\delta(x) = x$ for every $x \in \Lambda$. Since $\Lambda$ has full measure it is dense in $[0,1]$, using Lipschitz continuity of $\delta$ therefore yields $\delta = id_{[0,1]}$, which, in turn implies $E_\delta = M$ and completes the proof. □

Theorem 4.1 implies the following result:

**Corollary 4.2.** *The minimum copula $M$ is the only idempotent diagonal copula.*

## 5. Archimedean copulas

Recall that a function $\varphi : [0,1] \to [0,\infty]$ is called a *generator* (of a bivariate Archimedean copula, see [15]) if it is convex, strictly decreasing and fulfills $\varphi(1) = 0$. In the sequel we will, without loss of generality, assume that $\lim_{t\to 0+}\varphi(t) =: \varphi(0+) = \varphi(0)$ holds and that all generators $\varphi$ fulfill $\varphi(\frac{1}{2}) = 1$ which allows for a one-to-one correspondence between generators and Archimedean copulas (without these requirements the generator is only unique up to a positive multiplicative constant and in the case of $\varphi(0+) < \infty$ redefining $\varphi(0)$ as some value in $[\varphi(0+), \infty]$ does not change the copula). A generator $\varphi$ is called *strict* if $\varphi(0) = \infty$, in the other case $\varphi$ is referred to as *non-strict*. Every generator $\varphi$ induces a symmetric copula $A_\varphi$ via

$$A_\varphi(x, y) = \varphi^{-}(\varphi(x) + \varphi(y)) \tag{13}$$

where the pseudo inverse $\varphi^{-} : [0,\infty] \to [0,1]$ of $\varphi$ is defined by

$$\varphi^{-}(t) = \begin{cases} \varphi^{-}(t) & \text{if } t \in [0, \varphi(0)) \\ 0 & \text{if } t \geq \varphi(0). \end{cases}$$

If $\varphi$ is strict then $\varphi^{-}$ coincides with the standard inverse.

For every generator $\varphi$ we will let $D^+\varphi(x)$ $(D^-\varphi(x))$ denote the right-hand (left-hand) derivative of $\varphi$ at $x \in (0,1)$. Convexity of $\varphi$ implies that $D^+\varphi(x) = D^-\varphi(x)$ holds for all but at most countably many $x \in (0,1)$, i.e. $\varphi$ is differentiable outside a countable subset of $(0,1)$, and that $D^+\varphi$ is non decreasing and right-continuous (see, for instance, [10]).

For every $t \in (0,1]$ define $f^t : [0,1] \to [0,1]$ by $f^t(x) = \varphi^{-1}(\varphi(t) - \varphi(x))$ for $x \geq t$ and $f^t(x) = 1$ for $x < t$. Then according to [6] a Markov kernel $K_\varphi$ of $A_\varphi$ is given by

$$K_\varphi(x, [0,y]) = \begin{cases} 1 & \text{if } x \in \{0,1\} \\ \frac{D^+\varphi(x)}{(D^+\varphi)(A_\varphi(x,y))} & \text{if } x \in (0,1) \text{ and } y \geq f^0(x) \\ 0 & \text{if } x \in (0,1) \text{ and } y < f^0(x). \end{cases}$$

In [1] Albanese and Sempi showed that if $A_\varphi$ is an idempotent Archimedean copula whose generator $\varphi$ is even 3-monotone then $A_\varphi$ coincides with $\Pi$, i.e., assuming some additional mild regularity of the generator the only idempotent Archimedean copula is $\Pi$. In [1] the authors ask whether the same result holds for arbitrary Archimedean copulas and conjecture that $\Pi$ is indeed the only idempotent Archimedean copula. In what follows we will prove their conjecture in full generality and, as by-product, provide an alternative proof to the result established in [1].

To establish this main result we proceed in several steps and start with a simple lemma concerning the transformation $\mathcal{V}\colon \mathcal{C} \to \mathcal{C}$ by $\mathcal{V}(C) = C * C^t$ (also see [9]).

**Lemma 5.1.** *For every $C \in \mathcal{C}$ and every $x \in [0,1]$ we have $\delta_\Pi(x) \leq \delta_{\mathcal{V}(C)}(x)$.*

*Proof.* For $x \in [0,1]$ and $C \in \mathcal{C}$ applying Cauchy-Schwarz inequality and using eq. (7) yields

$$\begin{aligned} \delta_{\mathcal{V}(C)}(x) = C * C^t(x,x) &= \int_{[0,1]} \partial_2 C(x,s)\partial_1 C^t(s,x)\, d\lambda(s) = \int_{[0,1]} (\partial_2 C(x,s))^2\, d\lambda(s) \\ &\geq \left(\int_{[0,1]} \partial_2 C(x,s) d\lambda(s)\right)^2 = \left(\int_{[0,1]} K_{C^t}(s,[0,x]) d\lambda(s)\right)^2 \\ &= x^2 = \delta_\Pi(x). \end{aligned}$$

□

The previous lemma allows for a simple proof of the fact that an Archimedean copula can only be idempotent if the generator is strict.

**Lemma 5.2.** *There is no idempotent Archimedean copula with non-strict generator.*

*Proof.* If $A := A_\varphi$ is an idempotent Archimedean copula with generator $\varphi$ then obviously $\mathcal{V}(A) = A$, so Lemma 5.1 implies $x^2 \leq \delta_A(x)$ for every $x \in [0,1]$. If $\varphi$ is non-strict then setting $\varphi(0) =: b \in (0,\infty)$ and using continuity of $\varphi$ there exists some $x_0 \in (0,1)$ such that $2\varphi(x_0) > b$ holds, from which we get

$$\delta_A(x_0) = \varphi^-\left(2\varphi(x_0)\right) \leq \varphi^-(b) = 0 < x_0^2,$$

a contradiction. □

**Lemma 5.3.** *If $D^+\varphi$ has a discontinuity point $t \in (0,1)$ then the Archimedean copula $A_\varphi$ is not idempotent.*

*Proof.* Notice that according to [6] $t \in (0,1)$ is a discontinuity point of $D^+\varphi$ if and only if $K_{A_\varphi}(x, \{f^t(x)\}) > 0$ for $\lambda$-almost every $x \geq t$. Suppose that $A_\varphi$ is idempotent and let $t \in (0,1)$ be a discontinuity point of $D^+\varphi$. Then defining $\Lambda \in \mathcal{B}([0,1])$ by

$$\Lambda = \left\{x \in [t,1] : K_{A_\varphi}(x, \{f^t(x)\}) > 0 \text{ and } K_{A_\varphi} \circ K_{A_\varphi}(x, \cdot) = K_{A_\varphi}(x, \cdot)\right\}$$

it follows that $\lambda(\Lambda) = 1 - t$. Furthermore for $x \in \Lambda$ we get

$$\begin{aligned}
K_{A_\varphi}(x, \{x\}) &= K_{A_\varphi}(x, \{f^t \circ f^t(x)\}) = K_{A_\varphi} \circ K_{A_\varphi}(x, \{f^t \circ f^t(x)\}) \\
&= \int_{[0,1]} K_{A_\varphi}(y, \{f^t \circ f^t(x)\}) K_{A_\varphi}(x, dy) \\
&\geq K_{A_\varphi}\left(f^t(x), \{f^t \circ f^t(x)\}\right) K_{A_\varphi}(x, \{f^t(x)\}) > 0.
\end{aligned}$$

Having this, setting $\Gamma_t := \{(x,x) : x \in \Lambda\}$ and applying disintegration yields $\mu_{A_\varphi}(\Gamma_t) = \mu_{A_\varphi}^{abs}(\Gamma_t) > 0$. On the other hand (again see [6]), the number of discontinuity points $\mathrm{DC}(D^+\varphi)$ of $D^+\varphi$ is at most countably infinite, the measure $\mu_{A_\varphi}^{dis}$ concentrates its mass on the set

$$\Gamma = \bigcup_{s \in \mathrm{DC}(D^+\varphi)} \{(x, f^s(x)) : x \in [s,1]\},$$

and $\Gamma \cap \Gamma_t$ contains at most countably many points. The latter implies

$$\mu_{A_\varphi}^{abs}(\Gamma_t) = \mu_{A_\varphi}^{abs}(\Gamma \cap \Gamma_t) = 0,$$

a contradiction. □

**Lemma 5.4.** *Suppose that $\varphi$ is a strict generator which is continuously differentiable on $(0,1)$. If $A_\varphi$ is idempotent then $\varphi'$ is strictly increasing on $(0,1)$ and $C_\varphi$ has full support.*

*Proof.* Suppose that $\varphi$ fulfills the assumptions of the lemma and that $\varphi'$ is constant on an interval $[a,b] \subset (0,1)$ with $a < b$. Then the Kendall distribution function of $A_\varphi$ is continuous on $[0,1]$ and constant on the interval $[a,b]$ (see [6]), which implies $\mu_{A_\varphi}(E_{a,b}) = 0$, whereby

$$E_{a,b} = \{(x,y) \in [0,1]^2 : a \leq C_\varphi(x,y) \leq b\}.$$

Since we can obviously find a non-degenerated interval $J$ fulfilling $J\times J \subseteq E_{a,b}$ it follows that this interval fulfills $\lambda(J) > 0$ and $\mu_{A_\varphi}(J \times J) = 0$, a contradiction to Lemma 2.2. This proves the first assertion of the lemma.

To prove the second assertion notice that $\varphi'$ as well as $y \mapsto A_\varphi(x, y)$ are strictly increasing, so $y \mapsto K_{A_\varphi}(x, [0, y])$ is strictly increasing. As a direct consequence every non-degenerated square $S = [c, c + \delta] \times [d, d + \delta] \subseteq [0, 1]^2$ with $\delta > 0$ fulfills $\mu_{A_\varphi}(S) > 0$, so $\mu_{A_\varphi}$ has full support. □

In the following final steps we will use the results from [3] saying that idempotent copulas can be subdivided into three classes: (i) non-atomic ones, (ii) totally atomic ones, and (iii) atomic but not totally atomic ones. Before stating a nice and handy characterization of the just mentioned three classes going back to [3] recall that $h \in \mathcal{T}$ is called essentially invertible if, and only if, there exists some $g \in \mathcal{T}$ such that $g \circ h(x) = h \circ g(x) = x$ holds for $\lambda$-almost every $x \in [0, 1]$. For every essentially invertible $h \in \mathcal{T}$ it is straightforward to verify that $C_h^t * C_h = M = C_h * C_h^t$ as well as $C_h^t = C_g$ and $C_h = C_g^t$ hold. According to [3] the subsequent assertions hold for idempotent copulas $A$ (for the definition of an ordinal sum we refer to [4]):

(i) $A$ is non-atomic if, and only if, there exists some $h \in \mathcal{T}$ such that $A = C_h * C_h^t$ holds.

(ii) $A$ is totally atomic if, and only if, there exists essentially invertible $h \in \mathcal{T}$ and an ordinal sum $O$ of (finitely or countably infinitely many copies of) $\Pi$ such that $A = C_h * O * C_h^t$ holds.

(iii) $A$ is atomic but not totally atomic if, and only if, there exists some essentially invertible $h \in \mathcal{T}$ and an ordinal sum $O$ such that $A = C_h * O * C_h^t$ holds, whereby $O$ contains at least two 'blocks', all but one of which coincide with $\Pi$, and one block corresponds to a non-atomic idempotent copula $B$ (a more concise characterization will be given in the proof of Lemma 5.7).

It is straightforward to verify that every non-atomic idempotent copula $A$ is singular w.r.t. $\lambda_2$, i.e., $\mu_A = \mu_A^{sing} + \mu_A^{dis}$, that every totally-atomic idempotent copula $A$ is absolutely continuous, and that every atomic but not totally atomic idempotent copula $A$ fulfills that $\mu_A^{abs}$ as well as $\mu_A^{sing} + \mu_A^{dis}$ are non-degenerated. In fact, in the first case we may proceed as follows (the other two cases may be handled analogously): Fix $h \in \mathcal{T}$, set $A := C_h * C_h^t$ and choose a (version) $K_A(\cdot, \cdot)$ of the Markov kernel of $A$ fulfilling

$$K_A(x, E) = \int_{[0,1]} K_{C_h^t}(u, E) K_{C_h}(x, du) = K_{C_h^t}(h(x), E)$$

for every $x \in [0, 1]$ and every $E \in \mathcal{B}([0, 1])$. Letting $\Gamma(h) = \{(x, h(x)) : x \in [0, 1]\}$ denote the graph of $h$ using disintegration we have

$$1 = \mu_{C_h}(\Gamma(h)) = \mu_{C_h^t}(\Gamma(h)^t) = \int_{[0,1]} K_{C_h^t}(u, \underbrace{(\Gamma(h)^t)_u}_{=h^{-1}(\{u\})}) d\lambda(u) = \int_{[0,1]} K_{C_h^t}(u, h^{-1}(\{u\})) d\lambda(u),$$

implying that there exists a set $\Lambda \in \mathcal{B}([0, 1])$ fulfilling $\lambda(\Lambda) = 1$ such that $K_{C_h^t}(u, h^{-1}(\{u\})) = 1$ holds for every $u \in \Lambda$. Set $\Omega = h^{-1}(\Lambda)$. Then $\lambda(\Omega) = 1$ and for $x \in \Omega$ it follows that

$$K_A\left(x, h^{-1}(\{h(x)\})\right) = K_{C_h^t}\left(h(x), h^{-1}(\{h(x)\})\right) = 1.$$

Setting

$$\Psi = \{(x,y) \in [0,1]^2 : \, h(x) = h(y)\} \in \mathcal{B}([0,1]^2)$$

we obviously have $\lambda_2(\Psi) = 0$, considering $\Psi_x = \{y \in [0,1] : h(x) = h(y)\} = h^{-1}(\{h(x)\})$ we get $\mu_A(\Psi) = 1$, i.e., $A$ is singular.

**Lemma 5.5.** *Suppose that $\varphi$ is a strict generator which is continuously differentiable on $(0,1)$ and fulfills that $\varphi'$ is strictly increasing on $(0,1)$. If $A_\varphi$ is singular then $A_\varphi$ is not idempotent.*

*Proof.* If $A_\varphi$ is singular and idempotent then it is necessarily non-atomic and we can find some $h \in \mathcal{T}$ such that $A_\varphi = C_h * C_h^t$ holds. The assumptions on $\varphi$ imply that all conditional distribution functions $y \mapsto F_x^{A_\varphi}([0,y]) = K_{A_\varphi}(x,[0,y]) = \frac{\varphi'(x)}{\varphi'(A_\varphi(x,y))}$, $x \in (0,1)$, are continuous, singular and strictly increasing, so $K_{A_\varphi}(x,\cdot)$ is singular without discrete component and has support $[0,1]$. The fact that $A_\varphi = C_h * C_h^t$ holds implies

$$K_{A_\varphi}(x,\cdot) = \int_{[0,1]} K_{C_h^t}(y,\cdot) K_{C_h}(x,dy) = K_{C_h^t}(h(x),\cdot) \tag{14}$$

for $\lambda$-almost every $x \in (0,1)$. Additionally, considering $\mu_{C_h}(\Gamma(h)) = 1$ and using disintegration we get

$$1 = \mu_{C_h}(\Gamma(h)) = \int_{[0,1]} K_{C_h^t}(y, \Gamma(h)_{y,2}) d\lambda(y) = \int_{[0,1]} K_{C_h^t}(y, h^{-1}(\{y\})) d\lambda(y),$$

implying the existence of a set $\Lambda_1 \in \mathcal{B}([0,1])$ with $\lambda(\Lambda_1) = 1$ such that $K_{C_h^t}(y, h^{-1}(\{y\}) = 1$ holds for every $y \in \Lambda_1$. Letting $\Lambda_2$ denote the set of all $x \in (0,1)$ for which equation (14) as well as $K_{A_\varphi}(x, h^{-1}(\{h(x)\}) = 1$ holds, we have $\lambda(\Lambda_2) = 1$. Denoting by $\Lambda_3 \in \mathcal{B}([0,1])$ the set of all $x \in (0,1)$ with $K_{A_\varphi}(x, \Lambda_2) = 1$ and setting $\Lambda = \Lambda_2 \cap \Lambda_3$ yields $\lambda(\Lambda) = 1$.

Given $x, z \in (0,1)$ we will write $x \sim z$ if $F_x^{A_\varphi} = F_z^{A_\varphi}$. Obviously $\sim$ defines an equivalence relation on $(0,1)$. Considering the fact that $(x,y) \mapsto F_x^{A_\varphi}([0,y]) = K_{A_\varphi}(x,[0,y]) = \frac{\varphi'(x)}{\varphi'(A_\varphi(x,y))}$ is continuous on $(0,1) \times [0,1]$, the equivalence class $\langle x \rangle$ of every $x \in (0,1)$ is closed and has empty interior. In fact, in case $\langle x \rangle$ contained a non-degenerated open interval $(a,b)$, using singularity of $A_\varphi$ we could find a set $N \in \mathcal{B}([0,1])$ with $\lambda(N) = 0$ and $K_{A_\varphi}(x,N) = 1$, implying

$$0 = \lambda(N) \geq \int_{(a,b)} K_{A_\varphi}(t,N) d\lambda(t) = (b-a) K_{A_\varphi}(x,N) = (b-a) > 0.$$

Let $x, z \in \Lambda$ be arbitrary, but fixed. In case of $x \sim z$ we get

$$K_{A_\varphi}\big(x, h^{-1}(\{h(x)\})\big) = 1 = K_{A_\varphi}\big(z, h^{-1}(\{h(z)\})\big) = K_{A_\varphi}\big(x, h^{-1}(\{h(z)\})\big),$$

from which $h(x) = h(z)$ follows immediately. In other words, $\langle x \rangle \cap \Lambda \subseteq h^{-1}(\{h(x)\}) \cap \Lambda$ holds for every $x \in \Lambda$. Taking into account

$$\Lambda = \bigcup_{x \in \Lambda} (\langle x \rangle \cap \Lambda) \subseteq \bigcup_{x \in \Lambda} \big(h^{-1}(\{h(x)\}) \cap \Lambda\big) = \Lambda$$

for every $x \in \Lambda$ we even have $\langle x \rangle \cap \Lambda = h^{-1}(\{h(x)\}) \cap \Lambda$. For every $x \in \Lambda$ with $K_{A_\varphi}(x, \Lambda) = 1$ it follows that

$$K_{A_\varphi}(x, \langle x \rangle \cap \Lambda) = K_{A_\varphi}(x, h^{-1}(\{h(x)\}) \cap \Lambda) = 1,$$

implying $K_{A_\varphi}(x, \langle x \rangle) = 1$. Since $K_{A_\varphi}(x,\cdot)$ has full support we get $\langle x \rangle \cap U \neq \emptyset$ for every open interval $U \subseteq (0,1)$, implying that the topological closure $\overline{\langle x \rangle}$ of $\langle x \rangle$ coincides with $[0,1]$, a contradiction to the fact that $\langle x \rangle$ is closed without inner points. □

**Lemma 5.6.** *Suppose that $\varphi$ is a strict generator which is continuously differentiable on $(0,1)$ and fulfills that $\varphi'$ is strictly increasing on $(0,1)$. If $A_\varphi$ is absolutely continuous and idempotent then $A_\varphi = \Pi$.*

*Proof.* Assume that $A_\varphi$ is absolutely continuous and idempotent. Then there exists some essentially invertible $h \in \mathcal{T}$ and some ordinal sum $O$ of (copies of) $\Pi$ such that $A_\varphi = C_h * O * C_h^t = C_h * O * C_g$ holds, whereby $g \in \mathcal{T}$ fulfills $g \circ h(x) = h \circ g(x) = x$ for $\lambda$-almost every $x \in [0,1]$. Letting $(J_i)_{i \in I}$ with $I = \{1, \ldots, N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$ denote the family of pairwise disjoint open intervals corresponding to $O$ we have that $[0,1] \setminus \bigcup_{i \in I} J_i$ is finite our countably infinite (again see [4]). Considering

$$K_{C_h * O * C_h^t}(x, F) = K_O(h(x), g^{-1}(F))$$

there exists some $\Lambda \in \mathcal{B}([0,1])$ with $\lambda(\Lambda) = 1$ such that for every $x \in \Lambda$ we have

$$K_{A_\varphi}(x, F) = K_O\left(h(x), g^{-1}(F)\right)$$

holds for every $F \in \mathcal{B}([0,1])$. Setting $E_i = h^{-1}(J_i)$ for every $i \in I$ as well as $\Lambda' := \Lambda \cap \bigcup_{i \in I} E_i$ and considering $x \in \Lambda'$ therefore yields

$$\begin{aligned} K_{A_\varphi}(x, F) &= K_O\left(h(x), g^{-1}(F)\right) = \sum_{i \in I} \frac{\lambda(J_i \cap g^{-1}(F))}{\lambda(J_i)} \mathbf{1}_{J_i}(h(x)) \\ &= \sum_{i \in I} \frac{\lambda(E_i \cap h^{-1}(g^{-1}(F)))}{\lambda(E_i)} \mathbf{1}_{E_i}(x) = \sum_{i \in I} \frac{\lambda(E_i \cap F)}{\lambda(E_i)} \mathbf{1}_{E_i}(x). \end{aligned} \tag{15}$$

Since $K_{A_\varphi}(x, \cdot)$ has full support, for every $x \in \Lambda'$ and every open set $U \neq \emptyset$ using eq. (15) it follows that

$$0 < K_{A_\varphi}(x, U) = \sum_{i \in I} \frac{\lambda(E_i \cap U)}{\lambda(E_i)} \mathbf{1}_{E_i}(x).$$

The latter implies $\lambda(E_i \cap U) > 0$ for every $i \in I$ which shows that every $E_i$ is dense in $[0,1]$ since $U$ was arbitrary.

Suppose now that $I$ contains at least two elements and consider $x \in E_1 \cap \Lambda'$. Since according to eq. (15) the probability measures $K_{x,\cdot}$ are constant on each $E_i$ and since $E_2$ is dense in $[0,1]$ we can find a sequence $(x_n)_{n \in \mathbb{N}}$ in $E_2$ converging to $x$. Continuity of $(x,y) \mapsto K_{A_\varphi}(x, [0,y])$ on $(0,1)^2$ therefore yields

$$K_{A_\varphi}(x, \cdot) = K_{A_\varphi}(x_n, \cdot),$$

a contradiction, since $E_1 \cap E_2 = \emptyset$ and $K_{A_\varphi}(x, E_1) = 1 = K_{A_\varphi}(x_n, E_2)$. In other words: $I = \{1\}$ and the ordinal sum $O$ coincides with $\Pi$, which completes the proof since $A_\varphi = C_h * \Pi * C_h^t = \Pi$. □

**Lemma 5.7.** *Suppose that $\varphi$ is a strict generator which is continuously differentiable on $(0,1)$ and fulfills that $\varphi'$ is strictly increasing on $(0,1)$. If $A_\varphi$ fulfills that $\mu_{A_\varphi}^{abs}$ as well as $\mu_{A_\varphi}^{sing} + \mu_{A_\varphi}^{dis}$ are non-degenerated then $A_\varphi$ is not idempotent.*

*Proof.* Proceeding as in the totally atomic case (and using the same notation), in the current setting eq. (15) takes the form

$$\begin{aligned} K_{A_\varphi}(x, F) &= K_O\left(h(x), g^{-1}(F)\right) = \sum_{i \in I \setminus \{1\}} \frac{\lambda(J_i \cap g^{-1}(F))}{\lambda(J_i)} \mathbf{1}_{J_i}(h(x)) \\ &\quad + K_B\left(\varphi_1 \circ h(x), \varphi_1(g^{-1}(h^{-1}(J_1)))\right) \mathbf{1}_{J_1}(h(x)) \\ &= \sum_{i \in I \setminus \{1\}} \frac{\lambda(E_i \cap F)}{\lambda(E_i)} \mathbf{1}_{E_i}(x) + K_B\left(\varphi_1 \circ h(x), \varphi_1(g^{-1}(h^{-1}(J_1)))\right) \mathbf{1}_{E_1}(x) \end{aligned}$$

where $\varphi_1$ denotes the increasing affine transformation mapping $J_1$ onto $[0,1]$. As in the totally atomic case it follows that $E_2$ (in fact, all $E_i$ with $i \in I \setminus \{1\}$) are dense in $[0,1]$. Moreover, since $B$ is singular for almost every $u \in [0,1]$ the probability measure $K_B(u,\cdot)$ is singular. Choose $x \in E_1 \cap \Lambda'$ such that $K_B(\varphi_1(h(x)),\cdot)$ is singular. Then we can find a sequence $(x_n)_{n\in\mathbb{N}}$ in $E_2$ converging to $x$. Continuity of $(x,y) \mapsto K_{A_\varphi}(x,[0,y])$ on $(0,1)^2$ therefore yields

$$K_{A_\varphi}(x,\cdot) = K_{A_\varphi}(x_n,\cdot),$$

a contradiction since each $K_{A_\varphi}(x_n,\cdot)$ is absolutely continuous. □

Altogether we have proved the following main result of this section, which confirms the conjecture by Albanese and Sempi in [1].

**Theorem 5.8.** *The only idempotent Archimedean copula is* $\Pi$.

Suppose that $I = \{1,\ldots,N\}$ for some $N \in \mathbb{N}$ or $I = \mathbb{N}$. If the copula $O$ is the ordinal sum of copulas $(A_i)_{i\in I}$ with respect to the pairwise disjoint (open) intervals $(J_i)_{i\in I}$ then according to [1] $O * O$ is an ordinal sum of $(A_i * A_i)_{i\in I}$ with respect to $(J_i)_{i\in I}$. Since (see [11]) every associative copula can be expressed as ordinal sum of Archimedean copulas Theorem 5.8 implies the following:

**Theorem 5.9.** *Let $A$ be an arbitrary associative copula and suppose that $A$ can be expressed as ordinal sum of copulas $(A_i)_{i\in I}$ with respect to the pairwise disjoint open intervals $(J_i)_{i\in I}$. Then $A$ is idempotent if, and only if, $J_i = \Pi$ for every $i \in I$.*

**Acknowledgement**

The second author gratefully acknowledges the support of the WISS 2025 project 'IDA-lab Salzburg' (20204-WISS/225/197-2019 and 20102-F1901166-KZP).

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