On convergence and mass distributions of multivariate Archimedean copulas and their interplay with the Williamson transform

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Abstract

Motivated by a recently established result saying that within the class of bivariate Archimedean copulas standard pointwise convergence implies weak convergence of almost all conditional distributions this contribution studies the class $\mathcal{C}_{ar}^d$ of all d-dimensional Archimedean copulas with $d \geq 3$ and proves the afore-mentioned implication with respect to conditioning on the first $d-1$ coordinates. Several properties equivalent to pointwise convergence in $\mathcal{C}_{ar}^d$ are established and - as by-product of working with conditional distributions (Markov kernels) - alternative simple proofs for the well-known formulas for the level set masses $\mu_C(L_t)$ and the Kendall distribution function F_K^d as well as a novel geometrical interpretation of the latter are provided. Viewing normalized generators ψ of d-dimensional Archimedean copulas from the perspective of their so-called Williamson measures γ on $(0,\infty)$ is then shown to allow not only to derive surprisingly simple expressions for $\mu_C(L_t)$ and F_K^d in terms of γ and to characterize pointwise convergence in $\mathcal{C}_{ar}^d$ by weak convergence of the Williamson measures but also to prove that regularity/singularity properties of γ directly carry over to the corresponding copula $C_\gamma \in \mathcal{C}_{ar}^d$. These results are finally used to prove the fact that the family of all absolutely continuous and the family of all singular d-dimensional copulas is dense in $\mathcal{C}_{ar}^d$ and to underline that despite of their simple algebraic structure Archimedean copulas may exhibit surprisingly singular behavior in the sense of irregularity of their conditional distribution functions.

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1. Introduction

Archimedean copulas are a well-known family of dependence models whose popularity is mainly due to their simple algebraic structure: given a so-called (Archimedean, sufficiently monotone) generator $\psi : [0,\infty) \to [0,1] =: \mathbb{I}$ and letting φ denote its pseudo-inverse the

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Archimedean copula C_ψ is defined by

$$C_\psi(x_1, \ldots, x_d) = \psi(\varphi(x_1) + \cdots + \varphi(x_d)).$$

As a consequence, analytic, dependence, and convergence properties of Archimedean copulas can be characterized in terms of the corresponding generators (see, e.g., [2, 4, 13, 20] and [21, Chapter 4]). In particular, it was recently shown in [13] that within the class of *bivariate* Archimedean copulas pointwise convergence is equivalent to uniform convergence of the corresponding generators and, more importantly, even implies weak convergence of almost all conditional distributions (a.k.a weak conditional convergence, a concept generally much stronger than pointwise convergence). This result is surprising insofar that given samples $(X_1, Y_1), (X_2, Y_2), \ldots$ from some bivariate copula C the corresponding sequence of empirical copulas $(\hat{E}_n)_{n\in\mathbb{N}}$ does not necessarily converge weakly conditional to C.
The focus of the current paper is twofold: *on the one hand* we study convergence in the class $\mathcal{C}^d_{ar}$ of all d-dimensional Archimedean copulas, $d \geq 3$, and show that most results from the bivariate setting as established in [13] also hold in $\mathcal{C}^d_{ar}$, including the surprising fact that pointwise convergence implies weak conditional convergence (whereby we consider conditioning on the first $d-1$ coordinates). As a nice by-product of working with Markov kernels (conditional distributions) we obtain simple, alternative proofs of the well-known formulas for the Kendall distribution function F^d_K and the level set masses of Archimedean copulas. To the best of the authors' knowledge, working with so-called ℓ_1-norm symmetric distributions (as studied in [20]) nowadays seems to be the standard approach for deriving these formulas in the multivariate setting - we show that working with Markov kernels constitutes an interesting alternative and may provide additional insight. Additionally, motivated by [21, Chapter 4.3] and [4, Section 3] we offer a seemingly novel geometric interpretation of the level set masses in terms of ψ.
And, *on the other hand*, we revisit the close interrelation between Archimedean copulas and probability measures γ on $(0, \infty)$ via the so-called Williamson transform as studied in [20] (also see [24, Theorem 1.11]), characterize properties of the generator ψ in terms of normalized γ (to which we will refer to as Williamson measure) and then prove the fact that pointwise convergence in $\mathcal{C}^d_{ar}$ is equivalent to weak convergence of the corresponding probability measures on $(0, \infty)$. Moreover, we derive surprisingly simple expressions for the level set masses and the Kendall distribution functions in terms of γ and then show that singularity/regularity properties of γ directly carry over to the corresponding Archimedean copula $C_\gamma \in \mathcal{C}^d_{ar}$. This very property is then used, firstly, to show that the family of absolutely continuous as well as two disjoint subclasses of the family of all singular Archimedean copulas are dense in $\mathcal{C}^d_{ar}$ and, secondly, to illustrate the fact that despite their simple and handy algebraic structure Archimedean copulas may exhibit surprisingly irregular behavior by constructing elements of $\mathcal{C}^d_{ar}$ which have full support although being singular (see [4] for the already established bivariate results).

The rest of this contribution is organized as follows: Section 2 contains notation and preliminaries (in particular on Markov kernels) that are used throughout the text. Section 3 starts with deriving an explicit expression for the Markov kernel of d-dimensional Archimedean copulas, then restates the well-known formulas for the masses of level sets as well as the Kendall distribution function, and provides a geometric interpretation for the latter in terms of the generator ψ. In Section 4 we derive various characterizations of pointwise convergence in $\mathcal{C}^d_{ar}$ in several steps and prove the main result saying that pointwise convergence implies weak conditional convergence (with respect to the first $d-1$ coordinates). Turning to the Williamson transform, in Section 5 we first establish some complementing results on the interrelation of the generator ψ and the Williamson measure γ, then characterize pointwise convergence in $\mathcal{C}^d_{ar}$ in terms of weak convergence of the Williamson measures, and finally show how regularity/singularity properties of γ carry over to C_γ. These properties are then used to prove the afore-mentioned denseness results and to construct singular Archimedean copulas with full support. The Clayton and the Gumbel families of Archimedean copulas will serve as simple accompanying examples underlining the obtained results. Notice, however, that the real strength of the obtained results consists in their validity outside parametric classes of Archimedean copulas (for which the results are much more straightforward to derive). Several additional examples and graphics illustrate the chosen procedures and some underlying ideas.

2. Notation and preliminaries

In the sequel we will let $\mathcal{C}^d$ denote the family of all d-dimensional copulas for some fixed integer $d \geq 3$ and write vectors in bold symbols. For each copula $C \in \mathcal{C}^d$ the corresponding d-stochastic measure will be denoted by μ_C, i.e., $\mu_C([\mathbf{0},\mathbf{x}]) = C(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{I}^d$, whereby $[\mathbf{0},\mathbf{x}] := [0,x_1] \times [0,x_2] \times \ldots \times [0,x_d]$ and $\mathbb{I} := [0,1]$. To keep notation as simple as possible we will frequently write $\mathbf{x}_{1:m} = (x_1,\ldots,x_m)$ for $\mathbf{x} \in \mathbb{I}^d$ and $m \leq d$. Considering $1 \leq i < j \leq d$, the i-j-marginal of C will be denoted by C^{ij}, i.e., we have $C^{ij}(x_i,x_j) = C(1,\ldots,1,x_i,1,\ldots,1,x_j,1,\ldots,1)$. In order to keep notation as simple as possible for every $m < d$ the marginal copula of the first m coordinates will be denoted by $C^{1:m}$, i.e., $C^{1:m}(x_1,x_2,\ldots,x_m) = C(x_1,x_2,\ldots,x_m,1,\ldots,1)$. Considering the uniform metric d_∞ on $\mathcal{C}^d$ it is well-known that $(\mathcal{C}^d, d_\infty)$ is a compact metric space and that in $\mathcal{C}^d$ pointwise and uniform convergence are equivalent. For more background on copulas and d-stochastic measures we refer to [3, 21].

For every metric space (S,d) the Borel σ-field on S will be denoted by $\mathcal{B}(S)$ and $\mathcal{P}(S)$ denotes the family of all probability measures on $\mathcal{B}(S)$. The Lebesgue measure on $\mathcal{B}(\mathbb{I}^d)$ will be denoted by λ or (whenever particular emphasis to the dimension d is required) by λ_d. Furthermore δ_x denotes the Dirac measure in $x \in S$. Given another metric space (S',d'), a Borel-measurable transformation $T: S \to S'$ and some $\vartheta \in \mathcal{P}(S)$ the push-forward of ϑ via T will be denoted by ϑ^T, i.e., $\vartheta^T(F) = \vartheta(T^{-1}(F))$ for all $F \in \mathcal{B}(S')$.

In what follows Markov kernels will play a prominent role. Markov kernels are well known from the context of Markov chains in discrete time, but they are also key for describing the extent of dependence of a random vector $\mathbf{Y}$ on another random vector $\mathbf{X}$. More precisely, given a $(d-m)$-dimensional random vector $\mathbf{Y}$ and an m-dimensional random vector $\mathbf{X}$ on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ we are interested in the conditional distributions $\mathbb{P}^{\mathbf{Y}|\mathbf{X}=x}$ of $\mathbf{Y}$ given $\mathbf{X} = \mathbf{x}$ even if $\mathbb{P}(\mathbf{X} = \mathbf{x}) = 0$ holds since - intuitively speaking - the more 'different' the marginal distribution $\mathbb{P}^{\mathbf{Y}}$ of $\mathbf{Y}$ and the conditional distributions $\mathbb{P}^{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$ of $\mathbf{Y}$ given $\mathbf{X} = \mathbf{x}$ (for 'many' $\mathbf{x} \in \mathbb{R}^m$) the more information on $\mathbf{Y}$ is gained by knowing $\mathbf{X}$. If $\mathbf{Y}$ and $\mathbf{X}$ are independent then $\mathbf{X}$ provides no information on $\mathbf{Y}$ and we have $\mathbb{P}^{\mathbf{Y}} = \mathbb{P}^{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$. The direct opposite is the case of so-called complete dependence describing the situation in which $\mathbb{P}^{\mathbf{Y}|\mathbf{X}=\mathbf{x}}$ is degenerated for $\mathbb{P}^{\mathbf{X}}$-almost every $\mathbf{x}$, implying that $\mathbb{P}(\mathbf{Y} = h \circ \mathbf{X}) = 1$ holds for some measurable function $h : \mathbb{R}^m \to \mathbb{R}^{d-m}$, i.e., knowing $\mathbf{X}$ means knowing $\mathbf{Y}$. Considering that the afore-mentioned conditional distributions are formally Markov kernels triggered the seemingly natural idea introduced in [25] and extended in [9] (and the references therein) to quantify dependence as average distance of the conditional distributions to the marginal distribution.

Formally speaking, an *m-Markov kernel* from $\mathbb{R}^m$ to $\mathbb{R}^{d-m}$ is a mapping $K : \mathbb{R}^m \times \mathcal{B}(\mathbb{R}^{d-m}) \to \mathbb{I}$ fulfilling that for every fixed $E \in \mathcal{B}(\mathbb{R}^{d-m})$ the mapping $\mathbf{x} \mapsto K(\mathbf{x}, E)$ is $\mathcal{B}(\mathbb{R}^m)$-$\mathcal{B}(\mathbb{R}^{d-m})$-measurable and for every fixed $\mathbf{x} \in \mathbb{R}^m$ the mapping $E \mapsto K(\mathbf{x}, E)$ is a probability measure on $\mathcal{B}(\mathbb{R}^{d-m})$. Given $\mathbf{Y}$ and $\mathbf{X}$ as above we call a Markov kernel $K(\cdot,\cdot)$ a regular conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$ if for every fixed $E \in \mathcal{B}(\mathbb{R}^{d-m})$ the identity

$$K(\mathbf{X}(\omega), E) = \mathbb{E}(\mathbf{1}_E \circ \mathbf{Y}|\mathbf{X})(\omega)$$

holds for $\mathbb{P}$-almost every $\omega \in \Omega$. In this case we have $K(\mathbf{x}, E) = \mathbb{P}(\mathbf{Y} \in E\,|\mathbf{X} = \mathbf{x})$ for every $E \in \mathcal{B}(\mathbb{R}^{d-m})$ and $\mathbb{P}^{\mathbf{X}}$-almost every $\mathbf{x} \in \mathbb{R}^m$. It is well-known that for each pair of random vectors $(\mathbf{X}, \mathbf{Y})$ as above, a regular conditional distribution $K(\cdot,\cdot)$ of $\mathbf{Y}$ given $\mathbf{X}$ exists and is unique for $\mathbb{P}^{\mathbf{X}}$-almost all $\mathbf{x} \in \mathbb{R}^m$. In case $(\mathbf{X}, \mathbf{Y})$ has $C \in \mathcal{C}^d$ as distribution function (restricted to $\mathbb{I}^d$) we let $K_C : \mathbb{I}^m \times \mathcal{B}(\mathbb{I}^{d-m}) \to \mathbb{I}$ denote (a version of) the regular conditional distribution of $\mathbf{Y}$ given $\mathbf{X}$ and simply refer to it as *m-Markov kernel* of C. Defining the $\mathbf{x}$-section $G_{\mathbf{x}}$ of a set $G \in \mathcal{B}(\mathbb{I}^d)$ w.r.t. the first m coordinates by $G_{\mathbf{x}} := \{\mathbf{y} \in \mathbb{I}^{d-m} : (\mathbf{x}, \mathbf{y}) \in G\} \in \mathcal{B}(\mathbb{I}^{d-m})$ the well-known *disintegration theorem* implies

$$\mu_C(G) = \int_{\mathbb{I}^m} K_C(\mathbf{x}, G_{\mathbf{x}})\, \mathrm{d}\mu_{C^{1:m}}(\mathbf{x}). \tag{2.1}$$

It is well-known that the disintegration theorem also holds for general finite measures in which case the conditional measures $K(\cdot,\cdot)$ are not necessarily probability measures but general finite measures (sub- or super Markov kernels). For more background on conditional expectation and disintegration we refer to [11, Section 5] and [16, Section 8].

An Archimedean *generator* ψ is a continuous, non-increasing function $\psi : [0,\infty) \to [0,1]$ fulfilling $\psi(0) = 1$, $\lim_{z\to\infty} \psi(z) = 0 =: \psi(\infty)$ and being strictly decreasing on the interval $[0, \inf\{z \in [0,\infty] : \psi(z) = 0\}]$ (with the convention $\inf \emptyset := \infty$). For every Archimedean generator ψ we will let $\varphi : [0,1] \to [0,\infty]$ denote its pseudo-inverse defined by $\varphi(y) := \inf\{z \in [0,\infty] : \psi(z) \geq y\} = \inf\{z \in [0,\infty] : \psi(z) = y\}$ for every $y \in [0,1]$, where the second equality holds since ψ is decreasing and continuous. Obviously φ is strictly decreasing on $[0,1]$ and fulfills $\varphi(1) = 0$, moreover it is straightforward to verify that φ is right-continuous at 0 (for a short discussion of this property see Section 4 in [13]). If $\varphi(0+) = \infty$ (or, equivalently, if $\psi(z) > 0$ for every $z \geq 0$), we refer to ψ (and φ) as *strict* and as *non-strict* otherwise. A copula $C \in \mathcal{C}^d$ is called *Archimedean* (in which case we write $C \in \mathcal{C}^d_{\mathrm{ar}}$) if there exists some Archimedean generator ψ with

$$C_\psi(\mathbf{x}) = \psi(\varphi(x_1) + \cdots + \varphi(x_d))$$

for every $\mathbf{x} \in \mathbb{I}^d$. Following [20], $C_\psi(\mathbf{x}) = \psi(\varphi(x_1) + \cdots + \varphi(x_d))$ is a d-dimensional copula if, and only if, ψ is a d-monotone Archimedean generator on $[0, \infty)$, i.e., if, and only if, ψ is an Archimedean generator fulfilling that $(-1)^{d-2}\psi^{(d-2)}$ exists on $(0, \infty)$, is non-negative, non-increasing and convex on $(0, \infty)$, whereby, as usual, $g^{(m)}$ denotes the m-th derivative of a function g. Moreover (again see [20]) it is straightforward to verify that in the latter case $(-1)^m\psi^{(m)}$ exists on $(0, \infty)$, is non-negative, non-increasing and convex on $(0, \infty)$ for every $m \in \{0, \dots, d-2\}$. In the following we will sometimes simply write C instead of C_ψ when no confusion may arise. Furthermore we will simply refer to Archimedean generators as 'generators' and refer to C_ψ as strict if ψ is strict.

Letting D^-g and D^+g denote the left- and right- hand derivative of a function g, respectively, convexity of $(-1)^{d-2}\psi^{(d-2)}$ implies that both, $D^-\psi^{(d-2)}(z)$ and $D^+\psi^{(d-2)}(z)$ exist for every $z \in (0, \infty)$ and that the two derivatives coincide outside an at most countable set (see [12, Theorem 3.7.4] and [22, Appendix C]) - in fact, for every continuity point z of $D^-\psi^{(d-2)}$ we have $D^-\psi^{(d-2)}(z) = D^+\psi^{(d-2)}(z)$. Moreover, every d-monotone generator ψ fulfills $\lim_{z\to\infty} \psi^{(m)}(z) = 0$ for every $m \in \{0, \dots, d-2\}$ as well as $\lim_{z\to\infty} D^-\psi^{(d-2)}(z) = 0$. Indeed, $\lim_{z\to\infty} \psi'(z) = 0$ directly follows from monotonicity and convexity of ψ since d-monotonicity implies that $-\psi'$ is decreasing and convex too; proceeding iteratively yields the assertion. Notice that Lemma 5.4 yields an even simpler direct proof of this assertion. In the sequel we will also use the fact that (again by convexity, see [12, Theorem 3.7.4] and [22, Appendix C]) we can reconstruct the generator ψ from its derivatives in the sense that ($m \in \{1, \dots, d-2\}$)

$$\psi^{(m-1)}(z) = \int_{[z,\infty)} -\psi^{(m)}(s) \, \mathrm{d}\lambda(s), \quad \psi^{(d-2)}(z) = \int_{[z,\infty)} -D^-\psi^{(d-2)}(s) \, \mathrm{d}\lambda(s). \tag{2.2}$$

In order to have a one-to-one correspondence between copulas and their generator we follow [13] and from now on implicitly assume that all generators are normalized in the sense that $\varphi(\frac{1}{2}) = 1$, or equivalently, $\psi(1) = \frac{1}{2}$ holds. Notice that this can always be achieved by choosing a constant $b > 0$ with $b\,\varphi(\frac{1}{2}) = 1$ (or, equivalently) $\psi(b) = \frac{1}{2}$ and considering $\tilde{\varphi}(z) = b\varphi(z)$ and $\tilde{\psi}(z) = \psi(bz)$, respectively.

According to [20], an Archimedean copula $C \in \mathcal{C}^d_{\mathrm{ar}}$ is absolutely continuous if, and only if, $\psi^{(d-1)}$ exists and is absolutely continuous on $(0, \infty)$. In this case a version of the density c of C is given by

$$c(\mathbf{x}) = \mathbf{1}_{(0,1)^d}(\mathbf{x}) \prod_{i=1}^{d} \varphi'(x_i) \cdot D^-\psi^{(d-1)}\big(\varphi(x_1) + \cdots + \varphi(x_d)\big). \tag{2.3}$$

In the sequel we will use the handy consequence that lower dimensional marginals of d-dimensional Archimedean copulas are absolutely continuous ([20, Proposition 4.1]).

In the following we will use the Clayton and the Gumbel families of Archimedean copulas as accompanying examples illustrating the obtained theoretical results in a simple parametric setting. Recall that the class of all *d-dimensional Clayton copulas* $\mathcal{C}^d_{CL}$ consists of all Archimedean copulas with generators $\psi(z) = (\theta z + 1)^{-\frac{1}{\theta}}$, implying $\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, for $\theta > 0$ and $z \in [0, \infty)$. Considering the afore-mentioned normalization property $\psi(1) = \frac{1}{2}$

(to assure a one-to-one correspondence between the copula and its generator) we work with the normalized generators $\psi(z) = ((2^\theta - 1)z + 1)^{-\frac{1}{\theta}}$ and $\varphi(t) = \frac{t^{-\theta}-1}{2^\theta-1}$ in the following.
The family of all *d-dimensional Gumbel copulas* $\mathcal{C}_{GU}^d$ contains all Archimedean copulas with generators $\psi(z) = \exp(-z^{\frac{1}{\alpha}})$ (hence $\varphi(t) = (-\log t)^\alpha$) for $\alpha \geq 1$ and $z \in [0, \infty)$. According to our normalization assumption we will therefore work with $\psi(z) = \exp(-\log(2)z^{\frac{1}{\alpha}})$ and $\varphi(t) = \frac{(-\log(t))^\alpha}{\log(2)^\alpha}$ in the sequel. Notice that both, generators of Clayton as well as generators of Gumbel copulas are strict.

3. Markov kernel, mass distribution and Kendall distribution function of multivariate Archimedean copulas

In the proceedings contribution [14] the authors derive an explicit expression for (a version of) the $(d-1)$-Markov kernel of d-variate Archimedean copulas which, in turn, allows to derive the well-known formulas for level-set mass and the Kendall distribution function of d-variate Archimedean copulas in an alternative way. Considering that Markov kernels are key for the rest of this paper we introduce a (slightly modified) version of the Markov kernel considered in [14] and prove its properties in detail. Furthermore, we restate the already known formulas for the Kendall distribution function and level-set mass, add a new geometric interpretation, and, for the sake of completeness, include the corresponding purely Markov kernel-based proofs in the Appendix.

For every $t \in (0, 1]$ define the t-level set of $C \in \mathcal{C}_{\mathrm{ar}}^d$ by

$$\begin{aligned} L_t &:= \big\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : C(\mathbf{x}, y) = t\big\} \\ &= \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) = \varphi(t)\right\} \end{aligned} \tag{3.1}$$

and for $t = 0$ set

$$\begin{aligned} L_0 &:= \big\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : C(\mathbf{x}, y) = 0\big\} \\ &= \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \geq \varphi(0)\right\}. \end{aligned} \tag{3.2}$$

Subsequently we will work with the level sets of the $(d-1)$-dimensional marginal of C defined analogously and denote them by $L_t^{1:d-1}$ and $L_0^{1:d-1}$, respectively. As in the bivariate setting (see [13]) we can define functions f^t whose graph coincides with L^t for $t \in (0, 1]$ and with the boundary of L^0 for $t = 0$: In fact, for $t = 0$ defining the function $f^0 : \mathbb{I}^{d-1} \to \mathbb{I}$ by

$$f^0(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in L_0^{1:d-1} \\ \psi\left(\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i)\right) & \text{if } \mathbf{x} \notin L_0^{1:d-1} \end{cases}$$

with the conventions $\psi(\infty) = 0$ as well as $\psi(u) = 1$ for all $u < 0$ and for $t \in (0, 1]$, defining the upper t-cut of $C^{1:d-1}$ by $[C^{1:d-1}]_t = \{\mathbf{x} \in \mathbb{I}^{d-1} : C^{1:d-1}(\mathbf{x}) \geq t\}$ and considering the function $f^t : [C^{1:d-1}]_t \to \mathbb{I}$ given by

$$f^t(\mathbf{x}) := \psi\left(\varphi(t) - \sum_{i=1}^{d-1} \varphi(x_i)\right).$$

yields the above-mentioned property. It is straightforward to verify that for $\mathbf{x} \notin L_0^{1:d-1}$ and $y < f^0(\mathbf{x})$ we have $(\mathbf{x}, y) \in L_0$ and that for strict Archimedean copulas $\mathbf{x} \in L_0^{1:d-1}$ if, and only if, $M(\mathbf{x}) = 0$ where M denotes the d-dimensional minimum copula.

Theorem 3.1. *Suppose that $C \in \mathcal{C}_{ar}^d$ has generator ψ. Then setting*

$$K_C(\mathbf{x}, [0, y]) := \begin{cases} 1, & M(\mathbf{x}) = 1 \text{ or } \mathbf{x} \in L_0^{1:d-1} \\ 0, & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y < f^0(\mathbf{x}) \\ \frac{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)\right)}{D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi(x_i)\right)}, & M(\mathbf{x}) < 1, \mathbf{x} \notin L_0^{1:d-1}, y \geq f^0(\mathbf{x}) \end{cases} \tag{3.3}$$

yields (a version of) the $(d-1)$-Markov kernel of C.

Proof. First notice that absolute continuity of $C^{1:d-1}$ implies

$$0 = \mu_{C^{1:d-1}}(L_0^{1:d-1}) = \mu_C(L_0^{1:d-1} \times \mathbb{I}).$$

so $\mu_{C^{1:d-1}}(L_0^{1:d-1} \cup M^{-1}(\{1\})) = 0$, so the first condition on the right hand side of equation (3.3) only holds for a set of $\mu_{C^{1:d-1}}$-measure 0.

We start by showing that K_C defined according to equation (3.3) is indeed a $(d-1)$-Markov kernel and then prove that it is a Markov kernel of C. Fix $y \in \mathbb{I}$. Since measurability of f^0 and $D^-\psi^{(d-2)}$ are a direct consequence of the properties of ψ, using continuity of φ yields measurability of the mapping $\mathbf{x} \mapsto K_C(\mathbf{x}, [0, y])$. Building upon that, applying a standard Dynkin System argument (see [4, Theorem 2]) yields measurability of $\mathbf{x} \mapsto K_C(\mathbf{x}, E)$ for every Borel set $E \in \mathcal{B}(\mathbb{I})$.
For $\mathbf{x} = 1$ or fixed $\mathbf{x} \in L_0^{1:d-1}$ the map $y \mapsto K_C(\mathbf{x}, [0, y])$ is obviously a univariate distribution function. Considering $\mathbf{x} \notin L_0^{1:d-1}$, monotonicity and left-continuity of $(-1)^{d-2} D^-\psi^{(d-2)}$ implies that $y \mapsto K_C(\mathbf{x}, [0, y])$ is increasing and right-continuous. Moreover we obviously have $K_C(\mathbf{x}, [0, 1]) = 1$, so $y \mapsto K_C(\mathbf{x}, [0, y])$ is a univariate distribution function and it remains to show that it is a $(d-1)$-Markov kernel of C, i.e., that

$$C(\mathbf{x}, y) = \int_{[\mathbf{0}, \mathbf{x}]} K_C(\mathbf{s}, [0, y]) \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{s}) \tag{3.4}$$

holds for all $\mathbf{x} \in \mathbb{I}^{d-1}$ and every $y \in \mathbb{I}$.
The case $y = 1$ is trivial and for $y = 0$ we have that $K_C(\mathbf{x}, \{0\}) = 0$ for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x}$, implying that equation (3.4) holds. For $y \in (0, 1)$ considering that $L_0^{1:d-1}$ is a $\mu_{C^{1:d-1}}$-null set and using absolute continuity of $\mu_{C^{1:d-1}}$ yields $(\Upsilon := \mathbb{I}^{d-1} \setminus L_0^{1:d-1})$

$$
\begin{aligned}
I &:= \int_{[\mathbf{0},\mathbf{x}]\cap\Upsilon} K_C(\mathbf{s},[0,y])\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{s}) = \int_{[\mathbf{0},\mathbf{x}]\cap\Upsilon} K_C(\mathbf{s},[0,y]) c^{1:d-1}(\mathbf{s})\ \mathrm{d}\lambda(\mathbf{s}) \\
&= \int_{\Upsilon\cap\{\mathbf{t}\in(0,1)^{d-1}: y\geq f^0(\mathbf{t})\}\cap[\mathbf{0},\mathbf{x}]} \frac{D^-\psi^{(d-2)}(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y))}{D^-\psi^{(d-2)}(\sum_{i=1}^{d-1}\varphi(s_i))} \\
&\qquad \cdot \prod_{i=1}^{d-1}\varphi'(s_i) D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)\right)\ \mathrm{d}\lambda(\mathbf{s}) \\
&= \int_{\{\mathbf{t}\in(0,1)^{d-1}\setminus L_0^{1:d-1}:\, y\geq f^0(\mathbf{t})\}\cap[\mathbf{0},\mathbf{x}]} D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right) \\
&\qquad \cdot \prod_{i=1}^{d-1}\varphi'(s_i)\ \mathrm{d}\lambda(\mathbf{s}).
\end{aligned}
$$

Notice that on the one hand, for $\mathbf{s}\notin L_0^{1:d-1}$ we have $y<f^0(\mathbf{s})$ if, and only if, $\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)>\varphi(0)$, implying $D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right)=0$, and, on the other hand, $\mathbf{s}\in L_0^{1:d-1}$ also yields $D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right)=0$. Therefore in the strict and the non-strict case we get

$$
\begin{aligned}
I &= \int_{(\mathbf{0},\mathbf{x}]\setminus L_0^{1:d-1}} D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right)\cdot\prod_{i=1}^{d-1}\varphi'(s_i)\ \mathrm{d}\lambda(\mathbf{s}) \\
&= \int_{(\mathbf{0},\mathbf{x}]\cap(0,1)^{d-1}} D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right)\cdot\prod_{i=1}^{d-1}\varphi'(s_i)\ \mathrm{d}\lambda(\mathbf{s}) \\
&= \int_{(\mathbf{0},\mathbf{x}_{1:d-2}]}\prod_{i=1}^{d-2}\varphi'(s_i)\int_{(0,x_{d-1}]}\varphi'(s_{d-2}) \\
&\qquad \cdot D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(s_i)+\varphi(y)\right)\mathrm{d}\lambda(s_{d-1})\mathrm{d}\lambda(\mathbf{s}_{1:d-2}).
\end{aligned}
$$

Hence, using change of coordinates together with the fact that $\psi^{(m)}(\infty)=0$ holds for $m=1,2,\ldots,d-2$ it follows that

$$
\begin{aligned}
I &= \int_{(\mathbf{0},\mathbf{x}_{1:d-2}]}\prod_{i=1}^{d-2}\varphi'(s_i)\lim_{\Delta\to 0}\Bigg\{\psi^{(d-2)}\left(\sum_{i=1}^{d-2}\varphi(s_i)+\varphi(x_{d-1})+\varphi(y)\right) \\
&\qquad -\psi^{(d-2)}\left(\sum_{i=1}^{d-2}\varphi(s_i)+\varphi(\Delta)+\varphi(y)\right)\Bigg\}\mathrm{d}\lambda(\mathbf{s}_{1:d-2}) \\
&= \int_{(\mathbf{0},\mathbf{x}_{1:d-2}]}\prod_{i=1}^{d-2}\varphi'(s_i)\psi^{(d-2)}\left(\sum_{i=1}^{d-2}\varphi(s_i)+\varphi(x_{d-1})+\varphi(y)\right)\mathrm{d}\lambda(\mathbf{s}_{1:d-2}).
\end{aligned}
$$

Proceeding analogously $d-3$ times finally yields

$$I = \int_{(0,x_1]} \varphi'(s_1)\psi'\left(\varphi(s_1) + \varphi(x_2) + \cdots + \varphi(x_{d-1}) + \varphi(y)\right) \mathrm{d}\lambda(s_1)$$
$$= \psi\left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)\right) = C(\mathbf{x}, y)$$

as desired. □

Remark 3.2. Note that our Markov kernel in equation (3.3) is a slightly modified version of the one considered in [14] and constructed in order to keep notation as simple as possible. Contrary to [14] we merged the cases where the Markov-kernel is equal to 1 and directly work with the full level set L_0^{d-1} instead of considering the interior $\mathrm{int} L_0^{d-1}$.

Remark 3.3. As mentioned in the proof above $\mu_{C^{1:d-1}}(L_0^{1:d-1}) = 0$, implying that in the first line in equation (3.3) we could substitute the univariate distribution function $y \mapsto 1$ by any other univariate distribution function F fulfilling $F(1) = 1$.

Remark 3.4. It also seems feasible to consider m-kernels for $m \in \{2, \ldots, d-2\}$ instead of $(d-1)$-kernels, i.e., to condition on m instead of $d-1$ coordinates. As shown by the subsequent results, however, although (d-1)-kernels involve the highest derivatives of the generator they are easy to handle and provide various new results, particularly with respect to the interplay with the Williamson transform as discussed in Section 5.

Example 3.5 (Clayton and Gumbel families, cont.)**.** We first derive the Markov-kernel of a three-dimensional Clayton copula $C \in \mathcal{C}_{CL}^3$ with parameter $\theta > 0$. Strictness implies that the zero level set of the marginal copula $C^{1:2}$ is given by $L_0^{1:2} = (\{0\} \times \mathbb{I}) \cup (\mathbb{I} \times \{0\})$ and that $f^0(x_1, x_2) = 0$ holds for all $(x_1, x_2) \notin L_0^{1:2}$. Calculating the derivatives of ψ and applying Theorem 3.1 yields that (we will write $K_C(x_1, x_2, \cdot)$ instead of $K_C((x_1, x_2), \cdot)$ to keep notation simple)

$$K_C(x_1, x_2, [0, y]) = \begin{cases} 1, & M(x_1, x_2) = 1 \text{ or } (x_1, x_2) \in L_0^{1:2} \\ \frac{(x_1^{-\theta} + x_2^{-\theta} + y^{-\theta} - 2)^{-\frac{1}{\theta} - 2}}{(x_1^{-\theta} + x_2^{-\theta} - 1)^{-\frac{1}{\theta} - 2}}, & M(x_1, x_2) < 1, (x_1, x_2) \notin L_0^{1:2}. \end{cases}$$

Turning towards the Gumbel family consider that $C \in \mathcal{C}_{GU}^3$ with parameter $\alpha \geq 1$. Calculating the second derivative yields

$$\psi''(z) = \left(\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) + \frac{1}{\alpha^2} z^{\frac{1}{\alpha}} \log(2)\right) \log(2) z^{\frac{1}{\alpha} - 2} \exp(-\log(2) z^{\frac{1}{\alpha}})$$

for $z > 0$ and we get

$$K_C(x_1, x_2, [0, y]) = \begin{cases} 1, & M(x_1, x_2) = 1 \text{ or } (x_1, x_2) \in L_0^{1:2} \\ \frac{A(x_1, x_2, y)}{A(x_1, x_2, 1)}, & M(x_1, x_2) < 1, (x_1, x_2) \notin L_0^{1:2}, \end{cases}$$

where A is given by

$$A(x_1,x_2,y) = \left(\left(\frac{1}{\alpha}-\frac{1}{\alpha^2}\right)+\frac{1}{\alpha^2}\left((-\log(x_1))^\alpha+(-\log(x_2))^\alpha+(-\log(y))^\alpha\right)^{\frac{1}{\alpha}}\right)$$
$$\left((-\log(x_1))^\alpha+(-\log(x_2))^\alpha+(-\log(y))^\alpha\right)^{\frac{1}{\alpha}-2}$$
$$\exp\left(-\left((-\log(x_1))^\alpha+(-\log(x_2))^\alpha+(-\log(y))^\alpha\right)^{\frac{1}{\alpha}}\right).$$

Figure 1 depicts the conditional distribution functions $y\mapsto K_C(x_1,x_2,[0,y])$ and $(x_1,x_2)=(0.3,0.7)$ for some Clayton and some Gumbel copulas.

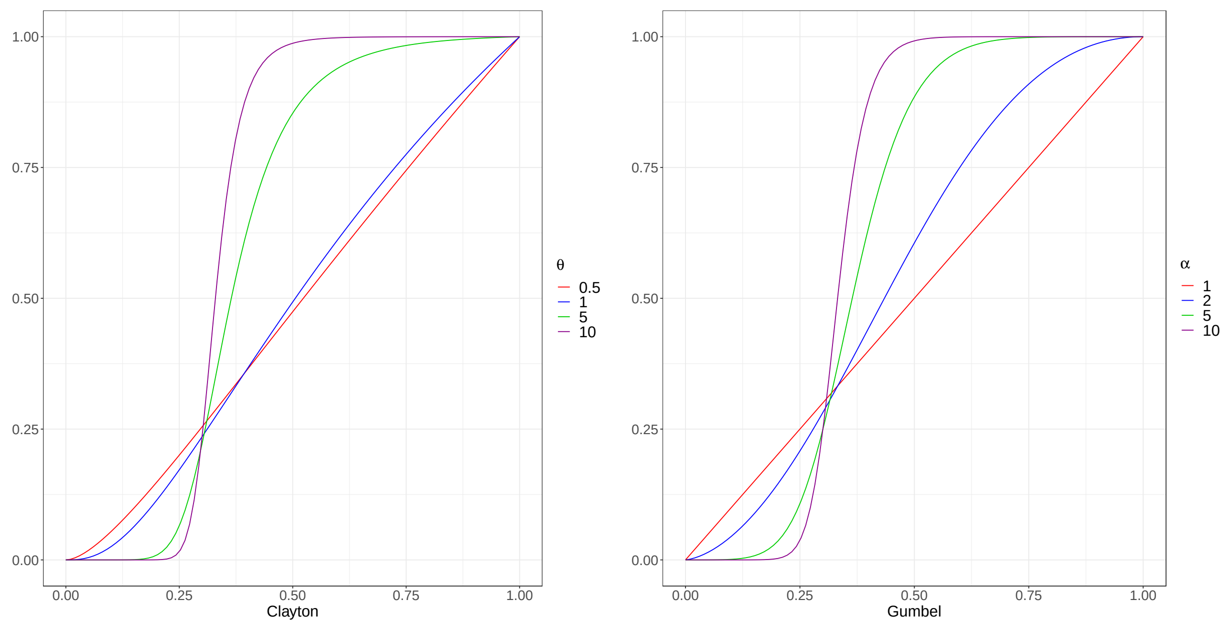

Figure 1: Conditional distribution functions $y\mapsto K_C(x_1,x_2,[0,y])$ for $x_1=0.3$ and $x_2=0.7$ of the Clayton copula with parameter $\theta=\frac{1}{2}$, $\theta=1$, $\theta=5$, $\theta=10$ (left panel) and of the Gumbel copula with parameter $\alpha=1$, $\alpha=2$, $\alpha=5$, $\alpha=10$ (right panel) as considered in Example 3.5.

Utilizing Theorem 3.1 allows to derive the well-known formulas (see [20]) for the level set masses and the Kendall distribution function of multivariate Archimedean copulas easily via Markov kernels and disintegration (see Propositions Appendix A.1 and Appendix A.2 in the Appendix). In fact, for $t\in(0,1]$ the identity

$$\mu_C(L_t)=\frac{(-\varphi(t))^{d-1}}{(d-1)!}\cdot\left(D^-\psi^{(d-2)}(\varphi(t))-D^-\psi^{(d-2)}(\varphi(t-))\right). \tag{3.5}$$

can be shown. Moreover, if C is strict then $\mu_C(L_0)=0$ and for non-strict C

$$\mu_C(L_0) \;=\frac{(-\varphi(0))^{d-1}}{(d-1)!}\cdot D^-\psi^{(d-2)}(\varphi(0)) \tag{3.6}$$

holds. Letting F_K^d denote the Kendall distribution function of C, for $t>0$ we have

$$F_K^d(t)=D^-\psi^{(d-2)}(\varphi(t))\frac{(-1)^{d-1}}{(d-1)!}\varphi(t)^{d-1}+\sum_{k=0}^{d-2}\psi^{(k)}(\varphi(t))\frac{(-1)^k}{k!}\varphi(t)^k. \tag{3.7}$$

Moreover for $t = 0$ and strict C we have $F_K^d(0) = 0$, and for non-strict C

$$F_K^d(0) = D^-\psi^{(d-2)}(\varphi(0)) \cdot \frac{(-1)^{d-1}}{(d-1)!} \cdot \varphi(0)^{d-1}. \tag{3.8}$$

holds.

Example 3.6 (Clayton and Gumbel families, cont.)**.** We calculate the level set masses and the Kendall distribution function for the Clayton and Gumbel families. Obviously generators ψ of Clayton/Gumbel copula are twice continuously differentiable on $(0, \infty)$, so

$$\mu_C(L_t) = 0$$

holds for every $C \in \mathcal{C}^3_{CL} \cup \mathcal{C}^3_{GU}$ and arbitrary $t \in (0, 1]$. Furthermore, strictness implies

$$\mu_C(L_0) = 0.$$

Having that, for $C \in \mathcal{C}^3_{CL}$ the Kendall distribution function F_K^3 is given by

$$\begin{aligned} F_K^3(t) &= \psi(\varphi(t)) - \psi'(\varphi(t))\varphi(t) + \frac{1}{2}\psi''(\varphi(t))\varphi(t)^2 \\ &= t + (t^{-\theta} - 1)t^{1+\theta}\tfrac{1}{\theta}\left[\tfrac{1}{2}(1 + \tfrac{1}{\theta})(1 - t^{\theta}) + 1\right] \end{aligned}$$

for $t > 0$ and for $t = 0$ we have $F_k^3(0) = 0$. Analogously, for $C \in \mathcal{C}^3_{GU}$ we obtain

$$\begin{aligned} F_K^3(t) &= \psi(\varphi(t)) - \psi'(\varphi(t))\varphi(t) + \frac{1}{2}\psi''(\varphi(t))\varphi(t)^2 \\ &= t + \frac{t\log(t)(\log(t) + 1 - 3\alpha)}{2\alpha^2} \end{aligned}$$

as well as $F_K^3(0) = 0$. Figure 2 depicts some Kendall distribution functions of Clayton and Gumbel copulas.

Remark 3.7. In the bivariate setting it is well-known that the formula for the level set masses can nicely be interpreted geometrically (see [4] and [21, Chapter 4.3]). In fact, following [4], given a discontinuity b of $D^+\varphi$ we have

$$\mu_C(L_b) = \varphi(b) \cdot \left(\frac{1}{D^-\varphi(b)} - \frac{1}{D^+\varphi(b)}\right),$$

i.e., $\mu_C(L_b)$ corresponds to the length of the line segment on the x-axis generated by the left-hand and right-hand tangents of φ at b (see Figure 1 in [4]). Translating from φ and x-axis to ψ and y-axis yields as special case of equation (3.5)

$$\mu_C(L_b) = \varphi(b) \cdot \left(D^-\psi(\varphi(b)) - D^-\psi(\varphi(b-))\right).$$

To establish the multivariate geometric analogue, rather than tangent lines we can consider the left and right hand Taylor polynomials of ψ of order $d-1$ at $a \in [0, \infty)$, i.e.,

$$T^{\pm}_{d-1}\psi(z, a) := D^{\pm}\psi^{(d-2)}(a) \cdot \frac{(z-a)^{d-1}}{(d-1)!} + \sum_{k=0}^{d-2} \psi^{(k)}(a) \cdot \frac{(z-a)^k}{k!}.$$

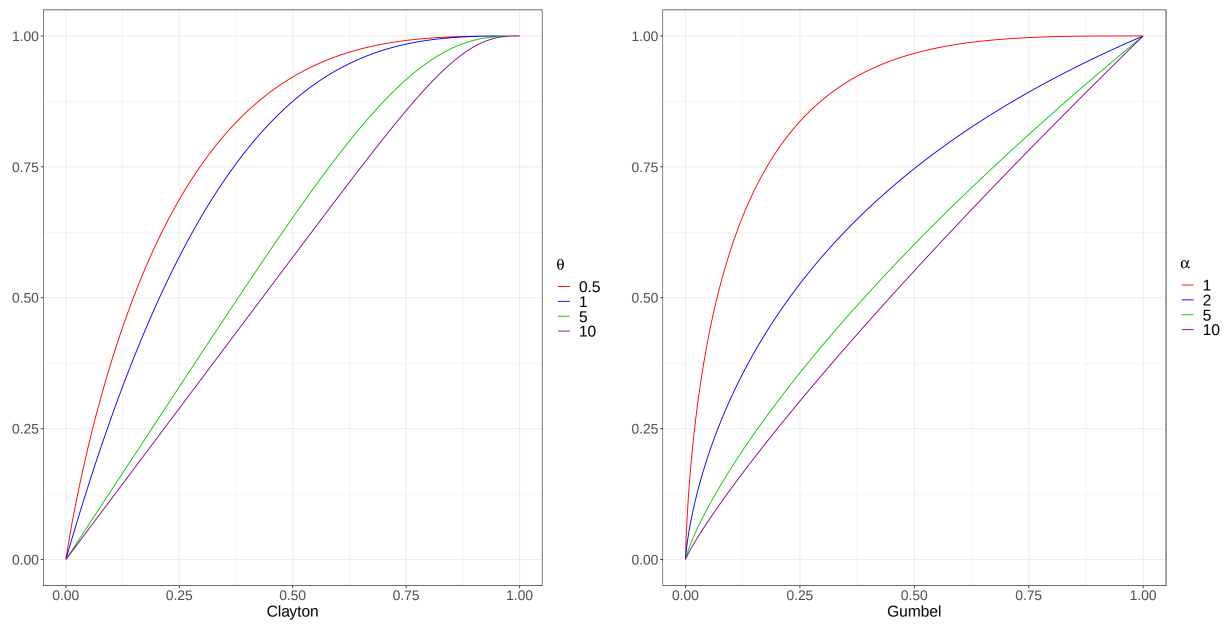

Figure 2: Kendall distribution function F_K^3 of a Clayton copula for parameters $\theta = \frac{1}{2}$, $\theta = 1$, $\theta = 5$, $\theta = 10$ (left panel) and of a Gumbel copula for parameters $\alpha = 1$, $\alpha = 2$, $\alpha = 5$, $\alpha = 10$ (right panel).

Having that yields

$$\begin{aligned} T_{d-1}^{-}\psi(0,\varphi(t)) &= D^{-}\psi^{(d-2)}(\varphi(t))\frac{(-1)^{d-1}\varphi(t)^{d-1}}{(d-1)!} + \sum_{k=0}^{d-2}\psi^{(k)}(\varphi(t))\frac{(-1)^k\varphi(t)^k}{k!} \\ &= F_K^d(t) \end{aligned}$$

and (3.5) reduces to

$$\mu_C(L_t) = T_{d-1}^{-}\psi(0,\varphi(t)) - T_{d-1}^{+}\psi(0,\varphi(t)) = F_K^d(t) - F_K^d(t-),$$

i.e., the t-level set mass corresponds to the difference of y-intercepts of the left and right hand Taylor polynomials of ψ of order $d-1$ at $\varphi(t)$. Panel 2 of Figure 3 in Example 3.8 illustrates this interpretation for $d = 3$.

Example 3.8. We construct an Archimedean generator ψ inducing some $C_\psi \in \mathcal{C}_{\mathrm{ar}}^3$ with $z_0 = 1$ being a discontinuity point of $(D^-\psi')$ and provide a geometric interpretation of $\mu_C(L_{z_0})$ in terms of ψ. As depicted in the left panel of Figure 3 we start with some non-negative, convex and decreasing function $(-1)\tilde{\psi}'$ (gray) with $D^+\tilde{\psi}'(z_0) \neq D^-\tilde{\psi}'(z_0)$. Considering

$$\tilde{\psi}(z) := \int_{[z,\infty)} (-1)\tilde{\psi}'(s)\ \mathrm{d}\lambda(s)$$

as well as setting $\psi(z) = \tilde{\psi}(z)/\tilde{\psi}(0)$ yields a generator of a three-dimensional Archimedean copula C_ψ (gray curve in the right panel). The length of the vertical red line segment formed by the left and right hand Taylor polynomials of order 2 at z_0 (right panel) coincides with $\mu_C(L_{z_0})$.

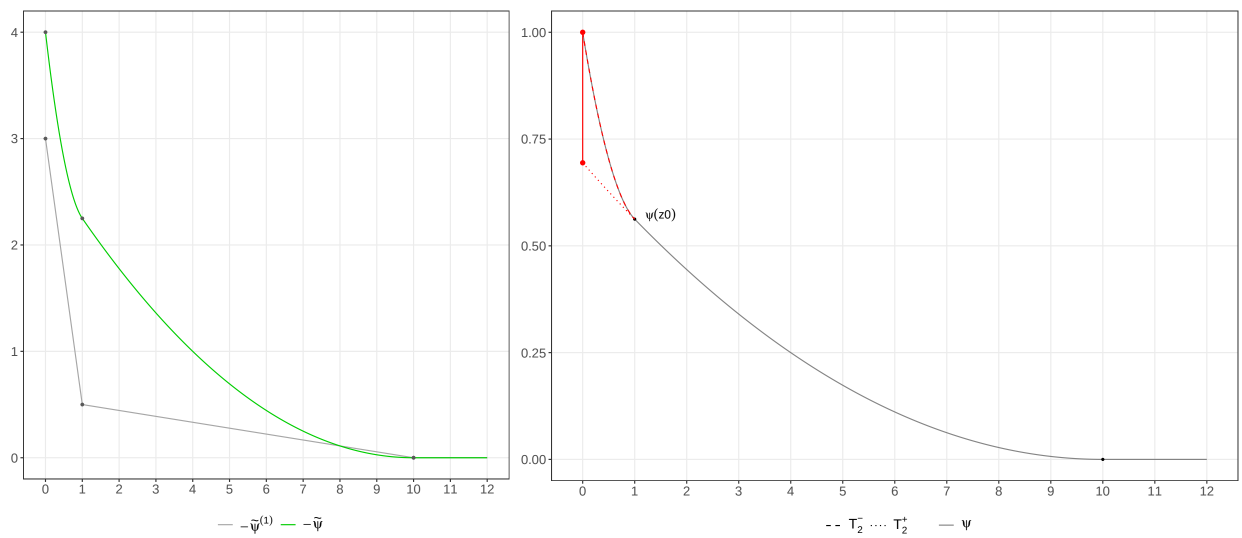

Figure 3: Stepwise construction of a non-strict 3-monotone Archimedean generator and geometric interpretation of the z_0-level set mass in terms of the vertical red line segment formed by the two Taylor parabolas in z_0 as considered in Example 3.8.

4. Characterizing pointwise convergence in $\mathcal{C}_{ar}^d$ and the interrelation with weak conditional convergence

In [2, Section 6] it was shown that within the class of bivariate Archimedean copulas pointwise convergence and weak conditional convergence (defined as weak conditional convergence of λ-almost all conditional distributions) are equivalent. Moreover, it was shown that in the general bivariate setting, weak conditional convergence implies pointwise convergence not necessarily vice versa. We now tackle the question whether analogous results hold in the general $d \geq 3$-dimensional setting. Considering that - contrary to the bivariate case $d = 2$ - for two copulas $A, B \in \mathcal{C}^d$ we do not necessarily have $A^{1:d-1} = B^{1:d-1}$ we first need to discuss potential extensions of the notion of weak conditional convergence to $\mathcal{C}^d$.

The seemingly most natural approach would be to say that a sequence $(A_n)_{n\in\mathbb{N}}$ of d-dimensional copulas converges *weakly conditional* to $A \in \mathcal{C}^d$ if, and only if, there exists a set Λ with $\mu_{A^{1:d-1}}(\Lambda) = 1$ such that for every $\mathbf{x} \in \Lambda$ the sequence $(K_{A_n}(\mathbf{x}, \cdot))_{n\in\mathbb{N}}$ of probability measures on $\mathcal{B}(\mathbb{I})$ converges weakly to the probability measure $K_A(\mathbf{x}, \cdot)$. As pointed out in [9], however, for $A, B \in \mathcal{C}^d$ we might even have that $A^{1:d-1}, B^{1:d-1}$ (or, more precisely, the measures $\mu_{A^{1:d-1}}, \mu_{B^{1:d-1}}$) are singular with respect to each other, making it unreasonable to compare $K_A(\mathbf{x}, \cdot)$ and $K_B(\mathbf{x}, \cdot)$ since they are only defined uniquely $\mu_{A^{1:d-1}}$-a.e. and $\mu_{B^{1:d-1}}$-a.e., respectively (see [9, Example 4.10] for an illustration of this scenario).

As a consequence, the afore-mentioned natural concept of weak conditional convergence does not yield a reasonable notion in full generality. For certain families of copulas such as classes of copulas with identical marginals, or for Archimedean copulas, however, considering weak conditional convergence does make sense. In fact, given $C_1, C_2 \in \mathcal{C}_{\mathrm{ar}}^d$ we already know that $C_1^{1:d-1}$ and $C_2^{1:d-1}$ are both not only absolutely continuous but according to equation (2.3) the corresponding densities $c_1^{1:d-1}, c_2^{1:d-1}$ fulfill $c_i^{1:d-1}(\mathbf{x}) > 0$ for every $\mathbf{x}$ outside of the respective zero set $L_0^{1:d-1}$, so it can not happen that $C_1^{1:d-1}$ and $C_2^{1:d-1}$ are singular w.r.t.

each other.

In what follows we will therefore work with the afore-mentioned notion of weak conditional convergence in $\mathcal{C}_{ar}^d$ and prove the following main result of this section in several steps:

Theorem 4.1. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$, respectively. Then the following assertions are equivalent (*$\mathrm{Cont}(g)$ *denotes the set of continuity points of a function g):*

1. $(C_n)_{n\in\mathbb{N}}$ *converges uniformly to C.*
2. $(C_n^{i:j})_{n\in\mathbb{N}}$ *converges uniformly to $C^{i:j}$ for all $i, j \in \{1, \ldots, d\}$ with $i \neq j$.*
3. $(\varphi_n)_{n\in\mathbb{N}}$ *converges pointwise to φ on* $(0, 1]$.
4. $(\psi_n)_{n\in\mathbb{N}}$ *converges uniformly to ψ on* $[0, \infty)$.
5. $(\psi_n^{(m)})_{n\in\mathbb{N}}$ *converges pointwise to $\psi^{(m)}$ on $(0, \infty)$ for every $m \in \{1, 2, \ldots, d-2\}$ and $(D^-\psi_n^{(d-2)})_{n\in\mathbb{N}}$ converges pointwise to $D^-\psi^{(d-2)}$ on the set* $\mathrm{Cont}(D^-\psi^{(d-2)})$.
6. $(c_n^{1:m})_{n\in\mathbb{N}}$ *converges to $c^{1:m}$ λ_m-almost everywhere in $\mathbb{I}^m$ for every $m \in \{2, 3, \ldots, d-1\}$.*

Furthermore, any of the six assertions implies weak conditional convergence of $(C_n)_{n\in\mathbb{N}}$ to C.

Obviously, assuming uniform convergence of $(C_n)_{n\in\mathbb{N}}$ to C yields convergence of all marginal copulas and, in particular, of the bivariate marginals. Thus, the equivalence of (1), (2) and (3) follows directly from the results in the two-dimensional setting established in [13].

Remark 4.2. Theorem 4.1 constitutes the natural extension of [13, Theorem 4.2] characterizing uniform convergence within the space of bivariate Archimedean copulas. In the multivariate setting, however, the afore-mentioned notion of weak conditional convergence in $\mathcal{C}_{ar}^d$ is a consequence of rather than an equivalence to uniform convergence. Slightly modifying the notion and incorporating the marginal densities, however, an equivalence can be established, see point (5) in Theorem 4.9 at the end of this section.

In what follows, we show the equivalence of the assertions (4), (5) and (6) and then conclude the section by deriving weak conditional convergence in several steps. We start with the following lemma clarifying the relationship between convergence of the generators and their pseudo-inverses. The proof of an analogous result in the context of t-norms with multiplicative generators can be found in [15, Theorem 8.14].

Lemma 4.3. *Suppose that $\psi, \psi_1, \psi_2, \ldots$ are Archimedean generators with pseudo-inverses $\varphi, \varphi_1, \varphi_2, \ldots$, respectively. Then $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to φ on* $(0, 1]$ *if, and only if, $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ uniformly on* $[0, \infty)$.

Proof. Defining

$$\tilde{\psi}(w) := \begin{cases} \psi(-w) & \text{if } w \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

yields a univariate distribution function $\tilde{\psi}$ on $\mathbb{R}$ whose pseudo-inverse $(\tilde{\psi})^- : (0,1] \to \mathbb{R}$ coincides with $-\varphi$. Considering that weak convergence of distribution functions is equivalent to weak convergence of their pseudo-inverses (see [26, Lemma 21.2]), using continuity of the involved functions it directly follows that pointwise convergence of $\tilde{\psi}_n$ to $\tilde{\psi}$ on $\mathbb{R}$ is equivalent to pointwise convergence of $-\varphi_n$ to $-\varphi$ on $(0,1]$ for $n \to \infty$. In other words: $\psi_n \to \psi$ pointwise on $[0,\infty)$ if, and only if, $\varphi_n \to \varphi$ pointwise on $(0,1]$. Finally, uniform convergence of $\tilde{\psi}_n$ to $\tilde{\psi}$ on $\mathbb{R}$ is a direct consequence of the fact that pointwise convergence of a sequence of univariate distribution functions to a continuous distribution function implies uniform convergence. Having that, obviously uniform convergence of $(\psi_n)_{n\in\mathbb{N}}$ to ψ on $[0,\infty)$ follows. □

The next lemma focuses on convergence of the derivatives of the Archimedean generators. Thereby we say that a sequence $(f_n)_{n\in\mathbb{N}}$ of real functions $(f_n)_{n\in\mathbb{N}}$ *converges continuously* to f if for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to $x \in \mathrm{Cont}(f)$ we have $\lim_{n\to\infty} f_n(x_n) = f(x)$ (cf. [17, Definition 3.18.1]).

Lemma 4.4. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$, respectively. If $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C then for every $m \in \{0,1,2,\ldots,d-2\}$ we have*

$$\lim_{n\to\infty} \psi_n^{(m)}(z) = \psi^{(m)}(z).$$

for every $z \in (0,\infty)$ and $\lim_{n\to\infty} D^-\psi_n^{(d-2)}(z) = D^-\psi^{(d-2)}(z)$ for every $z \in \mathrm{Cont}(D^-\psi^{(d-2)})$. Moreover, in both situations the convergence is continuous, i.e., for every sequence $(z_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ converging to $z \in \mathrm{Cont}(D^-\psi^{(d-2)})$ it holds that $\lim_{n\to\infty} D^-\psi_n^{(d-2)}(z_n) = D^-\psi^{(d-2)}(z)$. Vice versa, if $(\psi_n^{(m)})_{n\in\mathbb{N}}$ converges pointwise to $\psi^{(m)}$ on $(0,\infty)$ for some $m \in \{1,2,\ldots,d-2\}$ then $(C_n)_{n\in\mathbb{N}}$ converges pointwise to C and the same holds in case $\lim_{n\to\infty} D^-\psi_n^{(d-2)}(z) = D^-\psi^{(d-2)}(z)$ for every $z \in \mathrm{Cont}(D^-\psi^{(d-2)})$.

Proof. We already know that pointwise convergence of $(C_n)_{n\in\mathbb{N}}$ to C is equivalent to uniform convergence of $(\psi_n)_{n\in\mathbb{N}}$ to ψ on $[0,\infty)$. Convexity of generators implies that the sequence $(\psi_n')_{n\in\mathbb{N}}$ converges pointwise to ψ' on $\mathrm{Cont}(\psi') = (0,\infty)$ (see, e.g., [23, Theorem 25.7]). Considering that $(-1)^m\psi^{(m)}, (-1)^m\psi_1^{(m)}, (-1)^m\psi_2^{(m)}, \ldots$ are convex functions for every $m \in \{1,\ldots,d-2\}$ too, applying [23, Theorem 25.7], continuous convergence follows. The fact that $(D^-\psi_n^{(d-2)})_{n\in\mathbb{N}}$ converges continuously to $D^-\psi^{(d-2)}$ on $\mathrm{Cont}(D^-\psi^{(d-2)})$ is a consequence of Lemma Appendix A.3 in the Appendix.
We prove the reverse implication only for the case $d = 3$ and assume that $(D^-\psi_n^{(d-2)})_{n\in\mathbb{N}}$ converges to $D^-\psi^{(d-2)}$ on $\mathrm{Cont}(D^-\psi^{(d-2)})$ since the analogous statement for $m \in \{1,\ldots,d-2\}$ follows in the same manner and the extension to arbitrary $d \geq 3$ is obvious. For $z \in [0,\infty)$ according to equation (2.2) we have

$$\begin{aligned}\psi(z) &= \int_{[z,\infty)} -\psi'(s)\mathrm{d}\lambda(s) = \int_{[z,\infty)} - \int_{[s,\infty)} -D^-\psi'(t)\mathrm{d}\lambda(t)\mathrm{d}\lambda(s) \\ &= \int_{[z,\infty)} \int_{[s,\infty)} \underbrace{D^-\psi'(t)}_{\geq 0} \mathrm{d}\lambda(t)\mathrm{d}\lambda(s)\end{aligned}$$

and the same holds for every ψ_n. Using $\psi(0) = \psi_n(0) = 1$ we can interpret the functions $\iota_n, \iota : \Delta \to [0,\infty)$, defined by

$$\iota_n(t,s) := D^-\psi_n'(s), \quad \iota(t,s) := D^-\psi'(s)$$

as probability densities on the measure space $(\Delta, \mathcal{B}(\Delta), \lambda_2)$ with Δ denoting the closed set $\Delta := \{(x,y) \in [0,\infty)^2 : y \geq x\}$. By assumption, the sequence $(\iota_n)_{n\in\mathbb{N}}$ converges λ_2-almost everywhere on Δ to ι, so applying Scheffe's theorem (or Riesz' theorem, see [18]) yields

$$\lim_{n\to\infty} \int_\Delta |\iota_n(t,s) - \iota(t,s)| d\lambda_2(t,s) = 0.$$

Hence, for an arbitrary $z \in (0,\infty)$ using the triangle inequality it follows that

$$\begin{aligned}|\psi_n(z) - \psi(z)| &= \left| \int_{[z,\infty)} \int_{[s,\infty)} D^-\psi_n'(t) - D^-\psi'(t) \, \mathrm{d}\lambda(t)\mathrm{d}\lambda(s) \right| \\ &\leq \int_{\Delta\cap[z,\infty)^2} |\iota_n(t,s) - \iota(t,s)| \mathrm{d}\lambda_2(t,s) \leq \int_\Delta |\iota_n - \iota| \mathrm{d}\lambda_2 \to 0\end{aligned}$$

for $n \to \infty$, which completes the proof. □

Altogether we have already established the equivalence of the first five assertions in Theorem 4.1 and it remains to show the equivalence of the sixth assertion, which is tackled in the following lemma:

Lemma 4.5. *Suppose that $C, C_1, C_2, \dots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \dots$, respectively. Then pointwise convergence of $(C_n)_{n\in\mathbb{N}}$ to C is equivalent to pointwise convergence of $(c_n^{1:m})_{n\in\mathbb{N}}$ to $c^{1:m}$ almost everywhere in $\mathbb{I}^m$ for some $m \in \{2, 3, \dots, d-1\}$.*

Proof. We only prove the equivalence for $m = d-1$ since considering the fact that $\mathrm{Cont}(\psi^{(m)}) = (0,\infty)$ holds for $m = 1, 2, \dots, d-2$ all other cases follow analogously.
First observe that the set

$$\Gamma := \left\{ \mathbf{x} \in (0,1)^{d-1} : \sum_{i=1}^{d-1} \varphi(x_i) \in \mathrm{Cont}(D^-\psi^{(d-2)}) \right\} \in \mathcal{B}(\mathbb{I}^{d-1})$$

has full Lebesgue measure in $\mathbb{I}^{d-1}$. In fact, considering $\Gamma^c := (0,1)^{d-1} \setminus \Gamma$, applying disintegration and writing $\mathbf{x}_{1:d-2} = (x_1, x_2, \dots, x_{d-2})$ yields

$$\lambda_{d-1}(\Gamma^c) = \int\limits_{(0,1)^{d-2}} \lambda((\Gamma^c)_{\mathbf{x}_{1:d-2}}) \, \mathrm{d}\lambda_{d-2}(\mathbf{x}_{1:d-2}). \tag{4.1}$$

For arbitrary $\mathbf{x}_{1:d-2} \in (0,1)^{d-2}$ obviously the $\mathbf{x}_{1:d-2}$-cut $(\Gamma^c)_{\mathbf{x}_{1:d-2}}$ of Γ^c fulfills

$$(\Gamma^c)_{\mathbf{x}_{1:d-2}} = \left\{ x_{d-1} \in (0,1) : \sum_{i=1}^{d-1} \varphi(x_i) \notin \mathrm{Cont}(D^-\psi^{(d-2)}) \right\}.$$

Considering the fact that $D^{-}\psi^{(d-2)}$ has at most countably infinitely many discontinuities and that φ is strictly decreasing it follows that $(\Gamma^c)_{\mathbf{x}_{1:d-2}}$ is at most countably infinite and therefore has λ-measure 0. Applying equation (4.1) therefore directly yields $\lambda_{d-1}(\Gamma^c) = 0$, implying $\lambda_{d-1}(\Gamma) = 1$.
In case $\lim_{n\to\infty} d_\infty(C_n, C) = 0$ holds, for $\mathbf{x} \in \Gamma$ applying Lemma 4.4 yields

$$\begin{aligned}\lim_{n\to\infty} c_n^{1:d-1}(\mathbf{x}) &= \lim_{n\to\infty} \prod_{i=1}^{d-1} \varphi_n'(x_i) \cdot D^{-}\psi_n^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi_n(x_i)\right) \\ &= \prod_{i=1}^{d-1} \varphi'(x_i) \cdot D^{-}\psi^{(d-2)}\left(\sum_{i=1}^{d-1} \varphi(x_i)\right) = c^{1:d-1}(\mathbf{x}).\end{aligned}$$

Conversely, almost everywhere convergence of $(c_n^{1:m})_{n\in\mathbb{N}}$ to $c^{1:m}$ implies almost everywhere convergence of $(c_n^{1:2})_{n\in\mathbb{N}}$ to $c^{1:2}$. Using Scheffé's theorem we get that $(C_n^{1:2})_{n\in\mathbb{N}}$ converges pointwise to $C^{1:2}$. Since convergence of bivariate Archimedean copulas is equivalent to pointwise convergence of $(\varphi_n)_{n\in\mathbb{N}}$ to φ applying Lemma 4.3 yields uniform convergence of $(\psi_n)_{n\in\mathbb{N}}$ to ψ and the assertion follows from the already established equivalence of the first four assertions of Theorem 4.1. $\square$

Having proved the equivalence of the six conditions in Theorem 4.1 we now show that any of these properties implies weak conditional convergence. Notice that for the case that the (Archimedean) limit copula C is non-strict for every $z > \varphi(0)$ we obviously have

$$\lim_{n\to\infty} (-1)^{d-2} D^{-}\psi_n^{(d-2)}(z) = 0.$$

We now focus on 'good' $\mathbf{x} \in \mathbb{I}^{d-1}$ and $y \geq f^0(\mathbf{x})$ and show convergence of the Markov kernels. Notice that the subsequent lemma already establishes weak conditional convergence for the case of a strict d-dimensional limit copula C.

Lemma 4.6. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$ and Markov kernels $K_C, K_{C_1}, K_{C_2}, \ldots$, respectively. If $(C_n)_{n\in\mathbb{N}}$ converges uniformly to C then there exists a set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\mu_{C^{1:d-1}}(\Lambda) = 1$ such that for every $\mathbf{x} \in \Lambda$ the following assertion holds: there is some set $U^{\mathbf{x}} \subseteq [f^0(\mathbf{x}), 1]$ which is dense in $[f^0(\mathbf{x}), 1]$ and which fulfills that for every $y \in U^{\mathbf{x}}$*

$$\lim_{n\to\infty} K_{C_n}(\mathbf{x}, [0, y]) = K_C(\mathbf{x}, [0, y])$$

holds.

Proof. As shown in Lemma 4.5 the set

$$\Gamma = \left\{ \mathbf{x} \in (0,1)^{d-1} : \sum_{i=1}^{d-1} \varphi(x_i) \in \mathrm{Cont}(D^{-}\psi^{(d-2)}) \right\}$$

satisfies $\lambda_{d-1}(\Gamma) = 1$. Considering $\mu_{C^{1:d-1}}(\Gamma) = \int_\Gamma c^{1:d-1} \, \mathrm{d}\lambda_{d-1} = 1$ it follows that $\Lambda := \Gamma \setminus L_0^{1:d-1}$ fulfills $\mu_{C^{1:d-1}}(\Lambda)$=1. For fixed $\mathbf{x} \in \Lambda$ it follows by the same reasoning as in 4.5

that

$$U^{\mathbf{x}} := \left\{ y \in [f^0(\mathbf{x}), 1] : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \in \mathrm{Cont}(D^-\psi^{(d-2)}) \right\}$$

is of full λ-measure in $[f^0(\mathbf{x}), 1]$. Fixing $\mathbf{x} \in \Lambda$, $y \in U^{\mathbf{x}}$ and considering $z_n = \sum_{i=1}^{d-1} \varphi_n(x_i)$ and $z = \sum_{i=1}^{d-1} \varphi(x_i)$ as in Lemma 4.4, we have $z_n \overset{n\to\infty}{\longrightarrow} z$. Moreover, using $\mathbf{x} \in \Gamma$, it follows that $z = \sum_{i=1}^{d-1} \varphi(x_i) \in \mathrm{Cont}(D^-\psi^{(d-2)})$, so applying Lemma 4.4 directly yields

$$D^-\psi_n^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi_n(x_i) \right) \overset{n\to\infty}{\longrightarrow} D^-\psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) \right).$$

Analogously, using $\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \in \mathrm{Cont}(D^-\psi^{(d-2)})$ and applying Lemma 4.4 it follows that

$$D^-\psi_n^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi_n(x_i) + \varphi_n(y) \right) \overset{n\to\infty}{\longrightarrow} D^-\psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \right)$$

which altogether shows

$$\begin{aligned} \lim_{n\to\infty} K_{C_n}(\mathbf{x}, [0, y]) &= \lim_{n\to\infty} \frac{D^-\psi_n^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi_n(x_i) + \varphi_n(y) \right)}{D^-\psi_n^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi_n(x_i) \right)} \\ &= \frac{D^-\psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) \right)}{D^-\psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) \right)} = K_C(\mathbf{x}, [0, y]). \end{aligned}$$

□

It remains to show that we also have convergence for 'good' $\mathbf{x}$ and $y < f^0(\mathbf{x})$ (only relevant in the case of non-strict limit C.)

Lemma 4.7. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$ and Markov kernels $K_C, K_{C_1}, K_{C_2}, \ldots$, respectively. If $(C_n)_{n\in\mathbb{N}}$ converges uniformly to C then there exists a set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\mu_{C^{1:d-1}}(\Lambda) = 1$ such that for all $\mathbf{x} \in \Lambda$ we have*

$$\lim_{n\to\infty} K_{C_n}(\mathbf{x}, [0, y]) = 0 = K_C(\mathbf{x}, [0, y])$$

for every $y \in [0, f^0(\mathbf{x}))$.

Proof. Obviously it suffices to consider non-strict C. Let Λ be as in the proof of Lemma 4.6 and fix $\mathbf{x} \in \Lambda$ and $y \in (0, f^0(\mathbf{x}))$. Then, using the fact that ψ is invertible in $\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i)$ and $0 < y < f^0(\mathbf{x}) = \psi(\varphi(0) - \sum_{i=1}^{d-1} \varphi(x_i)$, yields that $\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) > \varphi(0)$ and we have

$$\lim_{n\to\infty} \sum_{i=1}^{d-1} \varphi_n(x_i) + \varphi_n(y) = \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) > \varphi(0).$$

Since $D^-\psi^{(d-2)}$ is left continuous, considering $\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)>\varphi(0)$ yield $\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)\in\mathrm{Cont}(D^-\psi^{(d-2)})$, so applying Lemma 4.4 (continuous convergence),

$$\lim_{n\to\infty} D^-\psi_n^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi_n(x_i)+\varphi_n(y)\right)=D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)\right)=0$$

as well as

$$\lim_{n\to\infty} D^-\psi_n^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi_n(x_i)\right)=D^-\psi^{(d-2)}\left(\sum_{i=1}^{d-1}\varphi(x_i)\right)\neq 0$$

follows. Having this, according to equation (3.3) the desired identity $\lim_{n\to\infty}K_{C_n}(\mathbf{x},[0,y])=0$ follows. Analogous arguments (or, simply using the fact that - by disintegration (2.1) - for every d-dimensional copula C we have $K_C(\mathbf{x},\{0\})=0$ for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x}\in\mathbb{I}^{d-1}$) also apply in the case $y=0$, so the proof is complete. □

Remark 4.8. The proof of Lemma 4.7 is also applicable in dimension $d=2$ and therefore provides an alternative simpler version of the rather involved approach followed in [13].

Theorem 4.1 has the following straightforward consequence (analogous to the bivariate statement considered in Theorem 4.2 in [13]):

Corollary 4.9. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$, respectively. Then the following assertions are equivalent:*

1. *$(C_n)_{n\in\mathbb{N}}$ converges uniformly to C.*
2. *$(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to φ on $(0,1]$.*
3. *$(\psi_n)_{n\in\mathbb{N}}$ converges uniformly to ψ on $[0,\infty)$.*
4. *$(\psi_n^{(m)})_{n\in\mathbb{N}}$ converges pointwise to $\psi^{(m)}$ on $(0,\infty)$ for every $m\in\{1,2,\ldots,d-2\}$ and $(D^-\psi_n^{(d-2)})_{n\in\mathbb{N}}$ converges pointwise to $D^-\psi^{(d-2)}$ on the set $\mathrm{Cont}(D^-\psi^{(d-2)})$.*
5. *$(C_n)_{n\in\mathbb{N}}$ converges weakly conditional to C and $(c_n^{1:d-1})_{n\in\mathbb{N}}$ converges to $c^{1:d-1}$ λ_{d-1}-almost everywhere in $\mathbb{I}^{d-1}$.*

Proof. We already know that the first four conditions are equivalent and that each of them implies the fifth assertion. On the other hand, if the conditions in (5) hold, then we also have λ_m-almost everywhere convergence of $(c_n^{1:m})_{n\in\mathbb{N}}$ to $c^{1:m}$ for every $m\in\{2,\ldots,d-2\}$ and applying Theorem 4.1 completes the proof. □

The next example illustrates that convergences of the parameters of Clayton or Gumbel copulas implies assertions (1) – (6) in Theorem 4.1 and hence also weak conditional convergence.

Example 4.10 (Clayton and Gumbel families, cont.)**.** It is straightforward to see that both, in the Clayton and in the Gumbel family, convergence of a sequence of parameters implies uniform convergence of the corresponding generators on $[0,\infty)$ and pointwise convergence of the corresponding pseudo-inverses. Considering, e.g., $\theta=0.3$ and $\theta_n=\frac{0.3n+5}{n}$ for every $n\in\mathbb{N}$ as parameters of 3-dimensional Clayton copulas with normalized pseudo-inverses $\varphi_n(t)=\frac{t^{-\theta_n}-1}{2^{\theta_n}-1}$ and $\varphi(t)=\frac{t^{-\theta}-1}{2^{\theta}-1}$, then obviously $\lim_{n\to\infty}\theta_n=\lim_{n\to\infty}\frac{0.3n+5}{n}=0.3=\theta$,

so $\varphi_n \overset{n\to\infty}{\longrightarrow} \varphi$ on $(0,1]$ and all assertions $(1)-(6)$ in Theorem 4.1 and weak conditional convergence follow. Convergence of the respective φ_n and ψ_n is illustrated in Figure 4.
Turning towards the Gumbel family and considering, e.g., $\alpha_n = \sqrt[n]{n}$ for $n \in \mathbb{N}$ and $\alpha = 1$ as parameters we obviously have $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sqrt[n]{n} = 1 = \alpha$ implying $\varphi_n \overset{n\to\infty}{\longrightarrow} \varphi$. Thus assertions $(1)-(6)$ in Theorem 4.1 and weak conditional convergence hold. Figure 5 illustrates the corresponding functions φ_n and ψ_n.

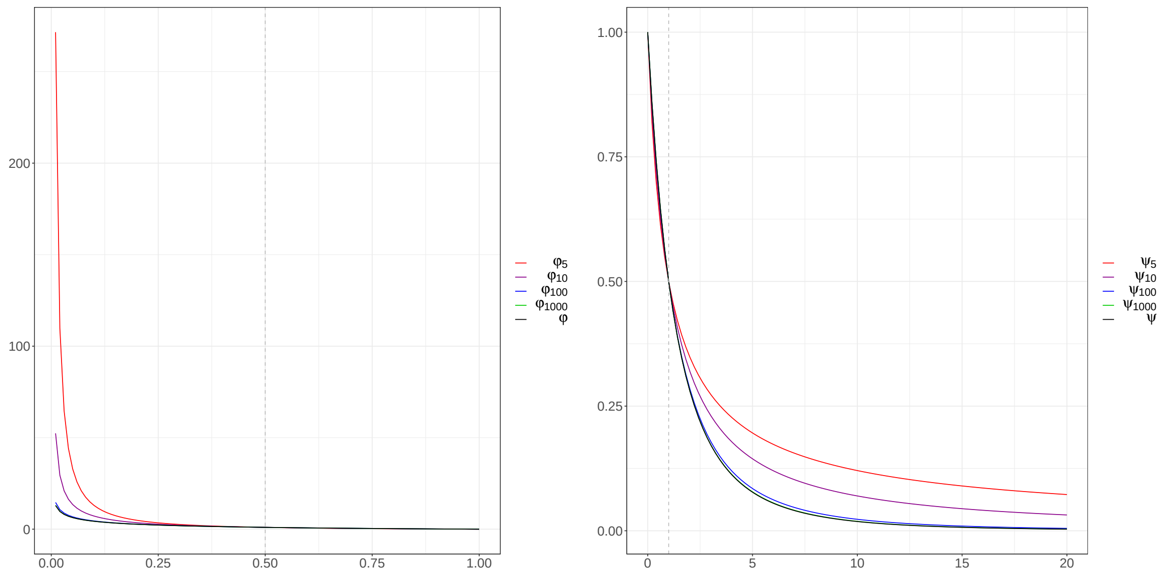

Figure 4: Generators of Clayton copulas; φ (black) and φ_n (left panel), ψ (black) and ψ_n (right panel) for $n = 5$, $n = 10$, $n = 100$ and $n = 1000$ as considered in Example 4.10.

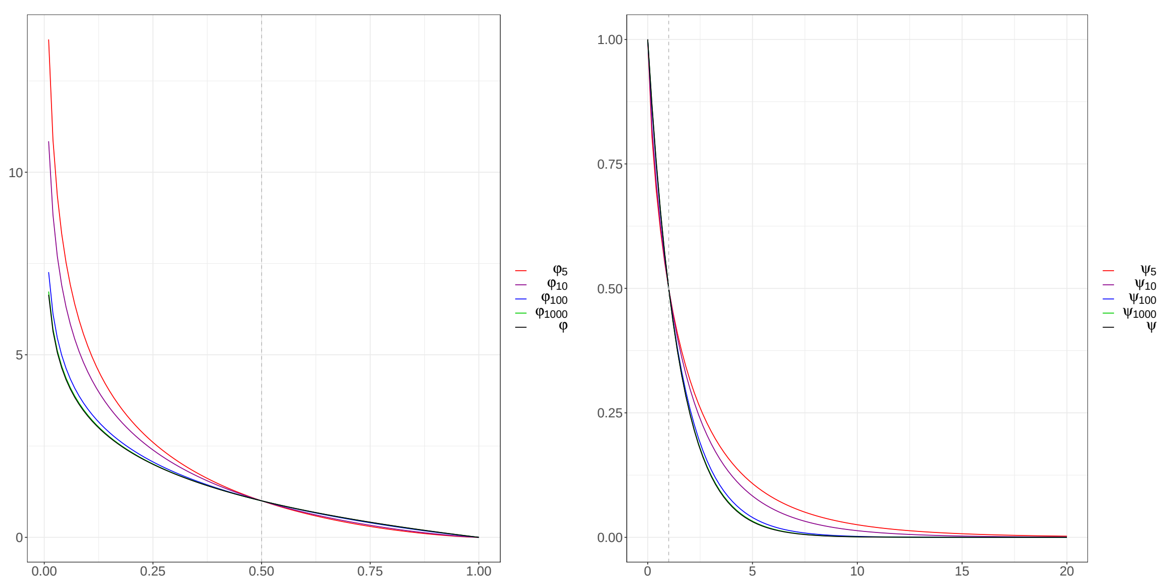

Figure 5: Generators of Gumbel copulas; φ (black) and φ_n (left panel) ,ψ (black) and ψ_n (right panel) for $n = 5$, $n = 10$, $n = 100$ and $n = 1000$ according to Example 4.10.

5. Archimedean copulas and the Williamson transform

Following [24, Theorem 1.11], every d-monotone function can be represented via the so-called Williamson transform of a unique probability measure on $[0,\infty)$. As a consequence, we may describe and handle d-dimensional Archimedean copulas $C = C_\psi$ in terms of their corresponding probability measure γ on $[0,\infty)$. In what follows we first establish some complementary useful results describing the interrelation between generator ψ and probability measures γ, express masses of level sets as well as the Kendall distribution function handily in terms of γ, show that regularity properties of the corresponding measure γ carry over to the Archimedean copula and prove the fact that pointwise convergence of Archimedean copulas $C_1, C_2, \ldots$ to an Archimedean copula C is equivalent to weak convergence of the corresponding probability measures $\gamma_1, \gamma_2, \ldots$ to the probability measure γ. Based on these facts we then finally show that both, the family of absolutely continuous and the family of singular Archimedian copulas is dense in $(\mathcal{C}^d_{ar}, d_\infty)$.

We start with recalling the following result (see [20] and [24, Theorem 1.11]) where we write f_+^m for the m-th power of the positive part f_+ of a function f, i.e., $f_+^m := (f_+)^m$:

Theorem 5.1. *Let $\psi\colon [0,\infty) \to \mathbb{I}$ be a function and $d \geq 2$. Then the following two conditions are equivalent:*

(1) ψ is the generator of a d-dimensional Archimedean copula C_ψ.

(2) There exists a unique probability measure γ on $\mathcal{B}([0,\infty))$ with $\gamma(\{0\}) = 0$ such that

$$\psi(z) = \int_{[0,\infty)} (1 - tz)_+^{d-1} \mathrm{d}\gamma(t) =: (\mathcal{W}_d\gamma)(z), \tag{5.1}$$

holds for every $z > 0$. In other words, ψ is the Williamson transform $\mathcal{W}_d\gamma$ of γ.

Obviously our assumed normalization property $\psi(1) = \frac{1}{2}$ translates to

$$\int_{\mathbb{I}} (1-t)^{d-1} \mathrm{d}\gamma(t) = \frac{1}{2}. \tag{5.2}$$

In what follows we therefore only consider measures γ fulfilling equation (5.2), let $\mathcal{P}_{\mathcal{W}_d}$ denote the family of all these measures, i.e.,

$$\mathcal{P}_{\mathcal{W}_d} = \left\{ \gamma \in \mathcal{P}([0,\infty)) : \gamma(\{0\}) = 0 \text{ and } \int_{\mathbb{I}} (1-t)^{d-1} \mathrm{d}\gamma(t) = \frac{1}{2} \right\} \tag{5.3}$$

and refer to $\mathcal{P}_{\mathcal{W}_d}$ as the family of all d-Williamson measures.

Again following [20] next we derive an explicit representation for the cumulative distribution function of γ in terms of the Archimedean generator ψ:

Lemma 5.2. *Let ψ be the generator of a d-dimensional Archimedean copula and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ its corresponding Williamson measure. Then*

$$\gamma([0,z]) = \sum_{k=0}^{d-2} \frac{(-1)^k \psi^{(k)}(\frac{1}{z})}{k!} \frac{1}{z^k} + \frac{(-1)^{d-1} D^- \psi^{(d-2)}(\frac{1}{z})}{(d-1)!} \frac{1}{z^{d-1}} \tag{5.4}$$

holds for every $z > 0$.

Proof. Defining $F\colon (0,\infty) \to \mathbb{I}$ by $F(z) := \gamma([0,z]) = \gamma((0,z])$ according to [27]

$$F(z) = \sum_{k=0}^{d-1} \frac{(-1)^k \psi^{(k)}(\frac{1}{z})}{k!} \frac{1}{z^k} \tag{5.5}$$

holds for every continuity point z of F, i.e., for every continuity point z of $D^-\psi^{(d-2)}$. Set $\mathrm{Cont}(D^-\psi^{(d-2)})$ and define the function $G : (0,\infty) \to \mathbb{I}$ by

$$G(z) := \sum_{k=0}^{d-2} \frac{(-1)^k \psi^{(k)}(\frac{1}{z})}{k!} \frac{1}{z^k} + \frac{(-1)^{(d-1)} D^-\psi^{(d-2)}(\frac{1}{z})}{(d-1)!} \frac{1}{z^{d-1}}.$$

Then the distribution function F and the right-continuous function G coincide on the set $\mathrm{Cont}(D^-\psi^{(d-2)})$ and since the latter is dense in $(0,\infty)$ equality $F = G$ follows and the proof is complete. $\square$

Example 5.3 (Clayton and Gumbel families, cont.)**.** For the Clayton copulas we obtain the following cumulative distribution function of the Williamson measure γ of $C \in \mathcal{C}^3_{CL}$ with parameter $\theta > 0$ (see left panel in Figure 6):

$$\begin{aligned}\gamma([0,z]) &= \psi(\tfrac{1}{z}) - \frac{\psi'(\frac{1}{z})}{z} + \frac{\psi''(\frac{1}{z})}{2z^2} \\ &= \left[1 + \frac{2^\theta - 1}{\theta(2^\theta - 1 + z)} + \frac{(2^\theta - 1)^2(\frac{1}{\theta} + 1)}{2\theta(2^\theta - 1 + z)^2}\right] \frac{1}{(\frac{2^\theta - 1}{z} + 1)^{\frac{1}{\theta}}},\end{aligned}$$

Proceeding analogously, calculating the required derivatives of ψ yields the following Williamson measure γ of $C \in \mathcal{C}^3_{GU}$ with parameter $\alpha \geq 1$ (see right panel in Figure 6):

$$\begin{aligned}\gamma([0,z]) &= \psi(\tfrac{1}{z}) - \frac{\psi'(\frac{1}{z})}{z} + \frac{\psi''(\frac{1}{z})}{2z^2} \\ &= \left[1 + \frac{\log(2) z^{-\frac{1}{\alpha}}}{\alpha} + \frac{\log(2)}{2}\left(\left(\frac{1}{\alpha} - \frac{1}{\alpha^2}\right) + \frac{\log(2)}{\alpha^2} z^{-\frac{1}{\alpha}}\right) z^{-\frac{1}{\alpha}}\right] \exp(-\log(2) z^{-\frac{1}{\alpha}}).\end{aligned}$$

The following lemma expresses $D^-\psi^{(d-2)}$ (appearing both in the numerator and the denominator of the Markov kernel $K_C(\cdot,\cdot)$ in Theorem 3.1) in terms of γ and will be useful in the sequel:

Lemma 5.4. *Let ψ be the generator of a d-dimensional Archimedean copula and $\gamma \in \mathcal{P}_{\mathcal{W}_d}$ be the corresponding Williamson measure. Then*

$$0 \geq G(z) := (-1)^{d-2} D^-\psi^{(d-2)}(z) = -(d-1)! \int_{(0,\frac{1}{z}]} t^{d-1} \mathrm{d}\gamma(t)$$

holds for every $z > 0$.

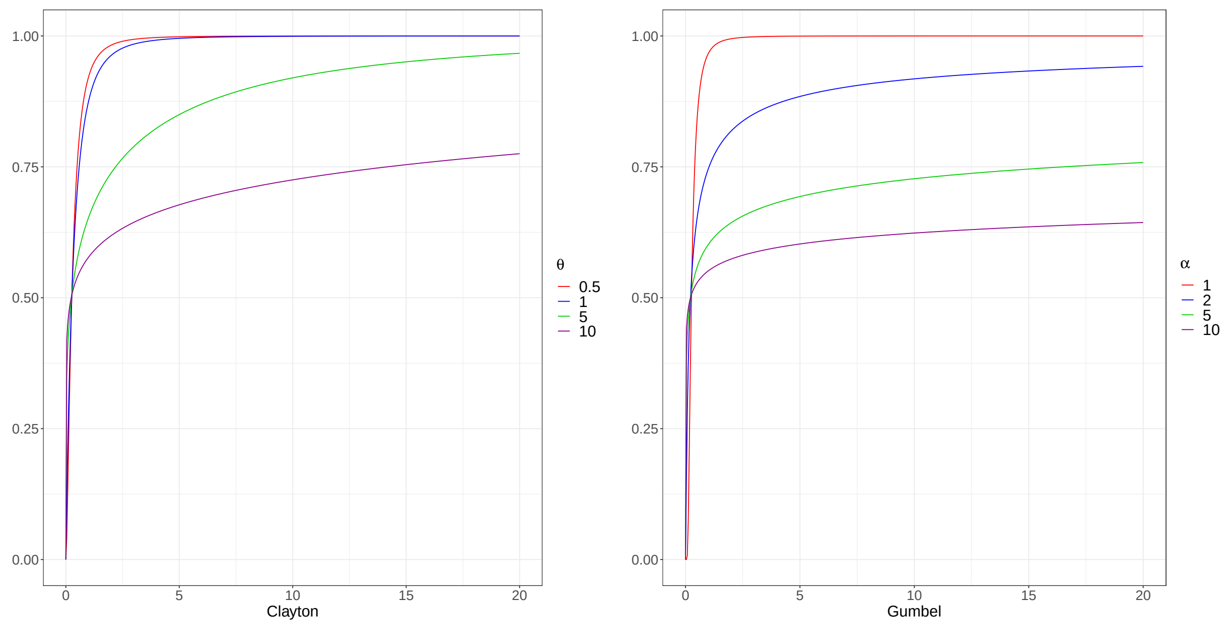

Figure 6: Cumulative distribution function of the Williamson measure γ of the Clayton copula with the parameter $\theta = \frac{1}{2}$, $\theta = 1$, $\theta = 5$, $\theta = 10$ (left panel) and of the Gumbel copula with parameter $\alpha = 1$, $\alpha = 2$, $\alpha = 5$, $\alpha = 10$ (right panel).

Proof. According to [24, Theorem 1.11] we already know that for every $z > 0$ we have

$$\begin{aligned}(-1)^{d-2}\psi^{d-2}(z) &= (d-1)! \int_{[0,\infty)} t^{d-2}(1-zt)_+ \mathrm{d}\gamma(t) \\ &= (d-1)! \int_{(0,\infty)} t^{d-2}(1-zt)_+ \mathrm{d}\gamma(t).\end{aligned}$$

Since $(-1)^{d-2}\psi^{d-2}$ is convex, the left hand derivative exists everywhere in $(0,\infty)$ and is left-continuous. For fixed $z > 0$ considering the left-hand difference quotient

$$Q_h(z) := (d-1)! \int_{(0,\infty)} t^{d-2} \frac{(1-t(z+h))_+ - (1-tz)_+}{h} \mathrm{d}\gamma(t)$$

for $h \in (-z, 0)$ we get

$$\begin{aligned}Q_h(z) &= \frac{(d-1)!}{h} \int_{(\frac{1}{z},\frac{1}{z+h})} t^{d-2}(1-t(z+h))\mathrm{d}\gamma(t) \\ &\quad + \frac{(d-1)!}{h} \int_{(0,\frac{1}{z}]} t^{d-2}[(1-t(z+h)) - (1-tz)]\mathrm{d}\gamma(t) \\ &= \underbrace{\frac{(d-1)!}{h} \int_{(\frac{1}{z},\frac{1}{z+h})} t^{d-2}(1-t(z+h))\mathrm{d}\gamma(t)}_{=:I_h} - (d-1)! \int_{[0,\frac{1}{z}]} t^{d-1}\mathrm{d}\gamma(t),\end{aligned}$$

and it suffices to show that I_h converges to 0. Using the monotonicity and non-negativity of the functions $t \mapsto t^{d-2}$ and $t \mapsto 1 - t(z+h)$ on $(\frac{1}{z}, \frac{1}{z+h})$ it follows that

$$\begin{aligned}|I_h| &\leq \frac{1}{|h|} \frac{1}{(z+h)^{d-2}} \left(1 - \frac{z+h}{z}\right) \gamma\left(\left(\frac{1}{z}, \frac{1}{z+h}\right)\right) \\ &= \frac{1}{(z+h)^{d-2}\, z}\, \gamma\left(\left(\frac{1}{z}, \frac{1}{z+h}\right)\right)\end{aligned}$$

from which the assertion follows immediately. □

As first application of the previous lemma we characterize strictness in terms of the Williamson measure γ:

Lemma 5.5. *Suppose that C is an Archimedean copula with Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Then C is strict if, and only if the support of γ contains 0, i.e., if $\gamma([0,r)) > 0$ for every $r > 0$.*

Proof. If the support of γ contains 0 then obviously $\int_{(0,r)} t^{d-1} \mathrm{d}\gamma(t) > 0$ for every $r > 0$, hence applying Lemma 5.4 directly yields $(-1)^{d-1} D^- \psi^{(d-2)}(z) > 0$ for every $z > 0$. Having this, strictness of ψ follows immediately.
On the other hand, if there exists some $r > 0$ with $\gamma([0,r)) = 0$ then considering $z_0 = \frac{1}{2r}$ and again using Lemma 5.4 we have $(-1)^{d-1} D^- \psi^{(d-2)}(z_0) = 0$. Since the function $z \mapsto (-1)^{d-1} D^- \psi^{(d-2)}(z)$ is non-negative, non-increasing and left-continuous it follows that $(-1)^{d-1} D^- \psi^{(d-2)}(z) = 0$ holds for every $z \geq z_0$. Hence $\psi(z_0) = 0$ and ψ is non-strict. □

We now return to the formulas for the level set masses and the Kendall distribution function of Archimedean copulas as already mentioned in Section 3 and reformulate them elegantly in terms of the Williamson measure γ. Although surprising, to the best of the authors' knowledge these formulas seem to be new:

Theorem 5.6. *Let C be a d-dimensional Archimedean copula with generator ψ and Williamson measure γ. Then (compare with equations (3.5) - (3.7)):*

$$\mu_C(L_t) = \gamma(\{\tfrac{1}{\varphi(t)}\}),\ t \in (0,1] \tag{5.6}$$

holds for every $t \in (0,1]$ and every C. Furthermore, for strict C we have $\mu_C(L_0) = 0$, and for non-strict C

$$\mu_C(L_0) = \gamma(\{\tfrac{1}{\varphi(0)}\}). \tag{5.7}$$

holds. Finally, the Kendall distribution function F_K^d of C fulfills

$$F_K^d(t) = \gamma([0, \tfrac{1}{\varphi(t)}]) \tag{5.8}$$

for every $t \in (0,1]$.

Proof. First of all notice that the expression for the Kendall distribution function follows immediately from equation (5.4). Furthermore for $t \in (0,1]$ considering

$$\mu_C(L_t) = F_K^d(t) - F_K^d(t-)$$

equation (5.6) follows immediately from equation (5.8). Finally, using equation (3.6) and incorporating Lemma 5.4 yields

$$\mu_C(L_0) = \frac{(-1)^{d-1}(\varphi(0))^{d-1}}{(d-1)!} D^-\psi^{(d-2)}(\varphi(0)) = \varphi(0)^{d-1} \int_{(0,\frac{1}{\varphi(0)}]} t^{d-1} \mathrm{d}\gamma(t).$$

Since equation (5.1) implies that for every $z_0 > 0$ we have that $\psi(z_0) = 0$ is equivalent to $\gamma((0, \frac{1}{z_0})) = 0$, the right-hand side of the last equation simplifies to $\gamma(\{\frac{1}{\varphi(0)}\})$ and the proof is complete. $\square$

Remark 5.7. Notice that equation (5.8) implies $F_K^d(\frac{1}{2}) = \gamma([0,1])$. More importantly, the (probably most) famous conjecture in the context of Archimedean copulas, saying that for every fixed $d \geq 3$ two Archimedean copulas $C, D \in \mathcal{C}_{ar}^d$ are identical if, and only if their Kendall distribution functions coincide (see [7] and [8]) would follow if it could be shown that the mapping assigning each Williamson measure γ the function $F_\gamma : \mathbb{I} \to \mathbb{I}$, defined by

$$F_\gamma(t) = \gamma\left(\left[0, \frac{1}{\varphi_\gamma(t)}\right]\right)$$

is injective, where φ_γ denotes the pseudo-inverse of the generator $\psi = \mathcal{W}_d\,\gamma$.

Example 5.8. The probability measure $\gamma = \frac{7}{8}\,\delta_{1/4} + \frac{1}{8}\,\delta_{3/4}$ obviously fulfills $\gamma \in \mathcal{P}_{\mathcal{W}_3}$. The induced generator ψ is given by

$$\psi(z) := \begin{cases} 1 - \frac{5z}{8} + \frac{z^2}{8} & \text{if } z < \frac{4}{3} \\ \frac{7}{8}\left(1 - \frac{z}{4}\right)^2 & \text{if } z \in \left[\frac{4}{3}, 4\right] \\ 0 & \text{otherwise} \end{cases}$$

and it is straightforward to verify that $\varphi(\frac{7}{18}) = \frac{4}{3}$ and $\varphi(0) = 4$ holds. Using Theorem 5.6 therefore yields $\mu_C(L_0) = \frac{7}{8}$ as well as $\mu_C(L_{7/18}) = \frac{1}{8}$. Figure 7 depicts the distribution function $z \mapsto \gamma([0,z])$ of γ (left panel), the induced generator ψ (middle) and the sets L_0 and $L_{7/18}$ carrying the mass (right panel).

The next result complements Theorem 4.1 and adds a seventh equivalent condition in terms of the corresponding Williamson measures. During the process of preparing this manuscript it has been brought to our attention that this very result was already established in [1]. Considering that the result is key especially for the subsequent regularity results and that the subsequent proof is simpler and less technical than the one given in [1] we include it for the sake of completeness.

Theorem 5.9. *Suppose that $C, C_1, C_2, \dots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \dots$ and let $\gamma, \gamma_1, \gamma_2, \dots$ denote the corresponding Williamson measures. Then the following assertions are equivalent:*

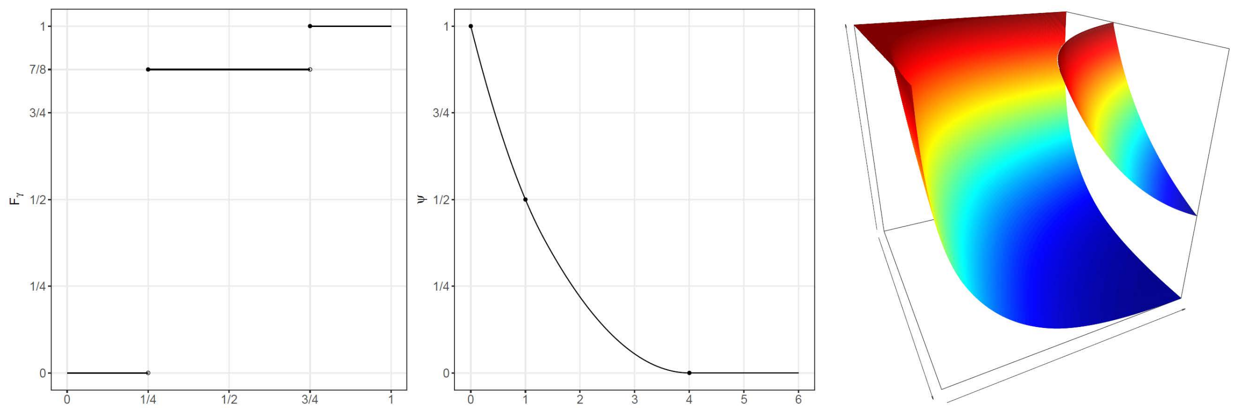

Figure 7: Distribution function of γ (left panel), induced generator ψ (middle) and the sets L_0 and $L_{7/18}$ (right panel) as considered in Example 5.8.

(1) $(C_n)_{n\in\mathbb{N}}$ *converges uniformly to* C.

(2) $(\gamma_n)_{n\in\mathbb{N}}$ *converges weakly on* $[0,\infty)$ *to* γ.

Proof. According to Theorem 4.1 the first assertion is equivalent to uniform convergence of $(\psi_n)_{n\in\mathbb{N}}$ to ψ. (i) If $(\gamma_n)_{n\in\mathbb{N}}$ converges weakly to γ, then applying Theorem 5.1 and using the fact that the function $t \mapsto (1-tz)_+^{d-1}$ is continuous and bounded

$$\psi_n(z) = \int_{[0,\infty)} (1-tz)_+^{d-1} d\gamma_n(t) \xrightarrow{n\to\infty} \int_{[0,\infty)} (1-tz)_+^{d-1} d\gamma(t) = \psi(z)$$

follows for every fixed but arbitrary $z \in [0,\infty)$. (ii) Vice versa, using Lemma 5.2 and Lemma 4.4 and considering $z \in (0,\infty)$ with $\frac{1}{z} \in \mathrm{Cont}(D^-\psi^{(d-2)})$ yields

$$\begin{aligned}\lim_{n\to\infty} \gamma_n([0,z]) &= \lim_{n\to\infty} \sum_{k=0}^{d-2} \frac{(-1)^k \psi_n^{(k)}(\frac{1}{z})}{k!} z^{-k} + \frac{(-1)^{d-1} D^-\psi_n^{(d-2)}(\frac{1}{z})}{(d-1)!} z^{-d+1} \\ &= \sum_{k=0}^{d-2} \frac{(-1)^k \psi^{(k)}(\frac{1}{z})}{k!} z^{-k} + \frac{(-1)^{d-1} D^-\psi^{(d-2)}(\frac{1}{z})}{(d-1)!} z^{-d+1} \\ &= \gamma([0,z]).\end{aligned}$$

This completes the proof since, firstly, $(0,1) \setminus \mathrm{Cont}(D^-\psi^{(d-2)})$ is at most countably infinite and, secondly, convergence of distribution functions on a dense set implies weak convergence. $\square$

Example 5.10 (Clayton and Gumbel families, cont.)**.** We illustrate Theorem 5.9 by considering the special situation of three-dimensional Gumbel and Clayton copulas: Considering $\theta = 0.3$ as well as $\theta_n = \frac{0.3n+5}{n}$ for every $n \in \mathbb{N}$, and using that $\theta_n \xrightarrow{n\to\infty} \theta$ yields that the induced three-dimensional Clayton copulas C_n converge uniformly to C. Therefore, according to Theorem 5.9 we have weak convergence of the corresponding Williamson measures γ_n to γ (see right panel of Figure 8). Analogously, for $\alpha_n = \sqrt[n]{n}$ and $\alpha = 1$ we have uniform convergence of the associated three-dimensional Gumbel copulas C_n to C and thus weak convergence of the corresponding Williamson measures γ_n to γ (see right panel in Figure 9).

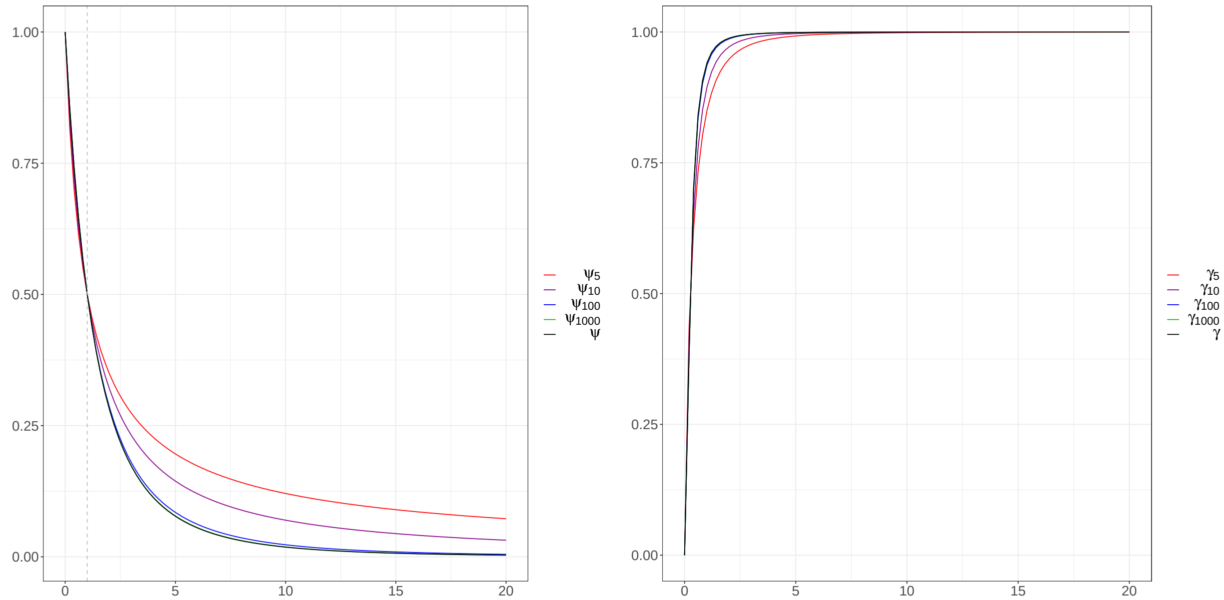

Figure 8: Generators and cumulative distribution functions of the Williamson measures of Clayton copulas; ψ (black) and ψ_n (left panel) and γ (black) and γ_n (right panel) for $n = 5$, $n = 10$, $n = 100$ and $n = 1000$ as considered in Example 5.10.

We now focus on studying how regularity/singularity properties of the Williamson measure carries over to regularity/singularity properties of the corresponding copula $C_\gamma \in \mathcal{C}^d_{ar}$ and first recall some basic notation. For every $m \in \{1, \dots, d\}$ we say that a finite measure ϑ on $\mathcal{B}(\mathbb{I}^m)$ is singular (with respect to λ_m) if there exists some $G \in \mathcal{B}(\mathbb{I}^m)$ fulfilling $\vartheta(G) = \vartheta(\mathbb{I}^m)$ and $\lambda_d(G) = 0$. A copula $C \in \mathcal{C}^m$ is called singular if the corresponding m-stochastic measure μ_C is singular.

For the bivariate setting singularity of a copula C is equivalent to singularity of λ-almost all conditional distributions $K_C(x, \cdot)$ (see Lemma 1 in [5]). A a fully analogous statement can not hold in general for arbitrary $d \geq 3$ - in fact, for example the copula C of a random vector (X, X, Y) with X, Y being independent and uniformly distributed on $[0, 1]$ is obviously singular but $\mu_{C^{1:2}} = \mu_M$-almost every conditional distribution $K_C(x, x, \cdot)$ coincides with λ and therefore is absolutely continuous. Assuming, however, absolute continuity of $C^{1:d-1}$ as it is the case for every $C \in \mathcal{C}^d_{ar}$, an analogue of the bivariate result remains valid:

Lemma 5.11. *Let C be a d-dimensional copula such that $C^{1:d-1}$ is absolutely continuous. Then C is singular if, and only if there exists some set $\Lambda \in \mathcal{B}(\mathbb{I}^{d-1})$ fulfilling $\mu_{C^{1:d-1}}(\Lambda) = 1$ such that $K_C(\mathbf{x}, \cdot)$ is singular for every $\mathbf{x} \in \Lambda$.*

Proof. If $C \in \mathcal{C}^d$ is singular then by definition there exists some set $G \in \mathcal{B}(\mathbb{I}^d)$ fulfilling $\mu_C(G) = 1$ as well as $\lambda_d(G) = 0$. Disintegration (see equation (2.1)) therefore yields the

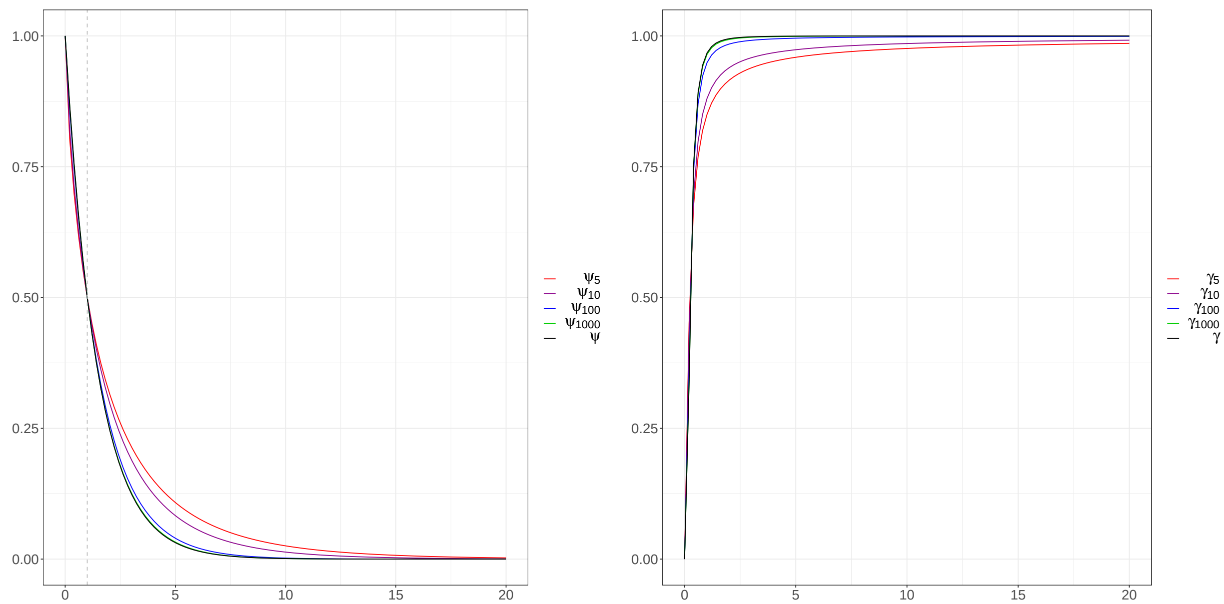

Figure 9: Generators and cumulative distribution functions of the Williamson measures of Gumbel copulas; ψ (black) and ψ_n (left panel) and γ (black) and γ_n (right panel) for $n = 5$, $n = 10$, $n = 100$ and $n = 1000$ according to Example 5.10.

existence of some set $\Lambda_1 \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\mu_{C^{1:d-1}}(\Lambda_1) = 1$ such that $K_C(\mathbf{x}, G_\mathbf{x}) = 1$ holds for all $\mathbf{x} \in \Lambda_1$. Moreover, again by applying disintegration, we have

$$0 = \lambda_d(G) = \int_{\mathbb{I}^{d-1}} \lambda_1(G_\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}),$$

so there exists some set $\Lambda_2 \in \mathcal{B}(\mathbb{I}^{d-1})$ fulfilling with $\lambda_{d-1}(\Lambda_2) = 1$ such that for all $\mathbf{x} \in \Lambda_2$ we have $\lambda_1(G_\mathbf{x}) = 0$. Since for every set $G \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\lambda_{d-1}(G) = 1$ we also have $\mu_{C^{1:d-1}}(G) = 1$, setting $\Lambda := \Lambda_1 \cap \Lambda_2$ yields $\mu_{C^{1:d-1}}(\Lambda) = 1$ and it follows that for every $\mathbf{x} \in \Lambda$ the $\mathbf{x}$-cut $G_\mathbf{x}$ of G fulfills $K_C(\mathbf{x}, G_\mathbf{x}) = 1$ as well as $\lambda(G_\mathbf{x}) = 0$. In other words, $K_C(\mathbf{x}, \cdot)$ is singular and the first implication is proved.

The reverse implication can be proved as follows. We show the contraposition and assume that μ_C is not singular with respect to λ_d, i.e., the absolutely continuous part μ_C^{abs} of the Lebesgue decomposition $\mu_C = \mu_C^{abs} + \mu_C^{sing}$ of μ_C with respect to λ_d is non-degenerated in the sense that $\mu_C^{abs}(\mathbb{I}^d) > 0$ holds. Let $G \in \mathcal{B}(\mathbb{I}^{d-1})$ with $\lambda_{d-1}(G) = 0$ be arbitrary but fixed. Then obviously

$$(\mu_C^{sing})^{1:d-1}(G) = \mu_C^{sing}(G \times \mathbb{I}) \leq \mu_C(G \times \mathbb{I}) = \mu_C^{1:d-1}(G) = \int_G c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) = 0,$$

so there exists a Radon-Nikodym derivative $f \colon \mathbb{I}^{d-1} \to [0, \infty)$ of $(\mu_C^{sing})^{1:d-1}$ with respect to λ_{d-1}. Letting $k \colon \mathbb{I}^d \to [0, \infty)$ denote the Radon-Nikodym derivative of μ_C^{abs} with respect to

λ_d, for arbitrary $E \in \mathcal{B}(\mathbb{I}^{d-1})$, $F \in \mathcal{B}(\mathbb{I})$ we get

$$\begin{aligned}
\int_E K_C(\mathbf{x}, F) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) &= \int_E K_C(\mathbf{x}, F) \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) \\
&= \mu_C(E \times F) = \mu_C^{abs}(E \times F) + \mu_C^{sing}(E \times F) \\
&= \int_E \left[\int_F k(\mathbf{x}, y) \mathrm{d}\lambda(y) \right] \mathrm{d}\lambda_{d-1}(\mathbf{x}) + \int_E H^{sing}(\mathbf{x}, F) \mathrm{d}(\mu_C^{sing})^{1:d-1}(\mathbf{x}) \\
&= \int_E \left[\int_F k(\mathbf{x}, y) \mathrm{d}\lambda(y) \right] \mathrm{d}\lambda_{d-1}(\mathbf{x}) + \int_E H^{sing}(\mathbf{x}, F) f(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \\
&= \int_E \left[\int_F k(\mathbf{x}, y) \mathrm{d}\lambda(y) + H^{sing}(\mathbf{x}, F) f(\mathbf{x}) \right] \mathrm{d}\lambda_{d-1}(\mathbf{x}),
\end{aligned}$$

where in the third line we used the disintegration theorem for arbitrary finite measures and $H^{sing}(\mathbf{x}, \cdot)$ denotes the conditional measure (sub- or super Markov kernel) of μ_C^{sing} given $\mathbf{x}$. Since $E \in \mathcal{B}(\mathbb{I}^{d-1})$ it follows that

$$K_C(\mathbf{x}, F) c^{1:d-1}(\mathbf{x}) = \int_F k(\mathbf{x}, y) \mathrm{d}\lambda(y) + H^{sing}(\mathbf{x}, F) f(\mathbf{x})$$

holds for λ_{d-1}-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$. Using the fact that $\mu_C^{1:d-1}$ is absolutely continuous and that obviously $\mu_{C^{1:d-1}}(\{\mathbf{x} \in \mathbb{I}^{d-1} \colon c^{1:d-1}(\mathbf{x}) = 0\}) = 0$ yields the identity

$$K_C(\mathbf{x}, F) = \int_F \frac{k(\mathbf{x}, y)}{c^{1:d-1}(\mathbf{x})} \mathrm{d}\lambda(y) + H^{sing}(\mathbf{x}, F) \frac{f(\mathbf{x})}{c^{1:d-1}(\mathbf{x})}$$

for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$. Since μ_C^{abs} is non-degenerated by assumption, there exists a set $\Upsilon \in \mathcal{B}(\mathbb{I}^{d-1})$ with $c^{1:d-1}(\mathbf{x}) > 0$ for every $\mathbf{x} \in \Upsilon$ and $\mu_{C^{1:d-1}}(\Upsilon) > 0$ such that for every $\mathbf{x} \in \Upsilon$ the absolutely continuous measure $F \mapsto \int_F \frac{k(\mathbf{x},y)}{c^{1:d-1}(\mathbf{x})} \mathrm{d}\lambda(y)$ is non-degenerated. This shows that for such $\mathbf{x}$ the measure $K_C(\mathbf{x}, \cdot)$ can not be singular and the proof is complete. $\square$

Following [19] every Markov kernel $K_C(\cdot, \cdot) \colon \mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$ can be decomposed into the sum of three sub- Markov kernels from $\mathbb{I}$ to $\mathcal{B}(\mathbb{I})$ as

$$K_C(\mathbf{x}, \cdot) = K_C^{dis}(\mathbf{x}, \cdot) + K_C^{sing}(\mathbf{x}, \cdot) + K_C^{abs}(\mathbf{x}, \cdot), \tag{5.9}$$

whereby each measure $K_C^{dis}(\mathbf{x}, \cdot)$ is discrete, each $K_C^{sing}(\mathbf{x}, \cdot)$ is singular and has no point masses and $K_C^{abs}(\mathbf{x}, \cdot)$ is absolutely continuous on $\mathcal{B}(\mathbb{I})$. Again assuming absolute continuity of $C^{1:d-1}$ and letting $c^{1:d-1}$ denote the corresponding density in what follows we will refer to the three measures $\mu_C^{dis}, \mu_C^{sing}, \mu_C^{abs}$, defined via disintegration by

$$\begin{aligned}
\mu_C^{dis}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{dis}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \\
\mu_C^{sing}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{sing}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \\
\mu_C^{abs}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{abs}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x})
\end{aligned} \tag{5.10}$$

for every $G \in \mathcal{B}(\mathbb{I})$ as the discrete, the singular, and the absolutely continuous component of μ_C.

We now show how singularity/regularity of γ carries over to the corresponding Archimedean copula.

Theorem 5.12. *Suppose that $C \in \mathcal{C}^d_{ar}$ has generator ψ and Williamson measure $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Then the following assertions hold:*

(1) If γ is absolutely continuous then $\mu_C^{abs}(\mathbb{I}^d) = 1$, i.e., C is absolutely continuous.

(2) If γ is discrete then $\mu_C^{dis}(\mathbb{I}^d) = 1$.

(3) If γ is singular without point masses then $\mu_C^{sing}(\mathbb{I}^d) = 1$.

Proof. (i) The first assertion has already been established in [20] and can alternatively be proved easily as follows: Suppose that γ is absolutely continuous with density f. Then using Lemma 5.4 we have

$$(-1)^{d-2} D^- \psi^{(d-2)}(z) = -(d-1)! \int_{(0,\frac{1}{z}]} t^{d-1} f(t) \mathrm{d}\lambda(t).$$

Considering that the right-hand side is obviously continuous in z it follows that $\psi^{(d-1)}$ exists on the full interval $(0,\infty)$. Moreover, the right-hand side is easily seen to be absolutely continuous, hence Proposition 4.2 in [20] yields absolute continuity of C.
For the proof of the remaining two assertions first notice that it suffices to consider $\mathbf{x}$ fulfilling $M(\mathbf{x}) < 1$ and $\mathbf{x} \notin L_0^{1:d-1}$. Additionally, in this case it holds that $0 < \sum_{i=1}^{d-1} \varphi(x_i) < \varphi(0) \in (0,\infty]$ as well as $(-1)^{d-1} D^- \psi^{(d-2)} \left(\sum_{i=1}^{d-1} \varphi(x_i) \right) > 0$. Moreover, assuming $y \geq f^0(\mathbf{x})$ using Lemma 5.4 the Markov kernel $K_C(\cdot,\cdot)$ according to equation (3.3) can be expressed as

$$K_C(\mathbf{x}, [0,y]) = \frac{G\left(\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)\right)}{G\left(\sum_{i=1}^{d-1} \varphi(x_i)\right)} = \frac{\int_{I_y} t^{d-1} \mathrm{d}\gamma(t)}{\int_{I_1} t^{d-1} \mathrm{d}\gamma(t)} \tag{5.11}$$

with $I_y = \left(0, \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)}\right]$ for every $y \in \mathbb{I}$ and G as in Lemma 5.4.
(ii) Suppose now that γ is discrete. Then there exist $a_1, a_2, \ldots \in (0,\infty)$ and constants $\alpha_1, \alpha_2, \ldots \in \mathbb{I}$ with $\sum_{j=1}^{\infty} \alpha_j = 1$ such that $\gamma = \sum_{j=1}^{\infty} \alpha_j \delta_{a_j}$ holds, and equation (5.11) simplifies to

$$K_C(\mathbf{x}, [0,y]) = \frac{\sum_{j:a_j \in I_y} a_j^{d-1} \alpha_j}{\sum_{j:a_j \in I_1} a_j^{d-1} \alpha_j}. \tag{5.12}$$

Notice that we do not assume all α_j to be greater than zero, so the case of finitely many point masses is covered as well. Considering, firstly, that $K_C(\mathbf{x},[0,y]) = 0$ for $y < f^0(\mathbf{x})$ and that the condition $a_j \in I_y$ is equivalent to $y \geq \psi(\frac{1}{a_j} - \sum_{i=1}^{d-1} \varphi(x_i))$ the function $y \mapsto K_C(\mathbf{x},[0,y])$ can be written as

$$K_C(\mathbf{x}, [0,y]) = \sum_{j:\, a_j \in I_1} \beta_j \mathbf{1}_{[\psi(\frac{1}{a_j} - \sum_{i=1}^{d-1} \varphi(x_i)),1]}(y)$$

with $\beta_j = \frac{a_j^{d-1}\alpha_j}{\sum_{l:a_l \in I_1} a_l^{d-1}\alpha_l}$ and therefore it

is easily seen to be the distribution function of the discrete probability measure on $\mathbb{I}$ having point mass $\frac{a_j^{d-1}\alpha_j}{\sum_{l:a_l \in I_1} a_l^{d-1}\alpha_l}$ in $\psi(\frac{1}{a_j} - \sum_{i=1}^{d-1} \varphi(x_i))$ for every j with $a_j \in I_1$. In other words, $K_C(\mathbf{x}, \cdot)$ is a discrete probability measure, so $K_C(\mathbf{x}, \cdot) = K_C^{dis}(\mathbf{x}, \cdot)$ holds for all $\mathbf{x}$ fulfilling $M(\mathbf{x}) < 1$ and $\mathbf{x} \notin L_0^{1:d-1}$. Having this and using equation (5.10) $\mu_C^{dis}(\mathbb{I}^d) = 1$ follows.

(iii) Finally suppose that γ is singular without point masses and again consider some $\mathbf{x}$ fulfilling $M(\mathbf{x}) < 1$ and $\mathbf{x} \notin L_0^{1:d-1}$. To show that $\mu_C^{sing}(\mathbb{I}^d) = 1$, we prove that the distribution function $y \mapsto F_{\mathbf{x}}^C(y) = K_C(\mathbf{x}, [0, y])$ is continuous and has derivative 0 λ-almost everywhere which is equivalent to singularity of the Markov-kernel $K_C(\mathbf{x}, \cdot)$ with respect to λ. According to Theorem 5.6 we have $\mu_C(L_t) = \gamma(\{\frac{1}{\varphi(t)}\}) = 0$ for every $t \in \mathbb{I}$, hence the conditional distribution function $F_{\mathbf{x}}^C$ has no point masses and therefore $F_{\mathbf{x}}^C$ is continuous. It now suffices to show that the derivative $(F_{\mathbf{x}}^C)'$ fulfills $(F_{\mathbf{x}}^C)'(y) = 0$ for λ-almost every $y > f^0(\mathbf{x})$, which can be done as follows: As already mentioned before, our choice of $\mathbf{x}$ implies that $0 < \sum_{i=1}^{d-1} \varphi(x_i) < \varphi(0) \in (0, \infty]$, so in particular

$$\frac{1}{\sum_{i=1}^{d-1} \varphi(x_i)} > \frac{1}{\varphi(0)}$$

and $\gamma(I_1) = \gamma((0, \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i)}]) > \gamma((0, \frac{1}{\varphi(0)}]) \geq 0$. Defining the measure $m : \mathcal{B}((0, \infty)) \to [0, \infty]$ by

$$m(B) := \int_B t^{d-1} \mathrm{d}\gamma(t)$$

it follows that m is σ-finite (in fact, finite for every finite interval), singular with respect to λ, has no point masses and fulfills $0 < m(I_1) < \infty$. Letting $G_m : I_1 \to [0, \infty)$ denote the measure-generating function induced by m via $G_m(x) := m([0, x])$ singularity of m implies that $G_m' = 0$ λ-almost everywhere on $(0, \infty)$, hence considering

$$\Lambda := \{z \in I_1 : G_m'(z) = 0\}$$

yields $\lambda(\Lambda) = \lambda(I_1)$. Defining Υ by

$$\Upsilon := \left\{ y \in (f^0(\mathbf{x}), 1] : \frac{1}{\sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y)} \in \Lambda \right\} \in \mathcal{B}(\mathbb{I}),$$

using the fact that φ is differentiable and strictly decreasing on $(0, 1)$ with derivative bounded away from 0 on any compact interval $[a, b] \subseteq (0, 1)$ it follows (see [12, Lemma 7.1.29]) that $\lambda(\Upsilon) = \lambda((f^0(\mathbf{x}), 1])$. For every $y \in \Upsilon$, however, the chain rule together with (5.11),

$G_m\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right) = \int_{I_y} t^{d-1}\mathrm{d}\gamma$ and $m(I_1) = \int_{I_1} t^{d-1}\mathrm{d}\gamma(t)$ yields that

$$\begin{aligned}(F^C_{\mathbf{x}})'(y) &= \frac{1}{m(I_1)}\frac{\mathrm{d}}{\mathrm{d}y}G_m\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right)\\ &= \frac{1}{m(I_1)}\underbrace{G'_m\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right)}_{=0}\cdot\frac{\partial}{\partial y}\left(\frac{1}{\sum_{i=1}^{d-1}\varphi(x_i)+\varphi(y)}\right) = 0.\end{aligned}$$

Altogether we have shown that for arbitrary $\mathbf{x}$ fulfilling $M(\mathbf{x}) < 1$ and $\mathbf{x} \notin L_0^{1:d-1}$ the measure $K_C(\mathbf{x},\cdot)$ is singular without point masses, i.e., $K_C(\mathbf{x},\cdot) = K_C^{sing}(\mathbf{x},\cdot)$ holds and considering equation (5.10) again $\mu_C^{sing}(\mathbb{I}^d) = 1$ follows. □

Remark 5.13. The second assertion of Theorem 5.12 can be proved in the following alternative way (the afore-mentioned version was chosen in order to underline the similarity of the discrete and the singular case): Let $\gamma = \sum_{j\in J}\alpha_j\delta_{a_j}$ for some finite or countably infinite index set $J \subseteq \mathbb{N}$ where $\alpha_j > 0$ for every $j \in J$, and $\sum_{j\in J}\alpha_j = 1$ (without loss of generality we assume $a_i \neq a_j$ for $i \neq j$). Then for every $j \in J$ there exists a unique $t_j \in [0,1)$ with $\frac{1}{\varphi(t_j)} = a_j$ and according to Theorem 5.6 we have

$$\mu_C\left(\bigcup_{j\in J} L_{t_j}\right) = \sum_{j\in J}\mu_C(L_{t_j}) = \sum_{j\in J}\gamma\left(\left\{\frac{1}{\varphi(t_j)}\right\}\right) = 1.$$

The set $L := \bigcup_{j\in J} L_{t_j}$ is as at most countable union of Borel sets itself an element of $\mathcal{B}(\mathbb{I}^d)$. Applying disintegration (2.1) we have

$$1 = \mu_C(L) = \int_{\mathbb{I}^{d-1}} K_C(\mathbf{x}, L_{\mathbf{x}})\mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}),$$

from which $K_C(\mathbf{x}, L_{\mathbf{x}}) = 1$ for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x}$ follows. Considering that the $\mathbf{x}$-cut $L_{\mathbf{x}}$ of L is at most countably infinite we get $K_C^{dis}(\mathbf{x}, L_{\mathbf{x}}) = 1$ for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x}$ from which the desired result follows.

Theorem 5.12 has the following consequence, whereby we will let $\mathcal{C}^d_{ar,abs}$ denote the family of all absolutely continuous d-dimensional Archimedean copulas, $\mathcal{C}^d_{ar,dis}$ the family of all $C \in \mathcal{C}^d_{ar}$ with $\mu_C^{dis}(\mathbb{I}^d) = 1$, and $\mathcal{C}^d_{ar,sing}$ the family of all $C \in \mathcal{C}^d_{ar}$ with $\mu_C^{sing}(\mathbb{I}^d) = 1$.

Corollary 5.14. *$\mathcal{C}^d_{ar,dis}$, $\mathcal{C}^d_{ar,abs}$ and $\mathcal{C}^d_{ar,sing}$ are dense in $(\mathcal{C}^d_{ar}, d_\infty)$.*

Proof. Let C be an arbitrary Archimedean copula and γ denote its corresponding Williamson measure on $\mathcal{B}([0,\infty))$. The stated results now follow from the fact (see Theorem Appendix B.2) that γ is the weak limit of a sequence of discrete, of a sequence of absolutely continuous and of a sequence of singular Williamson measures in combination with Theorem 5.9 and Theorem 5.12. □

6. Singular Archimedean copulas with full support

The results established in the previous section allow to prove the existence of multivariate Archimedean copulas which, considering their handy analytic form, exhibit a surprisingly irregular behavior. In fact, we will construct singular d-dimensional Archimedean copulas with full support $\mathbb{I}^d$ and thereby extend the examples given in [4] to the multivariate setting. As in the previous section the representation in terms of Williamson measures will play a crucial role. We first focus on the construction of some $C \in \mathcal{C}_{ar}^d$ fulfilling that C has full support although $\mu_{C^{1:d-1}}$-almost every conditional distribution $K_C(\mathbf{x}, \cdot)$ is a singular measure without point masses and then discuss the discrete analogue.

Theorem 6.1. *There exists a copula $C \in \mathcal{C}_{ar}^d$ with the following properties:*

(1) $\mu_C^{sing}(\mathbb{I}^d) = 1$ and C has full support.

(2) For $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ the conditional distribution function $y \mapsto K_C(\mathbf{x}, [0, y])$ is continuous, strictly increasing and singular.

(3) All level sets L_t of C fulfill $\mu_C(L_t) = 0$.

(4) The Kendall distribution function F_K^d of C is continuous, strictly increasing and singular.

Proof. Suppose that h is a strictly increasing singular homeomorphism of $\mathbb{I}$, i.e., a strictly increasing bijective transformation mapping $\mathbb{I}$ to itself fulfilling $h'(x) = 0$ for λ-almost every $x \in \mathbb{I}$ (see, e.g., [6, 10] for several well-known examples). Defining $F : [0, \infty) \to [0, 1]$ by

$$F(x) = \frac{1}{2}\, h\left(\frac{x}{2}\right) \mathbf{1}_{[0,2)}(x) \; + \sum_{i=1}^{\infty} \left(1 - \frac{1}{2^i} + \frac{1}{2^{i+1}}\, h\left(\frac{x - 2^i}{2^i}\right)\right) \mathbf{1}_{[2^i,2^{i+1})}(x)$$

obviously yields a strictly increasing continuous function F which, by construction, fulfills $F' = 0$ λ-almost everywhere. Letting β denote the corresponding probability measure on $\mathcal{B}([0, \infty))$ it follows that β is singular without point masses. Furthermore, the support of β contains 0 but in general does not need to be an element of $\mathcal{P}_{\mathcal{W}_d}$, we only know that

$$\int_{\mathbb{I}} (1 - t)^{d-1} \mathrm{d}\beta(t) \in (0, 1).$$

Proceeding, however, like in the proof of Lemma Appendix B.1 we can find some constant $a \in (0, \infty)$ such that the push-forward $\gamma = \beta^{T_a}$ with $T_a(x) = ax$ fulfills $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Considering that γ is obviously singular (without point masses) too and that the support of γ coincides with $[0, \infty)$ using Lemma 5.5 as well as Theorem 5.12 it follows that the corresponding d-dimensional Archimedean copula $C = C_\gamma$ is strict and fulfills $\mu_{C_\gamma}^{sing}(\mathbb{I}^d) = 1$. Furthermore, according to the proof of Theorem 5.12 for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ the conditional distribution function $y \mapsto K_C(\mathbf{x}, [0, y])$ is continuous and singular. Hence, considering that γ has full support using equation (5.11) yields that $y \mapsto K_C(\mathbf{x}, [0, y])$ is also strictly increasing on $\mathbb{I}$.
Having that, showing that C has full support is straightforward: In fact, for every

$(\mathbf{x}, y) \in (0,1)^{d-1} \times (0,1)$ and every open rectangle $U = U_1 \times \cdots \times U_d$ with open non-empty intervals $U_1, \ldots, U_d \subseteq (0,1)$ fulfilling $(\mathbf{x}, y) \in U$ and $U \subseteq (0,1)^d$ we can proceed as follows: Considering that $C^{1:d-1}$ is absolutely continuous strictness of ψ implies that the density $c^{1:d-1}$ of $C^{1:d-1}$ fulfills $c^{1:d-1} > 0$ λ_{d-1}-almost everywhere in $(0,1)^{d-1}$ using disintegration and the fact that $K_C(\mathbf{x}, \cdot)$ has full support and hence fulfills $K_C(\mathbf{x}, U_d) > 0$ for $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ it follows that

$$\mu_C(U) = \int_{\times_{j=1}^{d-1} U_j} K_C(\mathbf{x}, U_d) \,\mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) > 0.$$

This shows that $(\mathbf{x}, y)$ is contained in the support of μ_C, since supports are closed the the support of μ_C is $\mathbb{I}^d$ and the first two assertions are proved. Since γ has no point masses the third assertion is an immediate consequence of Theorem 5.6 and it remains to prove the last assertion. Again according to Theorem 5.6

$$F_K^d(t) = \gamma([0, \tfrac{1}{\varphi(t)}]) = \gamma((0, \tfrac{1}{\varphi(t)}])$$

holds for every $t \in (0,1]$, implying that F_K^d is continuous and strictly increasing. Finally, using a chain rule argument similar to the one at the end of the proof of Theorem 5.12 shows that $(F_K^d)'(x) = 0$ holds for λ-almost every $x \in \mathbb{I}$ and the proof is complete. $\square$

Starting with the probability measure $\beta := \sum_{i=1}^{\infty} 2^{-i} \delta_{q_i}$ with $\{q_1, q_2, \ldots\}$ denoting an enumeration of the rationals in $(0, \infty)$ and proceeding analogously to the proof of the previous theorem yields the following discrete version of it:

Theorem 6.2. *There exists a copula $C \in \mathcal{C}_{ar}^d$ with the following properties:*

(1) $\mu_C^{dis}(\mathbb{I}^d) = 1$ and C has full support.

(2) For $\mu_{C^{1:d-1}}$-almost every $\mathbf{x} \in \mathbb{I}^{d-1}$ the conditional distribution function $y \mapsto K_C(\mathbf{x}, [0, y])$ is a strictly increasing step function.

(3) There exists a dense countable subset $\mathcal{Q}$ of $(0,1)$ such that $\mu_C(L_t) > 0$ if, and only if, $t \in \mathcal{Q}$.

(4) The Kendall distribution function F_K^d of C is a strictly increasing step function.

Appendix A. Level set mass and Kendall distribution function: Calculations

Recall from Section 3 that the t-level hypersurfaces f^t are defined on the upper t-cuts $[C^{1:d-1}]_t$ of the $(d-1)$-marginal. Using the notation $\mathbf{x}_m = (x_1, x_2, \ldots, x_m)$, $m \in \mathbb{N}$, for $\mathbf{x} \in [C^{1:d-1}]_t$ we have $x_1 \geq t, x_2 \geq \psi(\varphi(t) - \varphi(x_1)) =: f^t(x_1), x_3 \geq \psi(\varphi(t) - \varphi(x_1) - \varphi(x_2)) =: f^t(\mathbf{x}_2)$ and iteratively,

$$x_{d-1} \geq \psi\big(\varphi(t) - \sum_{i=1}^{d-2} \varphi(x_i)\big) =: f^t(\mathbf{x}_{d-2}).$$

Proposition Appendix A.1. *Suppose that $C \in \mathcal{C}_{ar}^d$ has generator ψ and let μ_C denote the corresponding d-stochastic measure. Then for every $t > 0$ we have*

$$\mu_C(L_t) = \frac{(-\varphi(t))^{d-1}}{(d-1)!} \cdot \left(D^-\psi^{(d-2)}(\varphi(t)) - D^-\psi^{(d-2)}(\varphi(t-))\right). \tag{A.1}$$

If C is strict then $\mu_C(L_0) = 0$ and for non-strict C,

$$\mu_C(L_0) \quad = \tfrac{(-\varphi(0))^{d-1}}{(d-1)!} \cdot D^-\psi^{(d-2)}(\varphi(0)). \tag{A.2}$$

Proof. We start with $t > 0$. Using disintegration (2.1), the definition of f^t and the fact that $C^{1:d-1}$ is absolutely continuous with density $c^{1:d-1}$ we get

$$\begin{aligned}
\mu_C(L_t) &= \int_{\mathbb{I}^{d-1}} K_C(\mathbf{s}, (L_t)_{\mathbf{s}}) \, \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{s}) \\
&= \int_{[t,1]\times[f^t(s_1),1]\times\ldots\times[f^t(\mathbf{s}_{d-2}),1]} K_C(\mathbf{s}, \{f^t(\mathbf{s})\}) \, \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{s}) \\
&= \int_{[t,1]} \cdots \int_{[f^t(\mathbf{s}_{d-2}),1]} \prod_{i=1}^{d-1} \varphi'(s_i) \cdot \left[D^-\psi^{(d-2)}(\varphi(t)) - D^-\psi^{(d-2)}(\varphi(t-))\right] \, \mathrm{d}\lambda(\mathbf{s}) \\
&= \left[D^-\psi^{(d-2)}(\varphi(t)) - D^-\psi^{(d-2)}(\varphi(t-))\right] \cdot \int_{[t,1]} \cdots \int_{[f^t(\mathbf{s}_{d-2}),1]} \prod_{i=1}^{d-1} \varphi'(s_i) \, \mathrm{d}\lambda(\mathbf{s}).
\end{aligned}$$

Letting (II) denote the iterated integrals in the previous line we have

$$(II) = \int_{[t,1]} \int_{[f^t(s_1),1]} \cdots \prod_{i=1}^{d-2} \varphi'(s_i) \int_{[f^t(\mathbf{s}_{d-2}),1]} \varphi'(s_{d-1})(-1)^0 \left[\varphi(t) - \sum_{i=1}^{d-1-0} \varphi(s_i)\right]^0 \, \mathrm{d}\lambda(\mathbf{s})$$

and the chain rule directly yields

$$\begin{aligned}
(II) = \int_{[t,1]} \cdots \int_{[f^t(\mathbf{s}_{d-4}),1]} \prod_{i=1}^{d-3} \varphi'(s_i) \int_{[f^t(\mathbf{s}_{d-3}),1]} \varphi'(s_{d-2}) \\
\cdot \frac{(-1)^1}{1} \cdot \left[\varphi(t) - \sum_{i=1}^{d-1-1} \varphi(s_i)\right]^1 \, \mathrm{d}\lambda(s_{d-2})\mathrm{d}\lambda(\mathbf{s}_{d-3}).
\end{aligned}$$

Proceeding analogously for s_{d-2} gives

$$(II) = \int_{[t,1]} \cdots \int_{[f^t(\mathbf{s}_{d-4}),1]} \prod_{i=1}^{d-3} \varphi'(s_i) \cdot \frac{(-1)^2}{1 \cdot 2} \cdot \left[\varphi(t) - \sum_{i=1}^{d-1-2} \varphi(s_i)\right]^2 \, \mathrm{d}\lambda(\mathbf{s}_{d-3})$$

and after finitely many steps we obtain

$$(II) = \int_{[t,1]} \varphi'(s_1) \cdot \frac{(-1)^{d-2}}{1 \cdot 2 \cdots (d-2)} \left[\varphi(t) - \varphi(s_1)\right]^{d-2} \, \mathrm{d}\lambda(s_1) = \frac{(-1)^{d-1}}{(d-1)!} \cdot \varphi(t)^{d-1}$$

as desired. For $t = 0$ and strict C we obviously have $\mu_C(L_0) = 0$. For non-strict C we have $K_C(\mathbf{s}, \{f^0(\mathbf{s})\}) = K_C(\mathbf{s}, [0, f^0(\mathbf{s})])$ and calculations as those above yield the result. □

Proposition Appendix A.2. *Suppose that $C \in \mathcal{C}^d_{ar}$ has generator ψ. Then for $t > 0$*

$$F^d_K(t) = D^-\psi^{(d-2)}(\varphi(t))\frac{(-1)^{d-1}}{(d-1)!}\varphi(t)^{d-1} + \sum_{k=0}^{d-2}\psi^{(k)}(\varphi(t))\frac{(-1)^k}{k!}\varphi(t)^k. \tag{A.3}$$

holds. For $t = 0$ and strict C we have $F^d_K(0) = 0$ and for non-strict C,

$$F^d_K(0) = D^-\psi^{(d-2)}(\varphi(0)) \cdot \frac{(-1)^{d-1}}{(d-1)!} \cdot \varphi(0)^{d-1}. \tag{A.4}$$

Proof. Applying disintegration (2.1) and decomposing $\mathbb{I}^{d-1} = [C^{1:d-1}]_t \cup [C^{1:d-1}]^c_t$ yields

$$\begin{aligned} F^d_K(t) &= \mu_C([C]^c_t) = \int_{\mathbb{I}^{d-1}} K_C\left(\mathbf{x}, ([C]^c_t)_{\mathbf{x}}\right)\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) \\ &= \int_{[C^{1:d-1}]_t} K_C\left(\mathbf{x}, ([C]^c_t)_{\mathbf{x}}\right)\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) + \int_{[C^{1:d-1}]^c_t} K_C\left(\mathbf{x}, ([C]^c_t)_{\mathbf{x}}\right)\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) \end{aligned}$$

Denoting by (III) and (IV) the first and the second of the above summands, respectively, analogously as in Proposition Appendix A.1 we obtain

$$(III) = D^-\psi^{(d-2)}(\varphi(t)) \cdot \frac{(-1)^{d-1}}{(d-1)!}\varphi(t)^{d-1}.$$

Regarding (IV), we have $\mathbf{x} \in [C^{1:d-1}]^c_t$ if, and only if, $([C]^c_t)_{\mathbf{x}} = \mathbb{I}$ and hence

$$(IV) = \int_{[C^{1:d-1}]^c_t} 1\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x}) = \mu_{C^{1:d-1}}([C^{1:d-1}]^c_t) = F^{d-1}_K(t).$$

Proceeding iteratively finally yields

$$F^d_K(t) = D^-\psi^{(d-2)}(\varphi(t)) \cdot \frac{(-1)^{d-1}}{(d-1)!} \cdot \varphi(t)^{d-1} + \sum_{k=1}^{d-2}\psi^{(k)}(\varphi(t))\frac{(-1)^k}{k!}\varphi(t)^k + t.$$

For $t = 0$ we have $F^d_K(t) = \int_{\{\mathbf{s}\in\mathbb{I}^{d-1}:\ \sum_{i=1}^{d-1}\varphi(s_i)\le\varphi(0)\}} K_C(\mathbf{x}, ([C]^c_0)_{\mathbf{x}})\ \mathrm{d}\mu_{C^{1:d-1}}(\mathbf{x})$ and the result follows in the same manner. □

Lemma Appendix A.3. *Suppose that $f, f_1, f_2, \dots$ are convex functions such that $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise on $(0,\infty)$. Then $\lim_{n\to\infty} D^-f_n(x) = D^-f(x)$ holds for every $x \in Cont(D^-f)$. Moreover, the sequence $(D^-f_n)_{n\in\mathbb{N}}$ converges continuously to D^-f on $Cont(D^-f)$, i.e., for every sequence $(z_n)_{n\in\mathbb{N}}$ with limit $z \in Cont(D^-f)$ we have*

$$\lim_{n\to\infty} D^-f_n(z_n) = D^-f(z).$$

Proof. Convexity implies that for every $h > 0$ we have

$$\limsup_{n\to\infty} D^+f_n(x) \le \limsup_{n\to\infty}\frac{f_n(x+h) - f_n(x)}{h} = \frac{f(x+h) - f(x)}{h},$$

which, considering $h \downarrow 0$, yields $\limsup_{n\to\infty} D^+ f_n(x) \leq D^+ f(x)$. The other inequality $\liminf_{n\to\infty} D^- f_n(x) \geq D^- f(x)$ follows in the same manner, so altogether we get

$$D^- f(x) \leq \liminf_{n\to\infty} D^- f_n(x) \leq \limsup_{n\to\infty} D^+ f_n(x) \leq D^+ f(x)$$

from which the fist assertion follows since for $x \in \mathrm{Cont}(D^- f)$ we have $D^+ f(x) = D^- f(x)$.

For the second part let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$ converging to some point $z \in \mathrm{Cont}(D^- f) \subseteq (0, \infty)$. We can find two strictly decreasing sequences $(a_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}}$ in $(0, \infty)$ converging to 0 with $z - a_k, z + b_k \in \mathrm{Cont}(D^- f)$ for every $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. Then there exists some index $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have that $z - a_k < z_n < z + b_k$ holds. Monotonicity of $D^- f_n$ implies

$$D^- f_n(z - a_k) \leq D^- f_n(z_n) \leq D^- f_n(z + b_k)$$

for every such $n \geq n_0$. Having this we get

$$\begin{aligned} \lim_{n\to\infty} D^- f_n(z - a_k) &\leq \liminf_{n\to\infty} D^- f_n(z_n) \\ &\leq \limsup_{n\to\infty} D^- f_n(z_n) \leq \lim_{n\to\infty} D^- f_n(z + b_k), \end{aligned}$$

from which the result follows since $z \in \mathrm{Cont}(D^- f)$ and therefore

$$\lim_{k\to\infty} \lim_{n\to\infty} D^- f_n(z - a_k) = D^- f(z) = \lim_{k\to\infty} \lim_{n\to\infty} D^- f_n(z + b_k)$$

holds. □

Appendix B. Approximations by discrete, absolutely continuous and singular Williamson measures

We now tackle Theorem Appendix B.2 already used in the proof of Corollary 5.14 and start with the following simple lemma simplifying the approximation procedure. Thereby, for every $a \in (0, \infty)$ we will let $T_a : (0, \infty) \to (0, \infty)$ denote the linear transformation $T_a(x) = ax$.

Lemma Appendix B.1. *Suppose that $\gamma \in \mathcal{P}_{\mathcal{W}_d}$, that $(\beta_n)_{n\in\mathbb{N}}$ is a sequence of probability measures on $\mathcal{B}([0, \infty)$ satisfying $\beta_n(\{0\}) = 0$ for every $n \in \mathbb{N}$ but not necessarily fulfilling equation (5.2), and that $(\beta_n)_{n\in\mathbb{N}}$ converges weakly to γ. Then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $(0, \infty)$ converging to $\frac{1}{2}$ such that the following properties hold:*

- *Each probability measure $\gamma_n := \beta_n^{T_{a_n}}, n \in \mathbb{N}$, fulfills $\gamma_n \in \mathcal{P}_{\mathcal{W}_d}$.*
- *$(\gamma_n)_{n\in\mathbb{N}}$ converges weakly on $[0, \infty)$ to γ.*

Proof. Let ψ_γ denote the normalized Archimedean generator corresponding to γ and ψ_{β_n} the (not necessarily normalized) generator induced via $\psi_{\beta_n} = \mathcal{W}_d(\beta_n)$. Then proceeding as in the first part of the proof of Theorem 5.9 it follows that $(\psi_{\beta_n})_{n\in\mathbb{N}}$ converges uniformly to ψ_γ. Letting a_n denote the unique element in $(0, \infty)$ fulfilling $\psi_{\beta_n}(a_n) = \frac{1}{2}$ using monotonicity

of generators and the fact that ψ_γ is normalized it is straightforward to verify that $(a_n)_{n\in\mathbb{N}}$ converges to 1.
The probability measure $\gamma_n := \beta_n^{T_{a_n}}$ obviously fulfills $\gamma_n(\{0\}) = 0$, moreover using change of coordinates yields

$$\begin{aligned}\frac{1}{2} &= \psi_{\beta_n}(a_n) = \int_{(0,\infty)} (1-ta_n)_+^{d-1}\,\mathrm{d}\beta_n(t) = \int_{(0,\infty)} (1-T_{a_n}(t))_+^{d-1}\,\mathrm{d}\beta_n(t)\\ &= \int_{(0,\infty)} (1-s)_+^{d-1}\,\mathrm{d}\gamma_n(t) = \int_{(0,1)} (1-s)^{d-1}\,\mathrm{d}\gamma_n(t),\end{aligned}$$

so $\gamma_n \in \mathcal{P}_{\mathcal{W}_d}$ and it remains to show weak convergence. Applying Lemma 5.2 and Lemma 4.4 (continuous convergence) yields

$$\begin{aligned}\lim_{n\to\infty} \gamma_n([0,z]) &= \lim_{n\to\infty} \beta_n\left(\left[0, \frac{z}{a_n}\right]\right)\\ &= \lim_{n\to\infty} \sum_{k=0}^{d-2} \frac{(-1)^k \psi_{\beta_n}^{(k)}(\frac{a_n}{z})}{k!}\left(\frac{a_n}{z}\right)^k + \frac{(-1)^{d-1} D^- \psi_{\beta_n}^{(d-2)}(\frac{a_n}{z})}{(d-1)!}\left(\frac{a_n}{z}\right)^{d-1} = \gamma([0,z])\end{aligned}$$

for every point $z \in (0,\infty)$ with $\gamma(\{z\}) = 0$ which completes the proof. □

Theorem Appendix B.2. *Suppose that $\gamma \in \mathcal{P}_{\mathcal{W}_d}$. Then there exists a sequence $(\gamma_n^1)_{n\in\mathbb{N}}$ of discrete measures in $\mathcal{P}_{\mathcal{W}_d}$, a sequence $(\gamma_n^2)_{n\in\mathbb{N}}$ of singular measures without point masses in $\mathcal{P}_{\mathcal{W}_d}$, and a sequence $(\gamma_n^3)_{n\in\mathbb{N}}$ of absolutely continuous measures $\mathcal{P}_{\mathcal{W}_d}$ that all converge weakly to γ on $[0,\infty)$.*

Proof. Let F denote the distribution function corresponding to γ, i.e., $F(z) = \gamma((0,z]) = \gamma([0,z])$ for every $z \in [0,\infty)$ and let $\mathcal{Q} := \{q_0, q_1, q_2, \ldots\}$ denote a countably infinite subset of $\mathrm{Cont}(F)$ which is dense in $[0,\infty)$. Without loss of generality we assume that $q_0 = 0$. Furthermore let the function $f : \mathbb{I} \to [0,1]$ be right-continuous, non-decreasing with $f(0) = 0, f(1) = 1$ and $g : [0,\infty) \to \mathbb{I}$ be right-continuous, non-decreasing with $g(0) = 0, g(\infty) = 1$. For every non-degenerated compact interval $[a,b] \subseteq [0,\infty)$ and compact interval $[c,d] \subseteq [0,1]$ define the rescaled version $f_{[a,b]}^{[c,d]} : [a,b] \to [c,d]$ of f to $[a,b], [c,d]$ by

$$f_{[a,b]}^{[c,d]}(x) = c + (d-c) f\left(\frac{x-a}{b-a}\right),$$

and for every interval $[a,\infty) \subseteq [0,\infty)$ and $[c,d] \subseteq \mathbb{I}$ define $g_{[a,\infty)}^{[c,d]} : [a,\infty) \to [c,d]$ of g to $[a,\infty), [c,d]$ by

$$g_{[a,\infty)}^{[c,d]}(x) = c + (d-c) g(x-a).$$

Using this notation define the distribution function $F_1 : [0,\infty) \to [0,1]$ by

$$F_1(x) = f_{[0,q_1]}^{[0,F(q_1)]}(x) \cdot \mathbf{1}_{[0,q_1]}(x) \,+\, g_{[q_1,\infty)}^{[F(q_1),1]}(x) \cdot \mathbf{1}_{(q_1,\infty)}(x)$$

and notice that F_1 fulfills $F_1(q_i) = F(q_i)$ for $i \in \{0, 1\}$. In the second step define the distribution function $F_2 : [0, \infty) \to [0, 1]$ by

$$F_2(x) = f_{[0,q^2_{(1)}]}^{[0,F(q^2_{(1)})]}(x) \cdot \mathbf{1}_{[0,q^2_{(1)}]}(x) + f_{[q^2_{(1)},q^2_{(2)}]}^{[F(q^2_{(1)}),F(q^2_{(2)})]}(x) \cdot \mathbf{1}_{(q^2_{(1)},q^2_{(2)}]}(x) + g_{[q^2_{(2)},\infty)}^{[F(q^2_{(2)}),1]}(x) \cdot \mathbf{1}_{(q^2_{(2)},\infty)}(x),$$

whereby $0 < q^2_{(1)} < q^2_{(2)}$ denotes the order statistics of $0, q_1, q_2$ (the exponent denotes the 'sample size'). Obviously F_2 fulfills $F_2(q_i) = F(q_i)$ for $i \in \{0, 1, 2\}$. Proceeding analogously yields a sequence $(F_n)_{n\in\mathbb{N}}$ of distribution functions on $[0, \infty)$ fulfilling that for every $i \in \mathbb{N}$ we have $F_n(q_i) = F(q_i)$ for every $n \geq i$. In other words, the sequence $(F_n)_{n\in\mathbb{N}}$ converges on a dense set to the distribution function F, implying that $(F_n)_{n\in\mathbb{N}}$ converges to F weakly. Notice that the just discussed construction works for arbitrary f, g fulfilling the aforementioned requirements. If we chose both f and g absolutely continuous then obviously each F_n is absolutely continuous, if we chose both f and g as step functions then F_n is a step function, and if we choose f, g to be continuous with $f' = 0$ and $g' = 0$ almost everywhere (one could use, for instance, the Cantor function or work with any other strictly increasing singular continuous distribution function, see [10]) then each F_n is continuous and obviously fulfills $F_n' = 0$ almost everywhere. The desired result now follows by considering the probability measures β_n corresponding to F_n and applying Lemma Appendix B.1. □

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