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A link between Kendall's τ, the length measure and the surface of bivariate copulas, and a consequence to copulas with self-similar support

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Abstract: Working with shuffles we establish a close link between Kendall's τ, the so-called length measure, and the surface area of bivariate copulas and derive some consequences. While it is well-known that Spearman's ρ of a bivariate copula A is a rescaled version of the volume of the area under the graph of A, in this contribution we show that the other famous concordance measure, Kendall's τ, allows for a simple geometric interpretation as well - it is inextricably linked to the surface area of A.

Keywords: Asymmetry, copula, dependence measure, exchangeability, Markov kernel

1 Introduction

Spearman's ρ of a bivariate copula A is a rescaled version of the volume below the graph of A (see [3, 14]) in the sense that

$$\rho(A) = 12 \int_{[0,1]^2} A \, d\lambda_2 - 3$$

holds. Letting $[A]_t := \{(x, y) \in [0,1]^2 : A(x, y) \geq t\}$ denote the lower t-cut of A for every $t \in [0,1]$ and applying Fubini's theorem directly yields

$$\rho(A) = 12 \int_{[0,1]} \lambda_2([A]_t) d\lambda(t) - 3,$$

which lead the authors of [1] to conjecturing that adequately rescaling the so-called length measure $\ell(A)$ of A, defined as the average arc-length of the contour lines of A, might result in a (new or already known) concordance measure. The conjecture was falsified in [1], only some but not all properties of a concordance measure are fulfilled, in particular, we do not have continuity with respect to pointwise convergence of copulas in general.

Motivated by the afore-mentioned facts, the objective of this note is two-fold: we first derive the somewhat surprising result that on a subfamily of bivariate copulas - the class $\mathcal{C}_{mcd}$ of all mutually completely dependent copulas (including all classical shuffles), which is dense in the class $\mathcal{C}$ of all bivariate copulas with respect to uniform convergence - the length measure is, in fact, an affine transformation of Kendall's τ and vice versa. As a consequence, the length measure restricted to $\mathcal{C}_{mcd}$ is continuous with respect to pointwise convergence of copulas. We then focus on the surface area of bivariate copulas and derive analogous statements, i.e., that on the class $\mathcal{C}_{mcd}$ the surface area is an affine transformation of Kendall's τ (and hence of the length measure)

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too. For obtaining both main results a simple geometric identity linking the length measure and the surface area with the area of the set $\Omega_{\sqrt{2}}$, given by

$$\Omega_{\sqrt{2}}^{A_h} = \left\{(x,y) \in [0,1]^2 : h(x) \leq y,\, h^{-1}(y) \leq x\right\}, \tag{1}$$

where h denotes the transformation corresponding to the completely dependent copula A_h, will be key.

The remainder of this note is organized as follows: In Section 2 we gather preliminaries and notations that will be used in the sequel. Section 3 is the core of the paper, derives the afore-mentioned identities linking Kendall's τ, the length measure and the surface of mutually completely dependent copulas, provides a geometrical interpretation, and shows that outside the class $\mathcal{C}_{mcd}$ these identities need not hold. As a small (but mathematically interesting) application of the established relationships, Section 4 derives simple formulas for the length measure and the surface area of completely dependent copulas with self-similar support which, without the afore-mentioned identities seem very hard to establish. Finally, Section 5 provides an outlook on related questions to be tackled in the future.

2 Notation and preliminaries

In the sequel we will let $\mathcal{C}$ denote the family of all bivariate copulas. For each copula $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C, i.e., $\mu_C([0,x] \times [0,y]) = C(x,y)$ holds for all $x, y \in [0,1]$. Considering the uniform metric d_∞ on $\mathcal{C}$ it is well-known that $(\mathcal{C}, d_\infty)$ is a compact metric space and that in $\mathcal{C}$ pointwise and uniform convergence are equivalent. For more background on copulas and doubly stochastic measures we refer to [3, 14].

For every metric space (Ω, d) the Borel σ-field in Ω will be denoted by $\mathcal{B}(\Omega)$. The Lebesgue measure on the Borel σ-field $\mathcal{B}([0,1]^2)$ of $[0,1]^2$ will be denoted by λ_2, the univariate version on $\mathcal{B}([0,1])$ by λ. Given probability spaces $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\Omega', \mathcal{A}', \mathbb{P})'$ and a measurable transformation $T : \Omega \to \Omega'$ the push-forward of $\mathbb{P}$ via T will be denoted by $\mathbb{P}^T$, i.e., $\mathbb{P}^T(F) = \mathbb{P}(T^{-1}(F))$ for all $F \in \mathcal{A}'$.

In what follows, Markov kernels will be a handy tool. A mapping $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called a Markov kernel from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ if the mapping $x \mapsto K(x,B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and the mapping $B \mapsto K(x,B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K\colon \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called regular conditional distribution of a (real-valued) random variable Y given (another random variable) X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbb{1}_B \circ Y | X)(\omega)$$

holds $\mathbb{P}$-a.s. It is well known that a regular conditional distribution of Y given X exists and is unique $\mathbb{P}^X$-almost surely. For every $A \in \mathcal{C}$ (a version of) the corresponding regular conditional distribution (i.e., the regular conditional distribution of Y given X in the case that $(X,Y) \sim A$) will be denoted by $K_A(\cdot,\cdot)$ and directly be interpreted as mapping from $K_A : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$. Note that for every $A \in \mathcal{C}$ and Borel sets $E, F \in \mathcal{B}([0,1])$ we have the following disintegration formulas:

$$\int\limits_E K_A(x,F)\,d\lambda(x) = \mu_A(E \times F) \quad \text{and} \quad \int\limits_{[0,1]} K_A(x,F)\,d\lambda(x) = \lambda(F) \tag{2}$$

For more details and properties of conditional expectations and regular conditional distributions we refer to [10, 12].

A copula $A \in \mathcal{C}$ will be called completely dependent (or functionally dependent) if there exists some λ-preserving transformation $h : [0,1] \to [0,1]$ (i.e., a transformation with $\lambda^h = \lambda$) such that $K(x,E) = \mathbf{1}_E(h(x))$ is a Markov kernel of A. The copula induced by h will be denoted by A_h, the class of all completely dependent copulas by $\mathcal{C}_{cd}$. A completely dependent copula A_h is called mutually completely dependent, if the transformation h is bijective. Notice that in this case the transpose A_h^t of A_h, defined by $A_h^t(x,y) = A_h(y,x)$, coincides with $A_{h^{-1}}$. The family of all mutually completely dependent copulas will be denoted by $\mathcal{C}_{mcd}$. Notice that mutually completely dependent copulas model the seemingly pathological case of pairs (X, Y) of uniform $[0, 1]$ random variables X, Y such that Y is a measurable function of X and vice versa. It is well known (see [3, 14]), however, that $\mathcal{C}_{mcd}$ is dense in $(\mathcal{C}, d_\infty)$, in fact even the family of all equidistant even shuffles (again see [3, 14]) is dense. This very observation led to the observation that d_∞ is not able to distinguish independence and complete dependence, which, in turn, triggered the study of stronger metrics overcoming that problem in [16]. For further properties of completely dependent copulas we refer to [16] and the references therein.

Turning towards the length profile introduced and studied in [1], let $\Gamma_{A,t}$ denote the boundary of the lower t-cut $[A]_t$ in $(0, 1)^2$ and $H_1(\Gamma_{A,t})$ it's arc-length. Then the length profile of A is defined as the function $L_A : [0, 1] \to [0, \infty)$, given by

$$L_A(t) = H_1(\Gamma_{A,t}).$$

It is easy to verify that

$$\sqrt{2}(1-t) \leq L_A(t) \leq 2(1-t) \tag{3}$$

holds for every $t \in (0, 1)$. Building upon L_A the so-called length measure $\ell(A)$ of A is defined as

$$\ell(A) = \int_{(0,1)} L_A(t) d\lambda(t) \tag{4}$$

and describes the average arc-length of upper t-cuts of A. Using ineq. (3) immediately yields

$$\ell(W) = \frac{1}{\sqrt{2}} \leq \ell(A) \leq 1 = \ell(M) \tag{5}$$

as well as $\ell(A) \in [\frac{1}{\sqrt{2}}, 1]$ holds (ineq. (5) was also one of the reasons for falsely conjecturing that the length measure might be transformable into a concordance measure).

In [1] it was shown that for mutually completely dependent copulas A_h the length profile allows for a simple calculation. In fact, using the co-area formula we have

$$\ell(A_h) = \int_{(0,1)^2} \|\nabla A_h(u, v)\|_2 \, d\lambda_2(u, v),$$

where ∇A_h denotes the gradient of A_h (whose existence λ_2-almost everywhere is assured by Rademacher's theorem and Lipschitz continuity, see [5]). The last equation simplifies to the nice identity

$$\ell(A_h) = 1 - (2 - \sqrt{2})\, \lambda_2(\Omega_{\sqrt{2}}), \tag{6}$$

with

$$\begin{aligned} \Omega_{\sqrt{2}}^{A_h} := \Omega_{\sqrt{2}} &= \Big\{(u, v) \in (0, 1)^2 : \|\nabla A_h(u, v)\|_2 = \sqrt{2}\Big\} \\ &= \Big\{(u, v) \in (0, 1)^2 : h(u) \leq v,\, h^{-1}(v) \leq u\Big\}. \end{aligned} \tag{7}$$

According to [1] (also see the proof of Lemma 3.4) the λ_2-measure of the set $\Omega_{\sqrt{2}}$ coincides with the one of the set Ω_0, defined by

$$\begin{aligned} \Omega_0^{A_h} := \Omega_0 &= \Big\{(u, v) \in (0, 1)^2 : \|\nabla A_h(u, v)\|_2 = 0\Big\} \\ &= \Big\{(u, v) \in (0, 1)^2 : h(u) > v,\, h^{-1}(v) > u\Big\}. \end{aligned} \tag{8}$$

Obviously $\Omega_{\sqrt{2}}$ is the set of all points $(u, v) \in [0, 1]^2$ 'above', and Ω_0 the set of all points $(u, v) \in [0, 1]^2$ 'below' the graphs of h and h^{-1}. Notice that for classical equidistant straight shuffles eq. (6) implies that $\ell(A_h)$ can be calculated by simply counting squares as Figure 2 illustrates in terms of two simple examples - one shuffle with three, and a second one with nice equidistant stripes. Throughout the rest of this note we will only write $\Omega_{\sqrt{2}}$ instead of $\Omega_{\sqrt{2}}^{A_h}$ as well as Ω_0 instead of $\Omega_0^{A_h}$ whenever no confusion will arise.

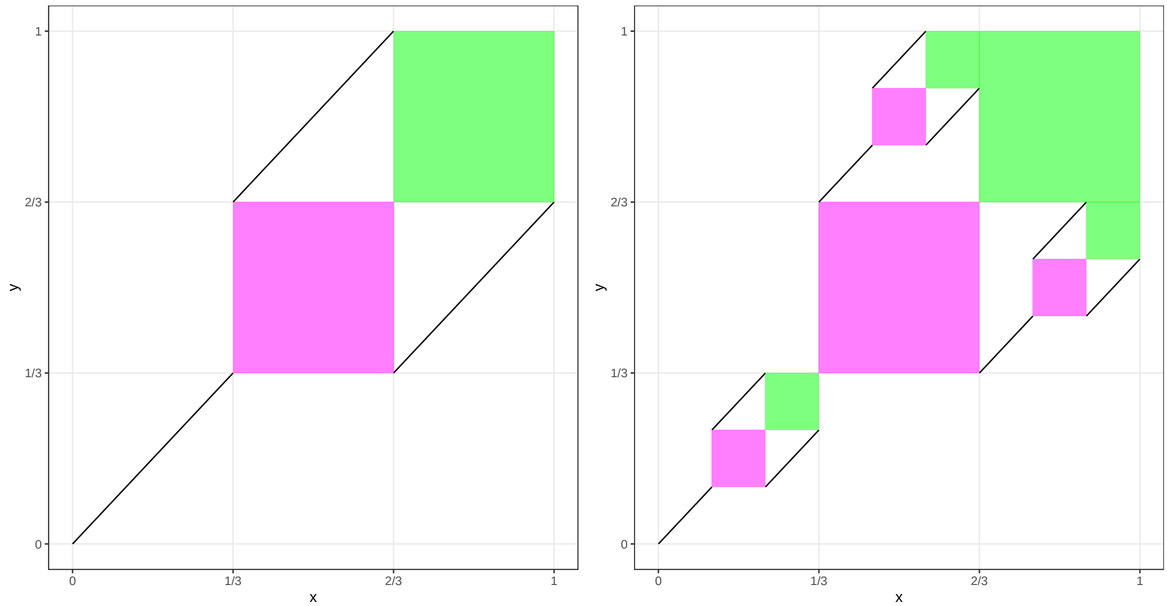

Fig. 1: The sets Ω_0 (in magenta) and $\Omega_{\sqrt{2}}$ (in green) for an even shuffles of three strips (left panel) and nine strips (right panel). In this case we have $\lambda_2(\Omega_{\sqrt{2}}) = \frac{1}{9}$ and $\ell(A_h) = 1 - (2-\sqrt{2})\frac{1}{9}$ for the first shuffle and $\lambda_2(\Omega_{\sqrt{2}}) = \frac{1}{9} + \frac{1}{27}$ as well as $\ell(A_h) = 1 - (2-\sqrt{2})\left(\frac{1}{9} + \frac{1}{27}\right)$ for the second one.

3 The interrelations

We now derive a simple formula linking Kendall's τ and the length measure for mutually completely dependent copula and start with some preliminary observations. Working with checkerboard copulas, using integration by parts (see [14]) and finally applying an approximation result like [11, Theorem 3.2] yields that for arbitrary bivariate copulas $A, B \in \mathcal{C}$ the following identity holds:

$$\tau(A) = 4 \int_{[0,1]^2} A d\mu_A - 1 = 4\left(\frac{1}{2} - \int_{[0,1]^2} K_A(x,[0,y])K_{A^t}(y,[0,x])d\lambda_2(x,y)\right) - 1. \tag{9}$$

For $A_h \in \mathcal{C}_{mcd}$ eq. (9) can be derived in the following simple alternative way, which we include for the sake of completeness: Using the fact that for $A_h \in \mathcal{C}_{mcd}$ and every $x \in [0,1]$ we have

$$\begin{aligned}
A(x,h(x)) &= \int_{[0,x]} K_{A_h}(t,[0,h(x)])d\lambda(t) = \int_{[0,x]} K_{A_h}(t,[0,h(x)))d\lambda(t) \\
&= \int_{[0,x]} \mathbf{1}_{[0,h(x))}(h(t))d\lambda(t) = \int_{[0,x]} (1 - \mathbf{1}_{[h(x),1]}(h(t))d\lambda(t) \\
&= x - \int_{[0,1]} \mathbf{1}_{[0,x]}(t)\mathbf{1}_{[h(x),1]}(h(t))d\lambda(t).
\end{aligned}$$

Using disintegration and change of coordinates directly yields

$$\int_{[0,1]^2} A d\mu_A = \int_{[0,1]} A(x,h(x))d\lambda(x) = \frac{1}{2} - \int_{[0,1]}\int_{[0,1]} \mathbf{1}_{[0,x]}(t)\mathbf{1}_{[h(x),1]}(h(t))d\lambda(t)d\lambda(x)$$

and hence proves eq. (9). The latter identity, however, boils down to an affine transformation of $\lambda_2(\Omega_{\sqrt{2}})$ by considering

$$\begin{aligned}\int_{[0,1]^2} A d\mu_A &= \frac{1}{2} - \int_{[0,1]}\int_{[0,1]} \mathbf{1}_{[0,x]}(h^{-1}\circ h(t))\mathbf{1}_{[h(x),1]}(h(t))d\lambda(t)d\lambda(x)\\ &= \frac{1}{2} - \int_{[0,1]}\int_{[0,1]} \mathbf{1}_{[0,x]}(h^{-1}(y))\mathbf{1}_{[h(x),1]}(y)d\lambda(y)d\lambda(x)\\ &= \frac{1}{2} - \lambda_2(\Omega_{\sqrt{2}}).\end{aligned}$$

Having this, the identity

$$\tau(A_h) = 4\left(\frac{1}{2} - \lambda_2(\Omega_{\sqrt{2}})\right) - 1 = 1 - 4\lambda_2(\Omega_{\sqrt{2}}). \tag{10}$$

follows immediately. Notice that eq. (10) implies that the area of $\Omega_{\sqrt{2}}$ coincides with the quantity inv(h) as studied in [15, Lemma 3.1]. Comparing eq. (6) and eq. (10) shows the existence of an affine transformation $a : [-1, 1] \to [\frac{1}{\sqrt{2}}, 1]$ such that

$$a\left(\tau(A_h)\right) = \ell(A_h)$$

holds for every $A_h \in \mathcal{C}_{mcd}$ - in other words, we have proved the subsequent result:

Theorem 3.1. *For every $A_h \in \mathcal{C}_{mcd}$ the following identity linking the length measure ℓ and Kendall's τ holds:*

$$\ell(A_h) = 1 - \frac{2-\sqrt{2}}{4}(1 - \tau(A_h)) \tag{11}$$

Theorem 3.1 provides an answer to the question posed in [1], 'whether there are links between the length of level curves and concordance measures' - even the conjectured 'weighting' mentioned in [1] is not necessary, in the class $\mathcal{C}_{mcd}$ all we need is a fixed affine transformation.
In [1] it was further shown that the length measure interpreted as function $\ell : \mathcal{C} \to [\frac{\sqrt{2}}{2}, 1]$ is not continuous w.r.t. d_∞. The previous result implies, however, that within the dense subclass $\mathcal{C}_{mcd}$ the length measure is indeed continuous:

Corollary 3.2. *The mapping $\ell : \mathcal{C}_{mcd} \to [\frac{1}{\sqrt{2}}, 1]$ is continuous with respect to d_∞.*

Proof. Suppose that $A_h, A_{h_1}, A_{h_2}, \ldots$ are mutually completely dependent copulas and that the sequence $(A_{h_n})_{n\in\mathbb{N}}$ converges to A_h pointwise. Being a concordance measure Kendall's τ is continuous with respect to d_∞, so we have $\lim_{n\to\infty} \tau(A_{h_n}) = \tau(A_h)$ and eq. (10) directly yields $\lim_{n\to\infty} \ell(A_{h_n}) = \ell(A_h)$. □

Corollary 3.3. *For every $z \in [\frac{1}{\sqrt{2}}, 1]$ there exists some mutually completely dependent copula A_h with $\ell(A_h) = z$. In other words, all values in $[\frac{1}{\sqrt{2}}, 1]$ are attained by ℓ.*

Proof. According to [15] for each (x, y) in the region determined by Kendall' τ and Spearman's ρ there exists some mutually completely dependent copula C_h fulfilling

$$(\tau(A_h), \rho(A_h)) = (x, y).$$

Having this, the result directly follows via eq. (10). □

Moving away from the length measure we now turn to the surface area of copulas, derive analogous statements and start with showing yet another simple formula for elements in $\mathcal{C}_{mcd}$. Considering that copulas are Lipschitz continuous, the surface area surf(A) of an arbitrary copula A is given by

$$\mathrm{surf}(A) = \int_{[0,1]^2} \sqrt{\left(\frac{\partial A}{\partial x}(x,y)\right)^2 + \left(\frac{\partial A}{\partial y}(x,y)\right)^2 + 1}\; d\lambda_2(x,y)$$

$$= \int_{[0,1]^2} \sqrt{K_A(x,[0,y])^2 + K_{A^t}(y,[0,x])^2 + 1}\, d\lambda_2(x,y). \tag{12}$$

Again working with mutually completely dependent copulas yields the following result:

Lemma 3.4. *For every $A_h \in \mathcal{C}_{mcd}$ the surface area of A_h is given by*

$$\mathit{surf}(A_h) = \sqrt{2} - \left(2\sqrt{2} - 1 - \sqrt{3}\right) \lambda_2(\Omega_{\sqrt{2}}). \tag{13}$$

Proof. For the case of a completely dependent copula A_h eq. (12) obviously simplifies to

$$\begin{aligned} \mathrm{surf}(A_h) &= \int_{[0,1]^2} \sqrt{\mathbf{1}^2_{[0,y]}(h(x)) + \mathbf{1}^2_{[0,x]}(h^{-1}(y)) + 1}\, d\lambda_2(x,y) \\ &= \int_{[0,1]^2} \sqrt{\mathbf{1}_{[0,y]}(h(x)) + \mathbf{1}_{[0,x]}(h^{-1}(y)) + 1}\, d\lambda_2(x,y). \end{aligned}$$

Considering that the latter integrand is a step function only attaining the values $1, \sqrt{2}$ and $\sqrt{3}$, defining

$$\Omega_0^{A_h} := \Omega_0 = \left\{(x,y) \in (0,1)^2 : h(x) > y,\, h^{-1}(y) > x\right\}$$

as well as ($\Omega_{\sqrt{2}}$ as before)

$$\Omega_1^{A_h} := \Omega_1 = [0,1]^2 \setminus (\Omega_{\sqrt{2}} \cup \Omega_0)$$

we therefore have

$$\mathrm{surf}(A_h) = \lambda_2(\Omega_0) + \sqrt{2}\,\lambda_2(\Omega_1) + \sqrt{3}\,\lambda_2(\Omega_{\sqrt{2}}).$$

The latter identity can be further simplified: The measurable bijection $\Psi_h : [0,1]^2 \to [0,1]^2$, defined by $\Psi_h(x,y) = (h^{-1}(y), h(x))$ obviously fulfills $\lambda_2^{\Psi_h} = \lambda_2$. Therefore using the fact that

$$\begin{aligned} \Psi_h^{-1}(\Omega_0) &= \{(x,y) \in [0,1]^2 : (h^{-1}(y), h(x)) \in \Omega_0\} \\ &= \{(x,y) \in [0,1]^2 : h(x) \le y, h^{-1}(y) \le x\} = \Omega_{\sqrt{2}} \end{aligned}$$

it follows that $\lambda_2(\Omega_0) = \lambda_2(\Omega_{\sqrt{2}})$ holds. This altogether yields

$$\begin{aligned} \mathrm{surf}(A_h) &= \lambda_2(\Omega_{\sqrt{2}}) + \sqrt{2}\,\lambda_2(\Omega_1) + \sqrt{3}\,\lambda_2(\Omega_{\sqrt{2}}) \\ &= (1+\sqrt{3})\,\lambda_2(\Omega_{\sqrt{2}}) + \sqrt{2}\left(1 - 2\lambda_2(\Omega_{\sqrt{2}})\right) \\ &= \sqrt{2} + \underbrace{\left(1 + \sqrt{3} - 2\sqrt{2}\right)}_{<0} \lambda_2(\Omega_{\sqrt{2}}), \end{aligned}$$

which completes the proof. □

Theorem 3.5. *For every $A_h \in \mathcal{C}_{mcd}$ the following identity linking the surface area and Kendall's τ holds:*

$$\mathit{surf}(A_h) = \sqrt{2} - \frac{2\sqrt{2} - 1 - \sqrt{3}}{4}(1 - \tau(A_h)) \tag{14}$$

As in the case of the length measure we have the following two immediate corollaries:

Corollary 3.6. *The mapping $\mathit{surf} : \mathcal{C}_{mcd} \to [\frac{1+\sqrt{3}}{2}, \sqrt{2}]$ is continuous with respect to d_∞.*

Corollary 3.7. *For every $z \in [\frac{1+\sqrt{3}}{2}, \sqrt{2}]$ there exists some mutually completely dependent copula A_h with $\mathit{surf}(A_h) = z$. In other words, all values in $[\frac{1+\sqrt{3}}{2}, \sqrt{2}]$ are attained by surf.*

Combining Theorem 3.1 and Theorem 3.5 yields the following identity linking the length measure and the surface area of mutually completely dependent copulas.

Corollary 3.8. *For every $A_h \in \mathcal{C}_{mcd}$ the following identity holds:*

$$\mathit{surf}(A_h) = \sqrt{2} - \frac{2\sqrt{2}-1-\sqrt{3}}{2-\sqrt{2}}\,(1-\ell(A_h)) \tag{15}$$

Remark 3.9. The afore-mentioned interrelations lead to the following seemingly new interpretation of the interplay between the two most well-known measures of concordance, Kendall's τ and Spearman's ρ, as studied in [2, 4, 15] (and the references therein): Within the dense class $\mathcal{C}_{mcd}$ maximizing/minimizing Kendall's τ for a given value of Spearman's ρ is equivalent to maximizing/minimizing the surface area of copulas for a given value of the volume. Determining the exact τ-ρ region (for which according to [15] considering all shuffles is sufficient) one might therefore be reminded of the famous isoperimetric inequality bounding the surface area of a set by a function of the volume (see [6]).

Example 3.10. Considering that all results established in this section - in particular Theorem 3.1 linking Kendall' τ with the length profile and Theorem 3.5 interrelating τ with the surface - have only been stated and proved for mutually completely dependent copulas, the question naturally arises, if they can be extended to classes larger than or different to $\mathcal{C}_{mcd}$. Since one naturally might conjecture that eq. (11) and eq. (14) additionally hold (at least) for sufficiently smooth copulas, complementing the results in [1] we now focus on the latter identity and show that it does not even hold for the product copula Π. First of all notice that because of $\tau(\Pi) = 0$ the right hand-side of eq. (14) simplifies to

$$\sqrt{2} - \frac{2\sqrt{2}-1-\sqrt{3}}{4}(1-\tau(\Pi)) = \frac{\sqrt{2}}{2} + \frac{1+\sqrt{3}}{4} \approx 1.3901 \in \left[\frac{1+\sqrt{3}}{2}, \sqrt{2}\right].$$

On the other hand, calculating $\mathit{surf}(\Pi)$ according to eq. (12) yields

$$\begin{aligned} \mathrm{surf}(\Pi) &= \int\limits_{[0,1]^2} \sqrt{K_\Pi(x,[0,y])^2 + K_{\Pi^t}(y,[0,x])^2 + 1}\; d\lambda_2(x,y) \\ &= \int\limits_{[0,1]^2} \sqrt{y^2+x^2+1}\; d\lambda_2(x,y). \end{aligned}$$

The latter integral can be calculated analytically and after some tedious but straightforward steps one finally gets

$$\mathrm{surf}(\Pi) = \underbrace{\frac{6\sqrt{3}-\pi}{18} + \operatorname{arsinh}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{36}\log\left(1351+780\sqrt{3}\right)}_{\approx 1.2808} < \frac{1+\sqrt{3}}{2} \approx 1.3660.$$

In other words: Eq. (14) does not hold for Π and, more surprisingly, there is no element of $\mathcal{C}_{mcd}$ whose surface is at least close to $\mathrm{surf}(\Pi)$ - for every element A_h of $\mathcal{C}_{mcd}$ the difference $\mathrm{surf}(A_h) - \mathrm{surf}(\Pi)$ is even larger than the diameter of the full range of surf restricted to $\mathcal{C}_{mcd}$. Notice that this observation does not contradict the fact that $\mathcal{C}_{mcd}$ is dense in $(\mathcal{C}, d_\infty)$ and that consequently for every copula $A \in \mathcal{C}$ there exists some sequence $(A_{h_n})_{n\in\mathbb{N}}$ in $\mathcal{C}_{mcd}$ with $\lim_{n\to\infty} d_\infty(A_{h_n}, A) = 0$ and $\lim_{n\to\infty} \tau(A_{h_n}) = \tau(A)$. In fact, pointwise convergence of copulas does not imply convergence of the sequence $(K_{A_{h_n}}(x,[0,y]))_{n\in\mathbb{N}}$ to $K_A(x,[0,y])$ for a sufficiently large set of $(x,y) \in [0,1]^2$ (see [16] for a related discussion), hence (as illustrated by Π above) the corresponding surface areas may be quite different.

Example 3.10 implies that the mapping $A \mapsto \mathrm{surf}(A)$ assigning each copula its surface area is not continuous on the full domain $\mathcal{C}$ with respect to d_∞; according to Corollary 3.6, however, its restriction to $\mathcal{C}_{mcd}$ is continuous. Intuitively this is not too surprising since - contrary to the general case where the integrand $\sqrt{K_A(x,[0,y])^2 + K_{A^t}(y,[0,x])^2 + 1}$ may attain arbitrary values in $[1, \sqrt{3}]$ - in the mutually completely dependent setting the kernels only attain the values 0 and 1, so the integrand is a step function only attaining the values $1, \sqrt{2}$ and $\sqrt{3}$.

One natural question therefore would be, whether there are stronger topologies $\mathcal{O}$ on $\mathcal{C}$ with respect to which the mapping $A \mapsto \text{surf}(A)$ is indeed continuous. And, in the positive case, if there are 'nice' subclasses which are dense in $(\mathcal{C}, \mathcal{O})$. We will provide a positive answer to both questions and start with recalling a stronger notion of convergence of copulas going back to [11] as well as the concept of checkerboard approximations.

Definition 3.11 ([11])**.** Suppose that $A, A_1, A_2, \ldots$ are copulas and let $K_A, K_{A_1}, K_{A_2}, \ldots$ be (versions of) the corresponding Markov kernels. We will say that $(A_n)_{n\in\mathbb{N}}$ converges *weakly conditional* to A if, and only if for λ-almost every $x \in [0,1]$ we have that the sequence $(K_{A_n}(x,\cdot))_{n\in\mathbb{N}}$ of probability measures on $\mathcal{B}([0,1])$ converges weakly to the probability measure $K_A(x,\cdot)$.

Following [13] we can define checkerboard copulas as follows: Fix $N \in \mathbb{N}$ and define the squares R_{ij}^N for $i, j \in \{1, \ldots, N\}$ by

$$R_{ij}^N = \left[\frac{i-1}{N}, \frac{i}{N}\right] \times \left[\frac{j-1}{N}, \frac{j}{N}\right].$$

Definition 3.12 ([13, 16])**.** A copula $A_N \in \mathcal{C}$ is called *N-checkerboard copula*, if A_N is absolutely continuous and (a version of) its density k_{A_N} is constant on the interior of each square R_{ij}^N. We refer to N as the resolution of A_N, denote the set of all $N-$checkerboard copulas by $\mathcal{CB}_N$, and set $\mathcal{CB} = \bigcup_{N=1}^{\infty} \mathcal{CB}_N$.

For $A \in \mathcal{C}$ and $N \in \mathbb{N}$ the (absolutely continuous) copula $CB_N(A) \in \mathcal{CB}_N$, defined by

$$CB_N(A)(x,y) := \int_0^x \int_0^y N^2 \sum_{i,j=1}^N \mu_A(R_{ij}^N) \mathbb{1}_{R_{ij}^N}(s,t)\, d\lambda(t) d\lambda(s) \tag{16}$$

is called *N-checkerboard approximation of* A or simply N-checkerboard of A.

Having that, the following approximation result for the mapping $A \mapsto \text{surf}(A)$, saying that checkerboard approximations also serve as surface approximations, can be proved.

Theorem 3.13. *Let $A \in \mathcal{C}$ be an arbitrary copula and $CB_N(A)$ the N-checkerboard approximation on A. Then the following identity holds:*

$$\lim_{N\to\infty} \mathit{surf}(CB_N(A)) = \mathit{surf}(A) \tag{17}$$

Proof. According to [13, Corollary 3.2] the sequence $(CB_N(A))_{N\in\mathbb{N}}$ converges weakly conditional to A for $N \to \infty$ and the same is true for the sequence $(CB_N(A^t))_{N\in\mathbb{N}}$ and A^t. Considering $CB_N(A^t) = (CB_N(A))^t$ we can proceed as follows: Let $\Lambda \in \mathcal{B}([0,1])$ denote a set with $\lambda(\Lambda) = 1$ such that for every $x \in \Lambda$ the sequence $(K_{CB_N(A)}(x,\cdot))_{N\in\mathbb{N}}$ converges weakly to $K_A(x,\cdot)$ and let $x \in \Lambda$ be arbitrary but fixed. Then we have

$$\lim_{N\to\infty} K_{CB_N(A)}(x,[0,y]) = K_A(x,[0,y])$$

for every continuity point $y \in [0,1]$ of the function $z \mapsto K_A(x,[0,z])$, so in particular for λ-almost every $y \in [0,1]$. Moreover, since (by disintegration) for λ-almost every $y \in [0,1]$ we have $K_{A^t}(y,\{x\}) = 0$ it follows that for λ-almost every $y \in [0,1]$ the point x is a continuity point of the function $z \mapsto K_{A^t}(y,[0,z])$, which implies that for λ-almost every $y \in [0,1]$

$$\lim_{N\to\infty} K_{CB_N(A)^t}(y,[0,x]) = K_{A^t}(y,[0,x])$$

holds. Having this, using Fubini's theorem and dominated convergence we conclude that

$$\begin{aligned}
&\lim_{N\to\infty} \int_{[0,1]} \sqrt{K_{CB_N(A)}(x,[0,y])^2 + K_{CB_N(A)^t}(y,[0,x])^2 + 1}\; d\lambda(y) \\
&\qquad = \int_{[0,1]} \sqrt{K_A(x,[0,y])^2 + K_{A^t}(y,[0,x])^2 + 1}\; d\lambda(y).
\end{aligned}$$

Considering that $x \in \Lambda$ was arbitrary and that $\lambda(\Lambda) = 1$, again using dominated convergence completes the proof. $\square$

Remark 3.14. Theorem 3.13 remains valid if instead of checkerboards we consider so-called checkmin copulas in which case shrunk versions of M instead of Π constitute the mass distributions in the little squares R_{ij}^N (again see [13]). Contrary to checkerboards, working with checkmin copulas the integrand in eq. (12) is again a step function (as it is the case for all shuffles). Checkmin copulas are closely liked to equidistant even shuffles (a.k.a. classical shuffles of M) - in fact, using Birkhoff's famous theorem (see [9]) it is straightforward to show that each checkmin copula with resolution N can be expressed as convex combination of finitely many equidistant even shuffles (with N stripes). The reason for considering checkerboards in Theorem 3.13 is that they are more commonly encountered than checkmins.

4 Calculating τ, ℓ and *surf* for mutually completely dependent copulas with self-similar support

Calculating even standard characteristics like Kendall's τ for copulas with fractal supports is a difficult endeavor. As a (playful) by-product of the identities established in the previous section we now show how simple formulas for τ, ℓ and *surf* of mutually completely dependent copulas with self-similar support can be derived. We first recall the notion of so-called transformation matrices and the construction of copulas with fractal/self-similar support, then use these tools to construct mutually completely dependent copulas with self-similar support and finally derive simple expressions for Kendall's τ and the length measure of copulas of this type.

Definition 4.1 ([8, 16, 17])**.** An $n \times m$- matrix $T = (t_{ij})_{i=1\dots n,\, j=1\dots m}$ is called *transformation matrix* if it fulfills the following four conditions: (i) $\max(n, m) \geq 2$, (ii), all entries are non-negative, (iii) $\sum_{i,j} t_{ij} = 1$, and (iv) no row or column has all entries 0.

In other words, a transformation matrix is a probability distribution τ on $(\mathcal{I}, 2^{\mathcal{I}})$ with $\mathcal{I} = I_1 \times I_2$, $I_1 = \{1, \dots, n\}$ and $I_2 = \{1, \dots, m\}$, such that $\tau(\{i\} \times I_2) > 0$ for every $i \in I_1$ and $\tau(I_1 \times \{j\}) > 0$ for every $j \in I_2$. Given a transformation matrix T define the vectors $(a_j)_{j=0}^m, (b_i)_{i=0}^n$ of cumulative column and row sums by

$$\begin{aligned} a_0 &= b_0 = 0 \\ a_j &= \sum_{j_0 \leq j} \sum_{i=1}^{n} t_{ij} \qquad j \in \{1, \dots, m\} \\ b_i &= \sum_{i_0 \leq i} \sum_{j=1}^{m} t_{ij} \qquad i \in \{1, \dots, n\}. \end{aligned} \tag{18}$$

Considering that T is a transformation matrix both $(a_j)_{j=0}^m$ and $(b_i)_{i=0}^n$ are strictly increasing. Consequently $R_{ji} := [a_{j-1}, a_j] \times [b_{i-1}, b_i]$ are compact non-empty rectangles for every $j \in \{1, \dots, m\}$ and $i \in \{1, \dots, n\}$. Defining the contraction $w_{ji} : [0, 1]^2 \to R_{ji}$ by

$$w_{ji}(x, y) = \big(a_{j-1} + x(a_j - a_{j-1})\,,\, b_{i-1} + x(b_i - b_{i-1})\big)$$

therefore yields the IFSP $\{[0, 1]^2, (w_{ji})_{j=1\dots m, i=1\dots n}, (t_{ij})_{j=1\dots m, i=1\dots n}\}$. The induced operator V_T on $\mathcal{P}_{\mathcal{C}}$, given by

$$V_T(\mu) := \sum_{j=1}^{m} \sum_{i=1}^{n} t_{ij}\, \mu^{w_{ji}} \tag{19}$$

is easily verified to be well-defined (i.e., it maps $\mathcal{P}_{\mathcal{C}}$ into itself, again see [17, 8, 16]) - in the sequel we will therefore also consider V_T as a transformation mapping $\mathcal{C}$ into itself. According to [16] for every transformation matrix T there exists a unique copula A^*_T with $V_T(A^*_T) = A^*_T$ such that

$$\lim_{n\to\infty} D_1(V^n_T(B), A^*_T) = 0 \tag{20}$$

holds for arbitrary $B \in \mathcal{C}$ (i.e., A^*_T is the unique, globally attractive fixed point of V_T).

Suppose now that $2 \leq N \in \mathbb{N}$ and let π be a permutation of $\{1,\ldots,N\}$. Then the matrix $T_\pi = (t_{ij})_{i=1\ldots N,\, j=1\ldots N}$, defined by

$$t_{i,j} = \frac{1}{N}\,\mathbf{1}_{\{j\}}(\pi(i)), \quad i,j \in \{1,\ldots,N\}$$

is obviously a transformation matrix. To simplify notation we will simply write $V_\pi := V_{T_\pi}$ as well as $A^*_{T_\pi} = A^*_T$ in the sequel. Obviously V_π does not only map $\mathcal{C}$ to $\mathcal{C}$ but also $\mathcal{C}_{mcd}$ to $\mathcal{C}_{mcd}$. Considering that (see [16]) $\mathcal{C}_{cd}$ is closed in $(\mathcal{C}, D_1)$ using eq. (20) it follows immediately that $A^*_\pi \in \mathcal{C}_{mcd}$, so there exists some λ-preserving bijection h^*_π with $A^*_\pi = A_{h^*}$. Since the support of A^*_π is self-similar it seems intractable to calculate $\ell(A^*_\pi), \mathrm{surf}(A^*_\pi)$ and $\tau(A^*_\pi)$ for general π. The results established in the previous section, however, make it possible to derive simple expressions for both quantities.

We start with a simple illustrative example and then prove the general result (in a different manner).

Example 4.2. Consider $N = 3$ and the permutation $\pi = (1,3,2)$. Since, firstly, $A^*_\pi \in \mathcal{C}_{mcd}$, secondly, A^*_π is globally attractive, and since, thirdly, $\lim_{n\to\infty} \tau(V^n_\pi(M)) = \tau(A^*_\pi)$ implies $\lim_{n\to\infty} \lambda_2(\Omega^{V^n_\pi(M)}_{\sqrt{2}}) = \lambda_2(\Omega^{A^*_\pi}_{\sqrt{2}})$, it suffices to calculate $\lambda_2\left(\Omega^{A_{\pi^*}}_{\sqrt{2}}\right)$ which can be done as follows: Obviously we have (see Figure 2 for an illustration of the steps 3-6 in the construction)

$$\begin{aligned}
\lambda_2\left(\Omega^{V_\pi(M)}_{\sqrt{2}}\right) &= \frac{1}{9}\\
\lambda_2\left(\Omega^{V^2_\pi(M)}_{\sqrt{2}}\right) &= \frac{1}{9} + 3\,\frac{1}{9^2} = \frac{1}{9}\left(1 + \frac{1}{3}\right)\\
\lambda_2\left(\Omega^{V^3_\pi(M)}_{\sqrt{2}}\right) &= \frac{1}{9} + 3\,\frac{1}{9^2} + 9\frac{1}{27^2} = \frac{1}{9}\left(1 + \frac{1}{3} + \frac{1}{3^2}\right)\\
\vdots\ &= \ \vdots\\
\lambda_2\left(\Omega^{V^n_\pi(M)}_{\sqrt{2}}\right) &= \frac{1}{9}\left(1 + \frac{1}{3} + \ldots + \frac{1}{3^{n-1}}\right)
\end{aligned}$$

which yields

$$\lambda_2\left(\Omega^{A_{\pi^*}}_{\sqrt{2}}\right) = \frac{1}{9}\frac{1}{1-\frac{1}{3}} = \frac{1}{6}.$$

Having that, using eqs. (6), (10) and (14) shows

$$\ell(A^*_\pi) = 1 - (2-\sqrt{2})\frac{1}{6}, \quad \tau(A^*_\pi) = 1 - 4\,\frac{1}{6} = \frac{1}{3}$$

as well as

$$\mathrm{surf}(A^*_\pi) = \sqrt{2} - \left(2\sqrt{2} - 1 - \sqrt{3}\right)\frac{1}{6}.$$

Theorem 4.3. *Suppose that $N \geq 2$ and that π is a permutation of $\{1,\ldots,N\}$. Then the following identities hold for the copula A^*_π with self-similar support:*

$$\begin{aligned}
\tau\left(A^*_\pi\right) &= 1 - 4\,\frac{N}{N-1}\lambda_2\left(\Omega^{V_\pi(M)}_{\sqrt{2}}\right) &(21)\\
&= 1 - 4\,\frac{1}{N(N-1)}\,\#\left\{(i,j) \in \{1,\ldots,N\}^2 : \pi(i) < j \text{ and } \pi^{-1}(j) < i\right\}\\
\ell\left(A^*_\pi\right) &= 1 - (2-\sqrt{2})\,\frac{N}{N-1}\lambda_2\left(\Omega^{V_\pi(M)}_{\sqrt{2}}\right) &(22)
\end{aligned}$$

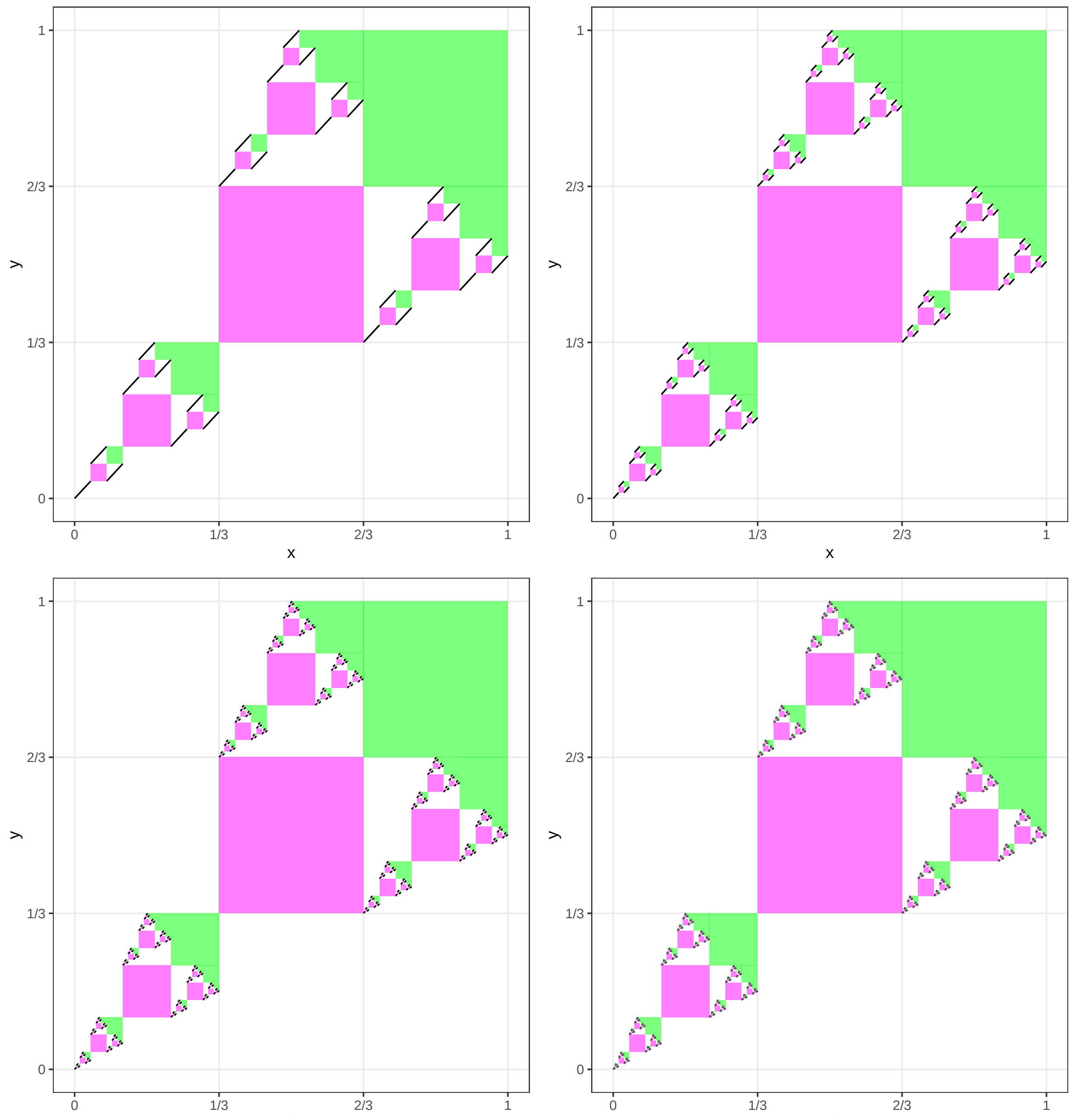

Fig. 2: Supports of the copulas $V_\pi^n(M)$ (black line segments) and the corresponding sets $\Omega_0^{V_\pi^n(M)}$ (magenta squares), $\Omega_{\sqrt{2}}^{V_\pi^n(M)}$ (green squares) for $n \in \{3, 4, 5, 6\}$ and $\pi = (1, 3, 2)$ as considered in Example 4.2.

$$
\begin{aligned}
&= 1 - (2 - \sqrt{2})\, \frac{1}{N(N-1)}\, \#\Big\{(i,j) \in \{1,\ldots,N\}^2 : \pi(i) < j \ \textit{and}\ \pi^{-1}(j) < i\Big\} \\
\textit{surf}\left(A_\pi^*\right) &= \sqrt{2} - \left(2\sqrt{2} - 1 - \sqrt{3}\right) \frac{N}{N-1} \lambda_2\left(\Omega_{\sqrt{2}}^{V_\pi(M)}\right) \qquad (23)
\end{aligned}
$$

Proof. First of all notice that for every $A_h \in \mathcal{C}_{mcd}$ we have

$$
\lambda_2\left(\Omega_{\sqrt{2}}^{V_\pi(A_h)}\right) = \lambda_2\left(\Omega_{\sqrt{2}}^{V_\pi(M)}\right) + \frac{1}{N}\lambda_2\left(\Omega_{\sqrt{2}}^{A_h}\right). \qquad (24)
$$

Since $A_\pi^* = A_{h^*}$ for some λ-preserving bijection h^* and since $V_\pi(A_\pi^*) = A_\pi^*$ holds, eq. (24) implies

$$
\lambda_2\left(\Omega_{\sqrt{2}}^{A_\pi^*}\right) = \lambda_2\left(\Omega_{\sqrt{2}}^{V_\pi(M)}\right) + \frac{1}{N}\lambda_2\left(\Omega_{\sqrt{2}}^{A_\pi^*}\right),
$$

from which we conclude

$$\lambda_2\left(\Omega^{A^*_\pi}_{\sqrt{2}}\right) = \frac{N}{N-1}\lambda_2\left(\Omega^{V_\pi(M)}_{\sqrt{2}}\right).$$

Having this, the desired identities follow by applying eqs. (6), (10), and (14). □

We conclude the paper with the following example.

Example 4.4. Consider $N = 4$ and the permutation $\pi = (3, 1, 4, 2)$. Figure 3 depicts the first four steps in the construction process of the corresponding copula A^*_π. Since in this case we have

$$\lambda_2\left(\Omega^{V_\pi(M)}_{\sqrt{2}}\right) = \frac{3}{16},$$

applying Theorem 4.3 directly yields

$$\tau(A^*_\pi) = 0, \quad \ell(A^*_\pi) = \frac{1}{2} + \frac{\sqrt{2}}{4}$$

as well as

$$\mathrm{surf}(A^*_\pi) = \frac{\sqrt{2}}{2} + \frac{1}{4} + \frac{\sqrt{3}}{4}.$$

5 Conclusion and outlook

As main results of the paper we have established affine interrelations between Kendall' τ, the length profile (as introduced in [1]), and the surface area of mutually completely dependent bivariate copulas. Additionally, we have shown that these interrelations do not carry over to the full class $\mathcal{C}$ of all bivariate copulas, in fact they do not even hold for the product copula Π. A small application of the derived identities to the calculation of Kendall's τ, the length measure and the surface of mutually completely dependent copulas with self-similar support rounds off the paper.
Related natural open questions which we plan to tackle in the future are in particular the following two:
(i) Is it possible to extend eq. (14) linking Kendall's τ and *surf* to the multivariate setting? Notice that for d-dimensional completely dependent copulas with $d \geq 3$ (see, e.g., [7] for a definition and some properties) the integrand of the surface integral is a step function again which will, however, attain up to $d+1$ distinct values, so expressing the surface area solely in terms of the Lebesgue measure of one single set will most likely not be possible.
(ii) According to Remark 3.9 the two most well-known concordance measures can (at least on a dense subclass) be interpreted geometrically in terms of surface and volume, respectively. It remains to be analyzed if other concordance measures allow for nice geometric interpretations as well.

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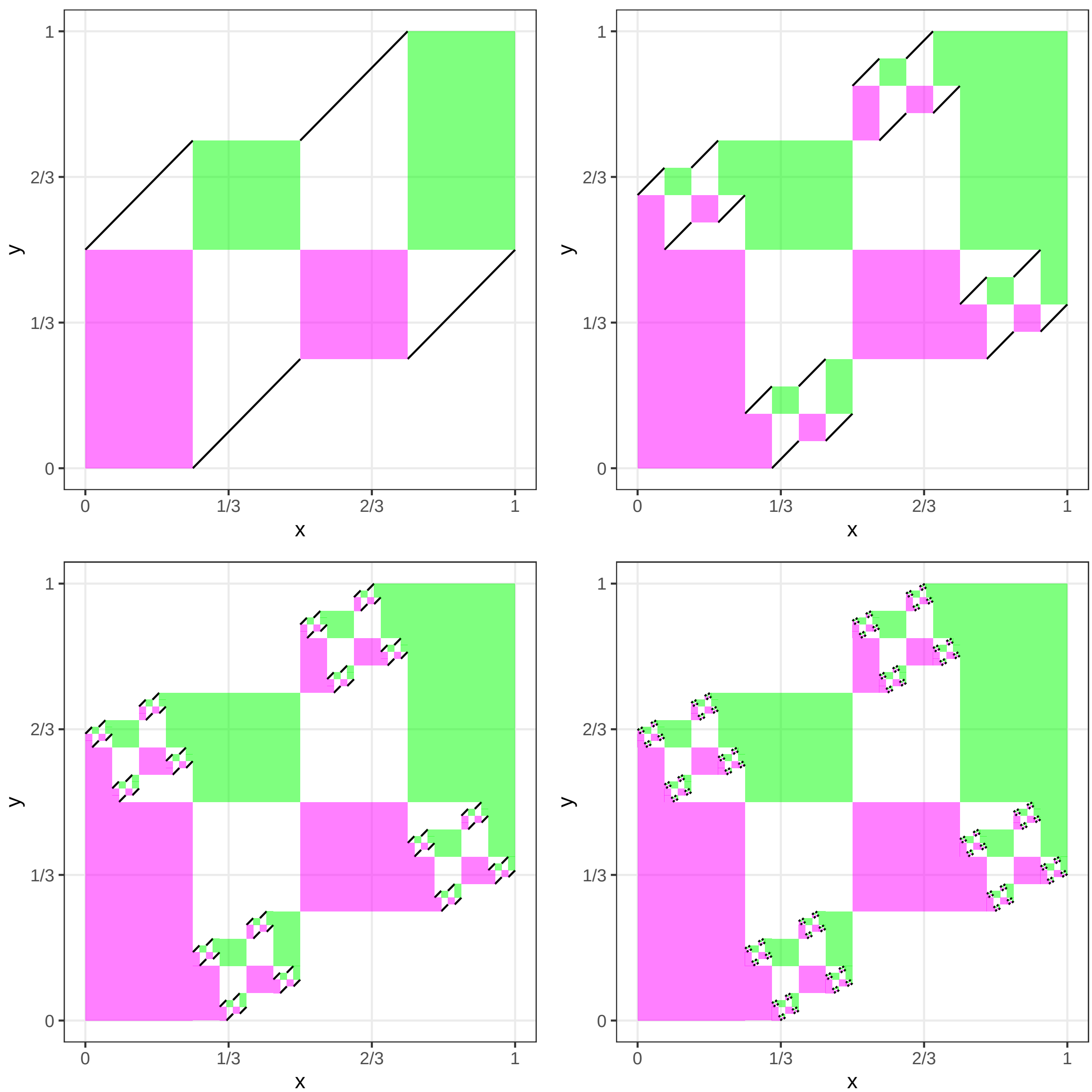

Fig. 3: Supports of the copulas $V_\pi^n(M)$ (black line segments) and the corresponding sets $\Omega_0^{V_\pi^n(M)}$ (magenta) and $\Omega_{\sqrt{2}}^{V_\pi^n(M)}$ (green) for $n \in \{1, 2, 3, 4\}$ and $\pi = (3, 1, 4, 2)$ as considered in Example 4.4.

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