On Cesáro convergence of iterates of the star product of copulas

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Abstract

For each copula A the iterates of the star product of A with itself are shown to be Cesáro convergent to an idempotent copula $\hat{A}$ with respect to the strong metric D_1. Sufficient conditions for $\hat{A} = \Pi$ are given.

Keywords: Copula, Idempotence, Markov Kernel, Markov operator

2010 MSC: 62H20, 60E05

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1. Introduction

Since its introduction by Darsow et el. (1992) the so-called star product of copulas has been studied in various papers. Olsen et al. (1996) showed that the space $(\mathcal{C}, *)$ of (two-dimensional) copulas with the star product as binary operation and the space $(\mathcal{M}, \circ)$ of Markov operators with the composition as binary operation are isomorphic (see Section 2) and that every copula $A \in \mathcal{C}$ can be written in the form $A = B^t * C$ whereby B, C are so-called completely dependent (or, equivalently, left invertible) copulas and B^t denotes the transpose of B. Using the above mentioned isomorphism Sempi (2002) showed that there is a one-to-one correspondence between the class of idempotent copulas $\mathcal{C}_{ip}$ (i.e. copulas with $A * A = A$) and the subclass of $\mathcal{M}$ consisting of conditional expectations. Durante et al. (2007) studied two product-like constructions for copulas, one being a generalized version of the star product. Darsow et al. (2010) classified idempotent copulas in

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non-atomic, atomic and totally atomic ones and, for each of the three types, gave a complete characterization of all of its members. Furthermore, based on these characterizations, they answered the question posed in Darsow et al. (1992) whether idempotent copulas are necessarily symmetric with 'yes'. In the current paper we will prove an ergodic-theorem-like result for iterates of the star product and show that for every copula $A \in \mathcal{C}$ there is a unique idempotent and symmetric copula $\hat{A}$ such that

$$\lim_{n\to\infty} D_1(s_{*n}(A), \hat{A}) = 0 \tag{1}$$

holds, whereby $s_{*n}(A) := \frac{1}{n}\sum_{i=1}^{n} A^{*i}$ for every $n \in \mathbb{N}$, A^{*i} is the i-times star product of A with itself, i.e. $A^{*1} = A$, $A^{*2} = A * A$, and $A^{*(n+1)} = A * A^{*n}$ for every $n \geq 2$, and D_1 is the metric introduced in Trutschnig (2011). Having this result proving symmetry for idempotent copulas is straightforward. We remark here that, in contrast to doubly stochastic idempotent matrices (see, for instance, Doob, 1942), idempotent copulas may be very complex objects. Using so-called Iterated Function Systems one may easily construct copulas with fractal support (see Fredricks et al., 2005, de Amo et al., 2012) that are at the same time idempotent (Trutschnig and Fernández Sánchez, 2012).
The above-mentioned convergence result (1) assures the existence of the Cesáro limit w.r.t. D_1 but it does not offer additional information about the limit $\hat{A}$. Since $A * \Pi = \Pi = \Pi * A$ holds for all $A \in \mathcal{C}$ (i.e. Π is the null element in $(\mathcal{C}, *)$) we will tackle the problem of finding general conditions under which $\hat{A}$ coincides with Π and prove that it is sufficient to know that the density of the absolutely continuous component of $s_{*j}(A)$ is strictly positive on the full interval $[0,1]^2$. Additionally we will give a necessary and sufficient condition for $\hat{A} = \Pi$ in the case of A being a completely dependent copula (see Lancaster, 1963, and Section 2).
The rest of this paper is organized as follows: Section 2 gathers some preliminaries and notations that will be used afterwards. Section 3 contains some examples and all above-mentioned results. Finally, Section 4 discusses open points and possible future work.

2. Notation and preliminaries

As already mentioned before $\mathcal{C}$ will denote the family of all (two-dimensional) *copulas*. For properties of copulas see Durante and Sempi (2010), Nelsen (2006), and Sempi (2011). For every $A \in \mathcal{C}$, μ_A will denote the corresponding *doubly stochastic measure*, $\mathcal{P}_\mathcal{C}$ the class of all these doubly stochastic measures. The Lebesgue measure on $[0,1]$ and $[0,1]^2$ will be denoted by λ and λ_2 respectively. For every metric space (Ω, d) the Borel σ-field on Ω will be denoted by $\mathcal{B}(\Omega)$. A *Markov kernel* from $\mathbb{R}$ to $\mathcal{B}(\mathbb{R})$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ such that $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and $B \mapsto K(x, B)$ is a probability measure for every fixed $x \in \mathbb{R}$. Suppose that X, Y are real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, then a Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \to [0,1]$ is called *regular conditional distribution of* Y *given* X if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ Y | X)(\omega) \tag{2}$$

holds $\mathcal{P}$-a.s. It is well know that for each pair (X, Y) of real-valued random variables a regular conditional distribution $K(\cdot, \cdot)$ of Y given X exists, that $K(\cdot, \cdot)$ is unique $\mathcal{P}^X$-a.s. (i.e. unique for $\mathcal{P}^X$-almost all $x \in \mathbb{R}$) and that $K(\cdot, \cdot)$ only depends on $\mathcal{P}^{X \otimes Y}$. Hence, given $A \in \mathcal{C}$ we will denote (a version of) the regular conditional distribution of Y given X by $K_A(\cdot, \cdot)$ and refer to $K_A(\cdot, \cdot)$ simply as *regular conditional distribution of* A or as *the Markov kernel of* A. Note that for every $A \in \mathcal{C}$, its conditional regular distribution $K_A(\cdot, \cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have

$$\int_{[0,1]} K_A(x, G_x)\, d\lambda(x) = \mu_A(G), \tag{3}$$

so in particular

$$\int_{[0,1]} K_A(x, F)\, d\lambda(x) = \lambda(F) \tag{4}$$

for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (4) induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (3). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see Kallenberg (1997) and Klenke (2007).

$\mathcal{T}$ will denote the family of all λ-preserving transformations $h : [0,1] \to [0,1]$ (see Walters, 2000). A copula $A \in \mathcal{C}$ will be called *completely dependent* if and only if there exists $h \in \mathcal{T}$ such that $K(x, E) := \mathbf{1}_E(hx)$ is a regular conditional distribution of A (see Lancaster, 1963, and Trutschnig, 2011, for equivalent definitions and main properties). For every $h \in \mathcal{T}$ the corresponding completely dependent copula will be denoted by C_h, the class of all completely dependent copulas by $\mathcal{C}_d$.

A linear operator T on $L^1([0,1]) := L^1([0,1], \mathcal{B}([0,1]), \lambda)$ is called *Markov* *operator* (Darsow et al., 1992; Olsen et al., 1996) if it fulfills the following three properties:

1. T is positive, i.e. $T(f) \geq 0$ whenever $f \geq 0$
2. $T(\mathbf{1}_{[0,1]}) = \mathbf{1}_{[0,1]}$
3. $\int_{[0,1]} (Tf)(x) d\lambda(x) = \int_{[0,1]} f(x) d\lambda(x)$

As mentioned in the introduction $\mathcal{M}$ will denote the class of all Markov operators on $L^1([0,1])$. It is straightforward to see that the operator norm of T is one, i.e. $\|T\| := \sup\{\|Tf\|_1 : \|f\|_1 \leq 1\} = 1$ holds. According to Olsen et al. (1996) *there is a one-to-one correspondence between* $\mathcal{C}$ *and* $\mathcal{M}$ - in fact, the mappings $\Phi : \mathcal{C} \to \mathcal{M}$ and $\Psi : \mathcal{M} \to \mathcal{C}$, defined by

$$\begin{aligned} \Phi(A)(f)(x) \quad &: \quad = (T_A f)(x) := \frac{d}{dx} \int_{[0,1]} A_{,2}(x,t) f(t) d\lambda(t), \\ \Psi(T)(x,y) \quad &: \quad = A_T(x,y) := \int_{[0,x]} (T\mathbf{1}_{[0,y]})(t) d\lambda(t) \end{aligned} \tag{5}$$

for every $f \in L^1([0,1])$ and $(x,y) \in [0,1]^2$ ($A_{,2}$ denoting the partial derivative w.r.t. y), fulfill $\Psi \circ \Phi = id_{\mathcal{C}}$ and $\Phi \circ \Psi = id_{\mathcal{M}}$. Note that in case of $f := \mathbf{1}_{[0,y]}$ we have $(T_A \mathbf{1}_{[0,y]})(x) = A_{,1}(x,y)$ λ-a.s. According to Trutschnig (2011) the first equality in (5) can be simplified to

$$(T_A f)(x) = \mathbb{E}(f \circ Y | X = x) = \int_{[0,1]} f(y) K_A(x, dy) \qquad \lambda\text{-a.s.} \tag{6}$$

Expressing copulas in terms of their corresponding regular conditional distributions the metric D_1 on $\mathcal{C}$ can be defined as follows:

$$D_1(A,B) := \int_{[0,1]} \int_{[0,1]} \big| K_A(x,[0,y]) - K_B(x,[0,y]) \big| d\lambda(x)\, d\lambda(y) \tag{7}$$

It can be shown that $(\mathcal{C}, D_1)$ is a complete and separable metric space and that, given copulas $A, A_1, A_2 \ldots$ and their corresponding Markov operators $T_A, T_{A_1}, T_{A_2} \ldots$, the following two conditions are equivalent:

(a) $\lim_{n\to\infty} D_1(A_n, A) = 0$

(b) $\lim_{n\to\infty} \|T_{A_n} f - T_A f\|_1 = 0$ for every $f \in L^1([0,1])$, i.e. D_1 is a metrization of the strong operator topology on $\mathcal{M}$ (see Trutschnig, 2011), which implies that D_1 is much stronger than the uniform metric d_∞ on $\mathcal{C}$ since, according to Olsen et al. (1996), d_∞ is a metrization of the weak operator topology on $\mathcal{M}$.

Given $A, B \in \mathcal{C}$ the *star product* $A * B \in \mathcal{C}$ is defined by (Darsow et al., 1992; Olsen et al., 1996)

$$(A * B)(x, y) := \int_{[0,1]} A_{,2}(x, t) B_{,1}(t, y) d\lambda(t) \tag{8}$$

and fulfills

$$T_{A*B} = \Phi_{A*B} = \Phi(A) \circ \Phi(B) = T_A \circ T_B, \tag{9}$$

so the mapping Φ in (5) actually is an isomorphism. Furthermore (Olsen et al., 1996) Φ maps transposes into adjoints, i.e. the adjoint operator $T_A^{adj} : L^\infty([0,1]) \to L^\infty([0,1])$ corresponds to the transposed copula A^t, defined by $A^t(x, y) := A(y, x)$ for all $x, y \in [0,1]$. Note that T_A^{adj} really corresponds to a copula although, by definition, it only lives on $L^\infty([0,1])$ - in fact T_A^{adj} has operator norm one on $L^\infty([0,1])$ and, using the fact that $L^\infty([0,1])$ is dense in $L^1([0,1])$ (see, for instance, Rudin, 2000), can easily be extended to a Markov operator on full $L^1([0,1])$.

A copula $A \in \mathcal{C}$ is called *idempotent* if $A * A = A$ holds, the family of all idempotent copulas will be denoted by $\mathcal{C}_{ip}$. $A \in \mathcal{C}$ is called *symmetric* if $A^t = A$. The following result, stating that the Markov kernel of $A * B$ is just the standard composition of the Markov kernels of A and B, will prove useful in the sequel:

Lemma 1 (Trutschnig and Fernández Sánchez, 2012). *Suppose that A, $B \in \mathcal{C}$ and let K_A, K_B denote regular conditional distributions of A and B. Then the Markov kernel $K_A \circ K_B$, defined by*

$$(K_A \circ K_B)(x, F) := \int_{[0,1]} K_B(y, F) K_A(x, dy), \tag{10}$$

is a regular conditional distribution of $A * B$.

3. Cesáro convergence of A^{*n}

We will state and prove the before mentioned ergodic-theorem-like result now, which, as byproduct, will provide an alternative quick proof of the fact that idempotent copulas are necessarily symmetric. For every copula $A \in \mathcal{C}$ and every $n \in \mathbb{N}$ as in the Introduction we set

$$s_{*n}(A) = \frac{1}{n} \sum_{i=1}^{n} A^{*i}. \tag{11}$$

Theorem 2 (Cesáro convergence of (iterates of) the star product). *For every copula A there exists a copula $\hat{A}$ such that*

$$\lim_{n\to\infty} D_1\big(s_{*n}(A), \hat{A}\big) = 0. \tag{12}$$

This copula $\hat{A}$ is idempotent, symmetric, and fulfills $\hat{A} * A = A * \hat{A} = \hat{A}$.

Proof: Fix $A \in \mathcal{C}$. Since T_A has operator norm one both on $L^1([0,1])$ and on $L^\infty([0,1])$ it follows from Riesz-Thorin's interpolation theorem (see Dunford and Schwartz, 1988) that $\|T_A\|_2 := \sup_{\|f\|_2 \leq 1} \|T_A f\|_2 = 1$. Hence Von Neumann's mean ergodic theorem for Hilbert spaces (Parry, 1981) implies that for every $f \in L^2([0,1])$ we have

$$\lim_{n\to\infty} \left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i f - \pi f \right\|_2 = 0 \tag{13}$$

whereby $\pi : L^2([0,1]) \to U_2$ denotes the orthogonal projection of $L^2([0,1])$ on the (closed) subspace $U_2 = \{g \in L^2([0,1]) : T_A\, g = g\}$. Furthermore we have the following: (i) $\pi \mathbf{1}_{[0,1]} = \mathbf{1}_{[0,1]}$ since $\mathbf{1}_{[0,1]} \in U_2$. (ii) π is obviously linear and, using (13), it is straightforward to see that π is positive too. (iii) π also preserves the integral of non-negative functions since, for every $f \in L^2([0,1])$ with $f \geq 0$, we have

$$\left| \int_{[0,1]} f\, d\lambda - \int_{[0,1]} \pi f\, d\lambda \right| \leq \left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i f - \pi f \right\|_1 \leq \left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i f - \pi f \right\|_2 \longrightarrow 0$$

for $n \to \infty$. We will now extend π to $L^1([0,1])$ by using the fact that $L^2([0,1])$ is dense in $L^1([0,1])$ in the standard way: For every $f \in L^1([0,1])$ choose a sequence $(f_n)_{n\in\mathbb{N}}$ of elements in $L^2([0,1])$ such that $\lim_{n\to\infty} \|f - f_n\|_1 = 0$ holds and set

$$\tilde{\pi} f := \lim_{n\to\infty} \pi f_n.$$

It is clear that $\tilde{\pi} : L^1([0,1]) \to L^1([0,1])$ is well-defined and it is straightforward to verify both that $\|\tilde{\pi}\|_1 = \sup_{\|f\|_1 \leq 1} \|\pi f\|_1 = 1$, and that $\tilde{\pi}$ maps

$L^1([0,1])$ into the closed subspace $U_1 := \{g \in L^1([0,1]) : T_A g = g\}$. Furthermore it follows directly from the construction that $\tilde{\pi}$ fulfills the above-mentioned points (i)-(iii) on $L^1([0,1])$, so $\tilde{\pi}$ is a Markov operator and there exists a unique copula $\hat{A} \in \mathcal{C}$ such that $\tilde{\pi} = T_{\hat{A}}$. Fix an arbitrary $f \in L^1([0,1])$. For every $\delta > 0$ there exists $g \in L^2([0,1])$ with $\|f - g\|_1 < \delta$, so, using the triangle inequality

$$\begin{aligned}\left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i f - T_{\hat{A}} f \right\|_1 &\leq 2\|f-g\|_1 + \left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i g - T_{\hat{A}} g \right\|_1 \\ &< 2\delta + \left\| \frac{1}{n} \sum_{i=1}^{n} T_A^i g - T_{\hat{A}} g \right\|_2 ,\end{aligned}$$

and therefore $\limsup_{n\to\infty} \|\frac{1}{n}\sum_{i=1}^{n} T_A^i f - T_{\hat{A}} f\|_1 < 2\delta$. Since $\delta > 0$ and f were arbitrary, using the fact that D_1 is a metrization of the strong operator topology, we have $\lim_{n\to\infty} D_1\big(s_{*n}(A), \hat{A}\big) = 0$, which completes the proof of the first part of the theorem.
The remaining assertions can be proved as follows: Equation (13) implies $T_A \circ T_{\hat{A}} f = T_{\hat{A}} f$ for every $f \in L^2([0,1])$, so again using the fact that $L^2([0,1])$ is dense in $L^1([0,1])$, we get $A * \hat{A} = \hat{A}$. Idempotence of $\hat{A}$ as well as $\hat{A} * A = \hat{A}$ follows analogously. Finally, in order to prove symmetry of $\hat{A}$, note that $\pi : L^2([0,1]) \to U_2$ as orthogonal projection is idempotent and self-adjoint. Hence, again using the fact that $L^2([0,1])$ is dense in $L^1([0,1])$, we get

$$\int_{[0,1]} (T_{\hat{A}} f)\, g \, d\lambda = \int_{[0,1]} f \, (T_{\hat{A}} g)\, d\lambda$$

for every $f \in L^1([0,1])$ and every $g \in L^\infty([0,1])$. Consequently, $T_{(\hat{A})^t} = (T_{\hat{A}})^{adj} = T_{\hat{A}}$ as well as $\hat{A} = (\hat{A})^t$ follows. ■

Theorem 2 allows for an alternative simple proof of symmetry for idempotent copulas - for the original proof we refer to Darsow and Olsen (2011), also see Sempi (2002):

Proposition 3. *Every idempotent copula is symmetric.*

Proof: If A is idempotent then $s_{*n}(A) = A$ for every $n \in \mathbb{N}$, so, with the notation of Theorem 13, $\hat{A} = A$ and A is symmetric. ■

Theorem 2 assures the existence of the Cesáro limit w.r.t. the metric D_1 but it does not provide any information about how the limit looks like. Since the product copula Π is the null element in $(\mathcal{C}, *)$ (see Darsow et al., 1992) and the star-product is smoothing in the sense that both $A * B$ and $B * A$ are absolutely continuous for every $B \in \mathcal{C}$ if A is absolutely continuous (Trutschnig, 2012) the natural question arises under which conditions $\hat{A} = \Pi$ holds. Before stating sufficient conditions for the general case we will answer the question for the class $\mathcal{C}_d$ - using results from ergodic theory (Lasota and Mackey, 1994; Walters, 2000) allows to prove the following theorem:

Theorem 4. *Suppose that $h \in \mathcal{T}$, then:*

1. $\lim_{n\to\infty} D_1(s_{*n}(C_h), \Pi) = 0$ *if and only if h is ergodic.*
2. $\lim_{n\to\infty} d_\infty(C_h^{*n}, \Pi) = 0$ *if and only if h is strong-mixing.*

Proof: Note that for completely dependent copulas we have $C_h^{*n} = C_{h^n}$ as well as $T_{C_{h^n}} f = f \circ h^n$ for every $f \in L^1([0,1])$, i.e. T_{C_h} coincides with the well-known Koopman operator (Lasota and Mackey, 1994; Walters, 2000). If h is ergodic then, using Von Neumann's L^1-ergodic theorem, it follows immediately that for each $f \in L^1([0,1])$ we have

$$\lim_{n\to\infty} \left\| T_{s_{*n}(C_h)} f - T_\Pi f \right\|_1 = \lim_{n\to\infty} \left\| \frac{1}{n}\sum_{i=1}^{n} f \circ h^i - \int_{[0,1]} f \, d\lambda \right\|_1 = 0 \tag{14}$$

from which $\lim_{n\to\infty} D_1(s_{*n}(C_h), \Pi) = 0$ follows. On the other hand, if $\lim_{n\to\infty} D_1(s_{*n}(C_h), \Pi) = 0$ holds and $E \in \mathcal{B}([0,1])$ fulfills $h^{-1}(E) = E$ then (14) holds for every $f \in L^1([0,1])$ and we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left\| T_{s_{*n}(C_h)} \mathbf{1}_E - T_\Pi \mathbf{1}_E \right\|_1 = \lim_{n\to\infty} \int_{[0,1]} \left| \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_E(h^i x) - \lambda(E) \right| d\lambda(x) \\ &= \lim_{n\to\infty} \int_{[0,1]} \left| \mathbf{1}_E(x) - \lambda(E) \right| d\lambda(x) = 2\lambda(E)(1-\lambda(E)), \end{aligned}$$

so $\lambda(E) \in \{0,1\}$ and h is ergodic.
To prove the second assertion we proceed in a similar way: Suppose that h is strong-mixing, then for every pair $x, y \in [0,1]$ it follows that (Walters, 2000)

$$\begin{aligned} C_h^{*n}(x,y) &= \int_{[0,x]} K_{C_{h^n}}(t, [0,y]) d\lambda(t) = \int_{[0,x]} \mathbf{1}_{[0,y]}(h^n t) \, d\lambda(t) \\ &= \int_{[0,1]} \mathbf{1}_{[0,x]}(t) \mathbf{1}_{[0,y]}(h^n t) \, d\lambda(t) \longrightarrow xy = \Pi(x,y) \end{aligned}$$

for $n \to \infty$. This completes the proof of one direction since x, y were arbitrary and pointwise and uniform convergence are equivalent in $\mathcal{C}$. Suppose now that $\lim_{n\to\infty} d_\infty(C_h^{*n}, \Pi) = 0$ holds, then for all intervals $E := [\underline{e}, \overline{e}]$ and $F := [\underline{f}, \overline{f}]$, we have

$$\begin{aligned} \lambda\big(E \cap h^{-n}(F)\big) &= \int_E \mathbf{1}_F(h^n t)d\lambda(t) = \int_E K_{C_{h^n}}(t, F)d\lambda(t) \\ &= \mu_{C_h^{*n}}(E \times F) \longrightarrow \mu_\Pi(E \times F) = \lambda(E)\lambda(F) \end{aligned}$$

for $n \to \infty$, which completes the proof since the class of compact intervals is a semi-algebra generating $\mathcal{B}([0,1])$ (see Walters, 2000). ■

Remark 5. Using Theorem 4 it is straightforward to give examples of copulas A for which the iterates A^{*n} are not even convergent to Π w.r.t. the uniform metric d_∞ (and consequently not w.r.t. D_1): one may simply consider C_h for the case that $h \in \mathcal{T}$ is not strong-mixing. Due to compactness of the metric space $(\mathcal{C}, d_\infty)$ we can always find a uniformly convergent subsequence of $(A^{*n})_{n\in\mathbb{N}}$ for every $A \in \mathcal{C}$, nevertheless, in general, $(A^{*n})_{n\in\mathbb{N}}$ is not convergent w.r.t. d_∞.

Example 1. Consider the λ-preserving transformation $h : [0,1] \to [0,1]$, defined by

$$hx := 2x \text{ (mod1)}.$$

Then h is exact (see Lasota and Mackey, 1994; Walters, 2000), hence strong-mixing and ergodic, and, according to Theorem 4, $\lim_{n\to\infty} d_\infty(C_h^{*n}, \Pi) = 0$ and $\lim_{n\to\infty} D_1(s_{*n}(C_h), \Pi) = 0$ holds. Nevertheless (Trutschnig, 2011), at the same time we have $D_1(C_h^{*n}, \Pi) = 1/3$ for all $n \in \mathbb{N}$. Figure 1 depicts image plots of $B_2 := C_h^{*2}, B_4 := C_h^{*4}$ and $B_{10} := C_h^{*10}$.

Example 2. As second example we consider the 'translation' $\tau_a \in \mathcal{T}$, defined by

$$\tau_a(x) = \begin{cases} 0 & \text{if } x = 0 \\ x + a & \text{if } x \in (0, 1-a] \\ x + a - 1 & \text{if } x \in (1-a, 1]. \end{cases}$$

It is well known that τ_a is ergodic (but not strong-mixing) if and only if a is irrational (see Lasota and Mackey, 1994; Walters, 2000). If we consider

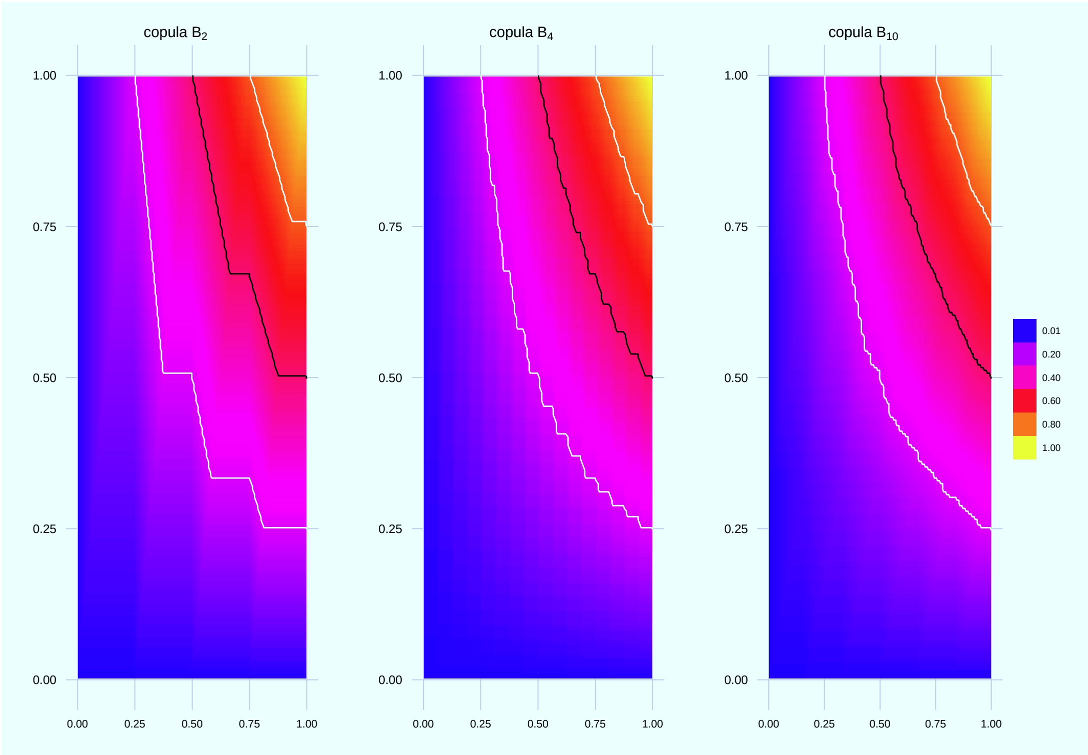

Figure 1: The copulas B_2, B_4, B_{10} in Example 1, colors ranging from blue (0) via red (0.5) to yellow (1); the three lines depict the $0.25, 0.5, 0.75$-contours

the case $a_0 = \sqrt{2}/2$ and set $\tau = \tau_{a_0}$ then, according to Theorem 4, we have $\lim_{n\to\infty} D_1(s_{*n}(C_\tau), \Pi) = 0$ but $\limsup_{n\to\infty} d_\infty(C_\tau^{*n}, \Pi) > 0$. Figure 2 depicts image plots of the kernels of $B_2 := s_{*2}(C_\tau), B_5 := s_{*5}(C_\tau)$ and $B_{20} := s_{*20}(C_\tau)$.

We will concentrate on the general case now and start with two lemmas. The first one has already been stated in a slightly different way in Trutschnig (2012), we only include the proof for the sake of completeness.

Lemma 6 (Trutschnig, 2012). *Suppose that $A \in \mathcal{C}$ is absolutely continuous with strictly positive density k_A on $[0,1]^2$, then we have $D_1(A * B_1, A * B_2) < D_1(B_1, B_2)$ for all $B_1, B_2 \in \mathcal{C}$ with $B_1 \neq B_2$. Furthermore Π is the only idempotent copula with strictly positive density on $[0,1]^2$.*

Proof: $\mathcal{D}([0,1])$ will denote the family of all probability densities in $L^1([0,1])$. Let $f_1, f_2 \in \mathcal{D}([0,1])$ and suppose that $\|f_1 - f_2\|_1 > 0$ and $\|T_A f_1 - T_A f_2\|_1 > 0$

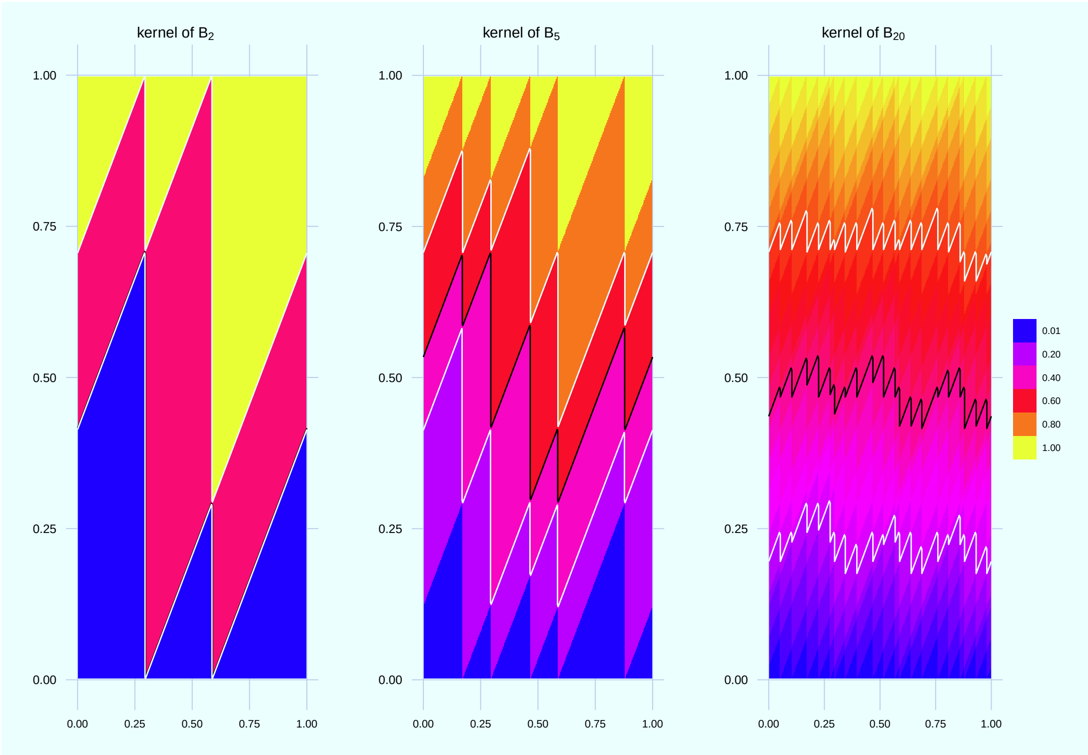

Figure 2: The functions $(x, y) \mapsto K_{B_2}(x, [0, y]), K_{B_5}(x, [0, y]), K_{B_{20}}(x, [0, y])$ for B_2, B_5, B_{20} in Example 2, colors ranging from blue (0) via red (0.5) to yellow (1); the three lines depict the $0.25, 0.5, 0.75$-contours

213 holds. Then, using Scheffé's theorem (see Scheffé, 1947) and setting $G :=$
214 $\{x \in [0, 1] : T_A f_1(x) > T_A f_2(x)\}$, we have

$$\begin{aligned}
\|T_A f_1 - T_A f_2\|_1 &= 2 \int_G \big(T_A f_1(x) - T_A f_2(x)\big)\, d\lambda(x) \\
&= 2 \int_G \left(\int_{[0,1]} (f_1(y) - f_2(y))\, k_A(x, y) d\lambda(y) \right) d\lambda(x) \\
&= 2 \int_{[0,1]} \left((f_1(y) - f_2(y)) \int_G k_A(x, y) d\lambda(x) \right) d\lambda(y) =: I.
\end{aligned}$$

215 Hence, using the fact that, by assumption $k_A(x, y) > 0$ for all $x, y \in [0, 1]$

$$I = 2 \int_{\{f_1 - f_2 > 0\}} \left((f_1(y) - f_2(y)) \underbrace{\int_G k_A(x, y) d\lambda(x)}_{\in (0,1) \text{ for every } y \in [0,1]} \right) d\lambda(y)$$

$$-2\int_{\{f_1-f_2<0\}}\left((f_2(y)-f_1(y))\underbrace{\int_G k_A(x,y)d\lambda(x)}_{\in(0,1)\text{ for every }y\in[0,1]}\right)d\lambda(y)$$

$$< \; 2\int_{\{f_1-f_2>0\}}(f_1(y)-f_2(y))d\lambda(y) = \|f_1-f_2\|_1$$

follows. Setting $f_i := T_{B_i}\mathbf{1}_{[0,y]}$ for $i \in \{1,2\}$ and $y \in (0,1)$, and assuming $B_1 \neq B_2$ this implies (both functions have the same integral so, up to a common scalar, they are densities)

$$\begin{aligned}
\Phi_{A*B_1,A*B_2}(y) &:= \int_{[0,1]}\big|K_{A*B_1}(x,[0,y]) - K_{A*B_2}(x,[0,y])\big|\,d\lambda(x) \\
&= \|T_Af_1 - T_Af_2\|_1 < \|f_1-f_2\|_1 \\
&= \|T_{B_1}\mathbf{1}_{[0,y]} - T_{B_2}\mathbf{1}_{[0,y]}\|_1 =: \Phi_{B_1,B_2}(y).
\end{aligned}$$

Hence (see Trutschnig, 2011) $D_1(A*B_1, A*B_2) < D_1(B_1,B_2)$ follows. To prove the last assertion suppose that $A \in \mathcal{C}$ is idempotent and that the density fulfills $k_A(x,y) > 0$ for all $x,y \in [0,1]$. If $D_1(A,\Pi) > 0$ then we would have

$$D_1(A,\Pi) = D_1(A*A, A*\Pi) < D_1(A,\Pi),$$

so $A = \Pi$ follows. ■

For every copula $A \in \mathcal{C}$ we will write $\mu_A = \mu_A^{abs} + \mu_A^{sing}$ for the Lebesgue decomposition of μ_A w.r.t. λ_2. Furthermore, the mass of the singular component μ_A^{sing} will be denoted by $sing(A)$, i.e.

$$sing(A) := \mu_A^{sing}([0,1]^2).$$

Note that, in general, neither $\frac{\mu_A^{abs}}{1-sing(A)}$ nor $\frac{\mu_A^{abs}}{sing(A)}$ is a copula, so A can not be expressed as convex combination of an absolutely continuous and a singular copula. The following lemma underlines the fact that the star product is smoothing once more:

Lemma 7. *Suppose that $A, B \in \mathcal{C}$, then:*

1. $sing(A*B) \leq \min\big\{sing(A), sing(B)\big\}$
2. *If the density k_B of μ_B^{abs} is strictly positive on $[0,1]^2$ and $sing(A) > 0$ holds then we have $sing(A*B) < sing(A)$.*

Proof: Choose versions K_A, k_A of the regular conditional distribution and the density of the absolutely continuous component of μ_A respectively. W.l.o.g. we may assume that for every $x \in [0,1]$ and every $E \in \mathcal{B}([0,1])$ $K_A(x,E) \geq \int_E k_A(x,y)d\lambda(y)$ holds. Consequently $H : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$, defined by

$$H_A(x,E) = K_A(x,E) - \int_E k_A(x,y)d\lambda(y)$$

is a sub-Markov kernel (i.e. measurable in x for fixed $B \in \mathcal{B}([0,1])$ and a measure with total mass at most one for fixed $x \in [0,1]$). Repeat this construction for copula B to get a sub-Markov kernel H_B. If $\lambda(E) = 0$ then it follows that $K_B(z,E) = 0$ for λ-almost all $z \in [0,1]$, so, using Lemma 1,

$$\begin{aligned}
K_{A*B}(x,E) &= \int_{[0,1]} K_B(z,E)K_A(x,dz) \\
&= \underbrace{\int_{[0,1]} K_B(z,E)k_A(x,z)d\lambda(z)}_{=0} + \int_{[0,1]} K_B(z,E)H_A(x,dz) \\
&= \int_{[0,1]} \underbrace{\int_E k_B(z,y)d\lambda(y)}_{=0 \text{ for every } z\in[0,1]} H_A(x,dz) + \int_{[0,1]} H_B(z,E)H_A(x,dz) \\
&= \int_{[0,1]} H_B(z,E)H_A(x,dz).
\end{aligned}$$

Suppose now that $N \in \mathcal{B}([0,1]^2)$ with $\lambda_2(N) = 0$. Then there exists a set $\Lambda \in \mathcal{B}([0,1])$ with $\lambda(\Lambda) = 1$ such that $\lambda(N_x) = 0$ for every $x \in \Lambda$, so, using disintegration, it follows that

$$\begin{aligned}
\mu_{A*B}(N) &= \int_{[0,1]} K_{A*B}(x,N_x)d\lambda(x) = \int_\Lambda K_{A*B}(x,N_x)d\lambda(x) \\
&= \int_\Lambda \int_{[0,1]} \underbrace{H_B(z,N_x)}_{\leq 1\ \forall z\in[0,1]} H_A(x,dz)d\lambda(x) \\
&\leq \int_{[0,1]} H_A(x,[0,1])d\lambda(x) = sing(A).
\end{aligned}$$

Since N was arbitrary $sing(A*B) \leq sing(A)$ follows, which, using the fact that $(A*B)^t = B^t * A^t$ and $sing(A) = sing(A^t)$ completes the proof of the first assertion. If we additionally assume that k_B is strictly positive on $[0,1]^2$ then we have $H_B(z, N_x) < 1$ for every $z \in [0,1]$ so the last inequality is strict and $\mu_{A*B}(N) < sing(A)$ follows. ■

We can now prove the following result:

Theorem 8. *Suppose that $A \in \mathcal{C}$ and that there exists an index $j \in \mathbb{N}$ such that the density $k_{s_{*j}(A)}$ of the absolutely continuous component of $\mu_{s_{*j}(A)}$ is strictly positive on $[0,1]^2$. Then the D_1-limit $\hat{A}$ of $s_{*n}(A)$ is Π.*

Proof: Since, according to Theorem 2, $A^{*i} * \hat{A} = \hat{A} = \hat{A} * A^{*i}$ holds for every $i \in \mathbb{N}$ we also have $s_{*j}(A) * \hat{A} = \hat{A} = \hat{A} * s_{*j}(A)$ If $sing(\hat{A}) > 0$ then Lemma 7 implies $sing(\hat{A}) = sing(\hat{A} * s_{*j}(A)) < sing(\hat{A})$, hence $sing(\hat{A}) = 0$ and $\hat{A}$ is absolutely continuous. W.l.o.g. we may assume that the density $k_{\hat{A}}$ fulfills

$$\int_{[0,1]} k_{\hat{A}}(x,y)d\lambda(x) = 1 \ \forall y \in [0,1] \text{ and } \int_{[0,1]} k_{\hat{A}}(x,y)d\lambda(y) = 1 \ \forall x \in [0,1]. \tag{15}$$

Again using the fact that $\hat{A} * s_{*j}(A) = \hat{A}$ it follows that

$$k_{\hat{A}}(x,y) \geq \int_{[0,1]} k_{s_{*j}(A)}(x,z)k_{\hat{A}}(z,y)d\lambda(y) > 0$$

for all $x, y \in [0,1]$. Applying Lemma 6 completes the proof. ■

Proposition 9. *Suppose that A is an absolutely continuous copula, that S is a singular copula, that $\alpha \in [0,1)$, and that there exists an index $j \in \mathbb{N}$ with $k_{A^{*j}} > 0$ on $[0,1]^2$. Then for $B := (1-\alpha)A + \alpha S$ the D_1-limit $\hat{B}$ of $s_{*n}(B)$ is Π.*

Example 3. The Marshall Olkin family $(M_{\alpha,\beta})_{(\alpha,\beta)\in[0,1]^2}$ of copulas (Nelsen, 2006) is defined by

$$M_{\alpha,\beta}(x,y) = \begin{cases} x^{1-\alpha}\, y & \text{if } x^\alpha \geq y^\beta \\ x\, y^{1-\beta} & \text{if } x^\alpha \leq y^\beta. \end{cases} \tag{16}$$

It contains Π ($\alpha = 0$ or $\beta = 0$) as well as M ($\alpha = \beta = 1$). Suppose that $\alpha, \beta > 0$ then a regular conditional distribution $K_{M_{\alpha,\beta}}(\cdot,\cdot)$ of $M_{\alpha,\beta}$ is given by ($x \in (0,1], y \in [0,1]$)

$$K_{M_{\alpha,\beta}}(x,[0,y]) = \begin{cases} (1-\alpha)x^{-\alpha}\, y & \text{if } y < x^{\frac{\alpha}{\beta}} \\ y^{1-\beta} & \text{if } y \geq x^{\frac{\alpha}{\beta}}. \end{cases} \tag{17}$$

It is straightforward to see that for $\alpha \in (0,1)$ and $\beta > 0$ the density of the absolutely continuous component of $M_{\alpha,\beta}$ is strictly positive (λ_2-almost everywhere) on $[0,1]$. Hence in this case, applying Theorem 8, we get

$$\lim_{n\to\infty} D_1(s_{*n}(M_{\alpha,\beta}),\Pi) = 0.$$

The same result also holds for $M_{1,\beta}$ with $\beta < 1$ since obviously the density of the absolutely continuous component of $M_{1,\beta}^{*2}$ is strictly positive on $[0,1]^2$.

We conclude this section with the following consequence of Theorem 8:

Proposition 10. *The mapping $\iota : \mathcal{C} \to \mathcal{C}_{ip}$, defined by $\iota(A) = \hat{A}$ is not continuous w.r.t. D_1.*

Proof: For every $n \geq 2$ define the copula A_n by

$$A_n = \frac{n-1}{n} M + \frac{1}{n}\Pi.$$

Then, on the one hand, we have $\lim_{n\to\infty} D_1(A_n, M) = 0$ and, on the other, according to Proposition 9, $\iota(A_n) = \hat{A}_n = \Pi$, so in particular (see Trutschnig, 2011), $D_1(\hat{A}_n, \hat{M}) = D_1(\Pi, M) = 1/3$ for all $n \geq 2$. ■

4. Future work

In this paper it has been proved that for each copula $A \in \mathcal{C}$ the iterated star products A^{*n} of A with itself are Cesáro convergent to an idempotent copula $\hat{A}$ w.r.t. the strong metric D_1. Furthermore sufficient conditions for $\hat{A}$ to coincide with Π have been given. The author conjectures that in many situations (like for instance the one in which the density of the absolutely continuous component of A^{*j} is strictly positive on $[0,1]^2$ for some index $j \in \mathbb{N}$) even $\lim_{n\to\infty} D_1(A^{*j},\Pi) = 0$ might hold.

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