Some members of the class of (quasi-)copulas with given diagonal from the Markov kernel perspective

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Abstract

Calculating Markov kernels of copulas allows not only for a precise description of the way Bertino- and diagonal copulas distribute mass, but also enables a simply proof of the fact that, for certain diagonals, both may degenerate to proper generalized shuffles of the minimum copula. After extending the kernel approach to the case of the maximum quasi-copula A_δ with given diagonal δ, a conjecture on singularity of A_δ by Nelsen et al. (2008) is established and an alternative simple and short proof of the result by Úbeda-Flores (2008) characterizing diagonals for which A_δ is a copula is given.

Keywords: Copula, Quasi-copula, Markov Kernel, Doubly Stochastic Measure, Shuffle

1 Introduction

The importance of copulas for fields like probability theory and statistics is underlined by Sklar's famous theorem (see Sklar, 1959), saying that the joint distribution function H of a pair (X, Y) of real-valued random variables and the (marginal) distribution functions F and G of X and Y respectively are linked by a copula C via $H(x, y) = C(F(x), G(y))$ for all $x, y \in \mathbb{R}$. If F and G are continuous, then the copula is unique; otherwise, the copula is only uniquely determined on $Range(F) \times Range(G)$ (see, for instance, de Amo et al. 2012). As pointed out by Jaworski (2009) there are various reasons for the interest in copulas with given diagonal δ - the facts that (i) tail dependence of a copula A only uses the copula's diagonal and (ii) that $(X, Y) \sim A$ implies $\max\{X, Y\} \sim \delta_A$ being two of the most important ones. It is well known that the class $\mathcal{C}_\delta$ of all copulas with given diagonal is non-empty for

every diagonal δ, that the Bertino copula B_δ is the lower bound in $\mathcal{C}_\delta$, and that the diagonal copula E_δ is the upper bound of the class of all symmetric elements in $\mathcal{C}_\delta$ (see Fredricks and Nelsen, 2002, Nelsen and Fredricks, 1997, Nelsen et al., 2008). Durante and Jaworski (2008) and Jaworski (2009) have given necessary and sufficient conditions for a diagonal δ to be the diagonal of an absolutely continuous copula. Durante et al. (2007) constructed asymmetric elements in $\mathcal{C}_\delta$ via patchworks, de Amo et al. (2013) analyzed the generalization of $\mathcal{C}_\delta$ to so-called sub- and super-diagonals.
In the current paper we will first take a closer look to Bertino- and diagonal copulas from the perspective of regular conditional distributions. We will calculate the Markov kernels both of Bertino- and of diagonal copulas. It will be shown that the kernel approach, firstly, provides a concise description of the way B_δ (E_δ) concentrates all mass on the union of the graphs of three (two) measurable functions implying singularity both of B_δ and of E_δ (loosely speaking we will also say the copulas 'live' on the graph of functions). And secondly, that it also serves as a handy tool for proving the existence of diagonals δ for which B_δ and E_δ degenerate to completely dependent copulas concentrating all mass on the graph of a Lebesgue-measure-preserving bijection $S : [0,1] \to [0,1]$ fulfilling $S \circ S = id_{[0,1]}$, which, however, is not monotonic on any interval. After that we focus on the maximum quasi-copula A_δ with diagonal δ introduced and studied by Úbeda-Flores (2008), as well as by Nelsen et al. (2008). We will construct a signed Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [-1,2]$ (see Section 2 for the definition of kernels) fulfilling

$$A_\delta(x,y) = \int_{[0,x]} K(t,[0,y])d\lambda(t) \tag{1}$$

for all $x, y \in [0,1]$, and, based on that, prove the existence of a doubly stochastic signed measure μ on the Borel σ-field $\mathcal{B}([0,1]^2)$ of $[0,1]^2$ fulfilling

$$A_\delta(x_2,y_2) - A_\delta(x_1,y_2) - A_\delta(x_2,y_2) + A_\delta(x_1,y_1) = \mu([x_1,x_2] \times [y_1,y_2])$$

for all intervals $[x_1,x_2], [y_1,y_2] \subseteq [0,1]$. Note that, according to Fernández Sánchez et al. (2010) and Nelsen et al. (2010), such a signed measure does not exist for every quasi-copula. Equation (1), together with the simple form of the signed Markov kernel $K(\cdot,\cdot)$, will also be useful for proving the fact that the (positive) measures μ^+, μ^- in the Hahn decomposition $\mu = \mu^+ - \mu^-$ of μ live on the graph of at most five functions, confirming singularity of μ as conjectured by Nelsen et al. (2008). Finally, usefulness of the kernel approach will be underlined once more by giving an alternative short and simple proof of the main result in Úbeda-Flores (2008) on the characterization of all diagonals for which the quasi-copula A_δ actually is a copula.
The rest of the paper is organized as follows: Section 2 gathers some preliminaries and notations. In Section 3 we construct a diagonal δ_0 for which the function $\hat{\delta}_0 : t \mapsto t - \delta_0(t)$ is not monotonic on any interval. Section 4 contains the calculation of Markov kernels of diagonal copulas E_δ and shows that E_{δ_0} is a proper generalized shuffle of M living on the graph of a Lebesgue-measure-preserving bijection $S : [0,1] \to [0,1]$ which is not monotonic on any interval. Section 5 contains the analogous results for Bertino copulas. Finally, the afore-mentioned quasi-copula A_δ is studied in Section 6.

2 Notation and preliminaries

Throughout the rest of the paper $\mathcal{B}([0,1])$ and $\mathcal{B}([0,1]^2)$ will denote the Borel σ-fields in $[0,1]$ and $[0,1]^2$ respectively, λ denotes the Lebesgue measure on $[0,1]$, and ϵ_x the Dirac measure at x. $\mathcal{T}$ will denote the class of all λ-preserving transformations $T:[0,1]\to[0,1]$, $\mathcal{T}_p$ the subset of all bijective $T\in\mathcal{T}$ fulfilling $T^{-1}\in\mathcal{T}$. $\mathcal{C}$ will denote the family of all copulas, i.e. the restrictions to $[0,1]^2$ of distribution functions with uniform $\mathcal{U}_{0,1}$-marginals. $\mathcal{Q}$ will denote the family of all quasi-copulas, i.e. the family of all functions $Q:[0,1]^2\to[0,1]$ fulfilling (i) $Q(x,0)=Q(0,x)=0$ and $Q(x,1)=Q(1,x)=x$ for all $x\in[0,1]$, (ii) Q is non-decreasing in each variable, and (iii) $|Q(x_1,y_2)-Q(x_2,y_2)|\le|x_1-x_2|+|y_1-y_2|$ for all $x_1,x_2,y_1,y_2\in[0,1]$. M will denote the minimum copula, W the copula defined by $W(x,y)=\max\{x+y-1,0\}$. For properties of copulas and quasi-copulas we refer to Genest et al (1999), Nelsen (2006), Durante and Sempi (2010), and Sempi (2011).

We will call a real-valued set function μ on a measurable space $(\Omega,\mathcal{A})$ *signed measure* if we can find finite positive measures μ^+,μ^- on $(\Omega,\mathcal{A})$ such that $\mu(F)=\mu^+(F)-\mu^-(F)$ holds for all $F\in\mathcal{A}$ (this is slightly more restrictive than the standard definition given, for instance, in Rudin, 1987, allowing at most one of μ^+,μ^- to be infinite, but sufficient for the rest of the paper). It is well known that for every signed measure μ we can choose μ^+,μ^- to be mutually singular w.r.t. each other - in this case we will refer to μ^+,μ^- as *Hahn decomposition* of μ (again see Rudin, 1987). In the sequel 'measure' always means non-negative measure - in case μ also assumes negative values we will explicitly mention the word 'signed'. A signed measure μ on $\mathcal{B}([0,1]^2)$ will be called *singular* if and only if both measures μ^+,μ^- in the Hahn decomposition are singular w.r.t. the Lebesgue measure λ_2. A (signed) measure μ on $\mathcal{B}([0,1]^2)$ will be called *doubly stochastic* if we have $\mu(E\times[0,1])=\mu([0,1]\times E)=\lambda(E)$ for every $E\in\mathcal{B}([0,1])$. For every copula $A\in\mathcal{C}$, setting $\mu_A([0,x]\times[0,y]):=A(x,y)$ for all $x,y\in[0,1]$ and extending μ_A to full $\mathcal{B}([0,1]^2)$ in the usual way yields a doubly stochastic measure μ_A. $\mathcal{P}_{\mathcal{C}}$ will denote the family of all doubly stochastic measures on $\mathcal{B}([0,1]^2)$.

Throughout this paper a *signed kernel* is a mapping $K:[0,1]\times\mathcal{B}([0,1])\to\mathbb{R}$ such that $x\mapsto K(x,B)$ is measurable for every fixed $B\in\mathcal{B}([0,1])$ and $B\mapsto K(x,B)$ is a signed measure for every fixed $x\in[0,1]$. $K(\cdot,\cdot)$ will simply be called *kernel* if it is a signed kernel assuming only non-negative values. A (signed) kernel $K(\cdot,\cdot)$ will be called *(signed) Markov kernel* if we have $K(x,[0,1])=1$ for every $x\in[0,1]$. Suppose that X,Y are $[0,1]$-valued random variables on a probability space $(\Omega,\mathcal{A},\mathcal{P})$, then a Markov kernel $K:[0,1]\times\mathcal{B}([0,1])\to[0,1]$ is called (a version of the) *regular conditional distribution of Y given X* if for every $B\in\mathcal{B}([0,1])$

$$K(X(\omega),B)=\mathbb{E}(\mathbf{1}_B\circ Y|X)(\omega) \tag{2}$$

holds $\mathcal{P}$-a.e. It is well known that for each pair (X,Y) of $[0,1]$-valued random variables a regular conditional distribution $K(\cdot,\cdot)$ of Y given X exists, that $K(\cdot,\cdot)$ is unique $\mathcal{P}^X$-almost everywhere (i.e. unique for $\mathcal{P}^X$-almost all $x\in[0,1]$) and that $K(\cdot,\cdot)$ only depends on $\mathcal{P}^{X\otimes Y}$.

Definition 2.1 *Suppose that $A\in\mathcal{C}$ and that the vector (X,Y) has joint distribution function A. Then we will denote (a version of) the regular conditional distribution of Y given X by $K_A(\cdot,\cdot)$ and refer to $K_A(\cdot,\cdot)$ simply as* regular conditional distribution of A *or as* Markov kernel of A.

Note that for every $A \in \mathcal{C}$, its conditional regular distribution $K_A(\cdot,\cdot)$, and every Borel set $G \in \mathcal{B}([0,1]^2)$ we have ($G_x := \{y \in [0,1] : (x,y) \in G\}$ denoting the x-section of G for every $x \in [0,1]$)

$$\int_{[0,1]} K_A(x, G_x)\, d\lambda(x) = \mu_A(G), \tag{3}$$

so in particular

$$\int_{[0,1]} K_A(x, F)\, d\lambda(x) = \lambda(F) \tag{4}$$

for every $F \in \mathcal{B}([0,1])$. On the other hand, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ fulfilling (4) obviously induces a unique element $\mu \in \mathcal{P}_{\mathcal{C}}([0,1]^2)$ via (3). For more details and properties of conditional expectation, regular conditional distributions, and disintegration see Kallenberg (1997) and Klenke (2007).
A copula $A \in \mathcal{C}$ will be called *completely dependent* if and only if there exists $T \in \mathcal{T}$ such that $K(x,E) := \mathbf{1}_E(Tx)$ is a regular conditional distribution of A (see Lancaster, 1963, and Trutschnig, 2011, for equivalent definitions and main properties). For every $T \in \mathcal{T}$ the induced completely dependent copula will be denoted by C_T throughout the rest of the paper. It is straightforward to verify (see Trutschnig, 2013) that the star product $C_T * C_S$ of C_T and C_S coincides with $C_{S\circ T}$. A copula A is called *shuffle* of the minimum copula M (see Durante et al., 2009, Nelsen, 2006) if $A = C_T$ for some interval-exchange transformation $T \in \mathcal{T}$. We will call $A \in \mathcal{C}$ *generalized shuffle of* M if $A = C_T$ for some $T \in \mathcal{T}_p$ (for a further generalization see Trutschnig and Fernández Sánchez, 2013).

Definition 2.2 *A function $\delta : [0,1] \to [0,1]$ will be called* diagonal *if and only if the following four conditions are fulfilled (i) $\delta(0) = 0, \delta(1) = 1$, (ii) δ is monotonically non-decreasing, (iii) δ is Lipschitz continuous with Lipschitz constant $L = 2$, and (iv) $\delta(t) \leq t$ for all $t \in [0,1]$. $\mathcal{D}$ will denote the family of all diagonals.*

$\mathcal{C}_\delta$ ($\mathcal{Q}_\delta$) will the family of all copulas (quasi-copulas) A fulfilling $A(x,x) = \delta(x)$ for all $x \in [0,1]$ and a given $\delta \in \mathcal{D}$. It is well known (see Durante et al., 2005, Durante and Jaworski, 2008, Nelsen, 2006, Úbeda-Flores, 2009) that $\mathcal{C}_\delta$ (and hence $\mathcal{Q}_\delta$) is non-empty for every $\delta \in \mathcal{D}$. For every diagonal δ the function $\hat{\delta} : [0,1] \to [0,1/2]$ is defined by $\hat{\delta}(t) = t - \delta(t)$. It is straightforward to verify that $\hat{\delta}(0) = \hat{\delta}(1) = 0$, that $\hat{\delta}$ is Lipschitz continuous with Lipschitz constant 1, and that $0 \leq \hat{\delta}(t) \leq \min\{t, 1-t\}$ for all $t \in [0,1]$. Figure 1 depicts two diagonals δ_1 and δ_2 which will serve as ongoing example in Sections 4 - 6. Due to Lipschitz continuity both δ and $\hat{\delta}$ are differentiable λ-almost everywhere and we can find Borel measurable functions $w_\delta : [0,1] \to [0,2]$ and $\hat{w}_\delta : [0,1] \to [-1,1]$ fulfilling $\delta'(x) = w_\delta(x)$ and $\hat{\delta}'(x) = \hat{w}_\delta(x)$ respectively for λ-almost every $x \in [0,1]$ (see Rudin, 1987). In the sequel we will refer to w_δ and $\hat{w}_\delta$ as (a versions of) the *derivative of δ and $\hat{\delta}$* respectively. Finally, when working with *Dini derivatives* in Section 5 - 6 we will write $D^+f(x), D_+f(x), D^-f(x), D_-f(x)$ for the upper right-, the lower right-, the upper left- and the lower left Dini derivative of a real-valued function f at x respectively, see Hewitt and Stromberg (1965).

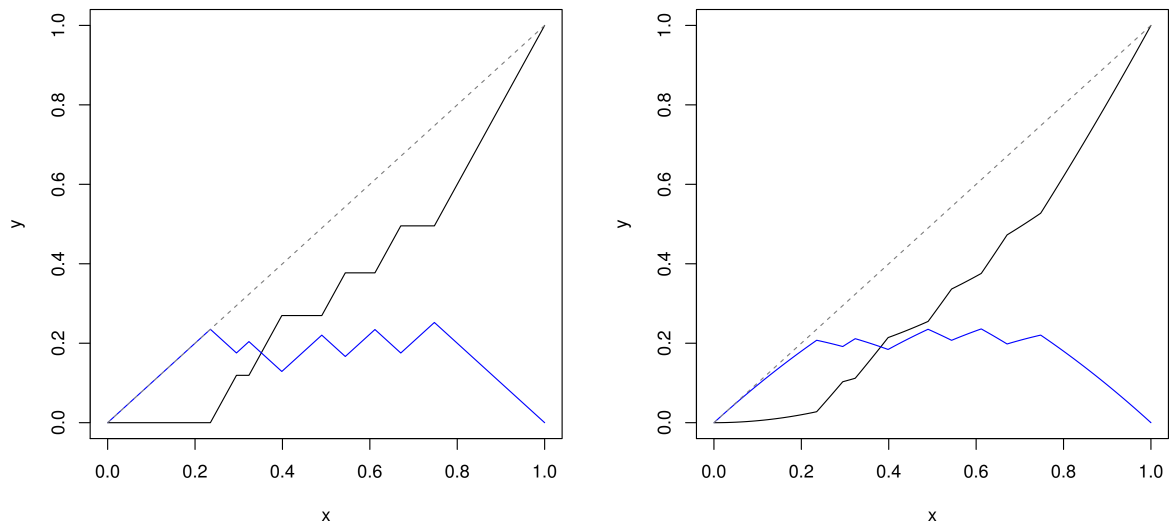

Figure 1: Piecewise linear diagonal δ_1 with $\delta_1' \in \{0,2\}$ λ-almost everywhere; δ_2 defined by $\delta_2(t) = \frac{1}{2}(\delta_1(t) + t^2)$; the blue lines depict the corresponding functions $\hat{\delta}_1, \hat{\delta}_2$.

3 A diagonal δ for which $\hat{\delta}$ is not monotonic on any interval

Although $\hat{\delta}$ is Lipschitz continuous it does not need to be monotonic on any interval - the following simple lemma provides the basis for one possible construction of such a $\hat{\delta}$.

Lemma 3.1 *There exists a Borel set $\Omega \in \mathcal{B}([0,1])$ with $\lambda(\Omega) = 1/2$ fulfilling the following conditions:*

(i) $\lambda((a,b) \cap \Omega) > 0$ *and* $\lambda((a,b) \cap \Omega^c) > 0$ *for every open non-empty interval* $(a,b) \subset [0,1]$.

(ii) $\lambda([0,x] \cap \Omega) > \frac{x}{2}$ *for every* $x \in (0,1)$.

Proof: Let $C_\infty \subseteq [0,1]$ denote the classical Smith-Volterra-Cantor set constructable as follows: Start with the unit interval $[0,1]$, remove an open interval of length $\frac{1}{4}$ around the mid point $\frac{1}{2}$ and let C_1 denote the remaining compact set. Remove an interval of length $\frac{1}{4^2}$ centered at the mid points of the 2 intervals constituting C_1, denote the remaining set by C_2 and proceed analogously, i.e. remove intervals of length $\frac{1}{4^{n+1}}$ centered at the mid points of the 2^n intervals constituting C_n. It is easily verified that $C_\infty := \bigcap_{n=1}^\infty C_n$ is a totally disconnected compact set fulfilling $\lambda(C_\infty) = \frac{1}{2}$.
In the next step we will paste affine copies of C_∞ into the closures $(J_{1,n})_{n\in\mathbb{N}}$ of the countably many pairwise disjoint open intervals $(U_{1,n})_{n\in\mathbb{N}}$ constituting C_∞^c. Doing so we use the

following notation: For each compact interval $J = [a,b] \subseteq [0,1]$ let $S_J : [0,1] \to J$ denote the function $S_J(x) = a + (b-a)x$. Set $H_1 = C_\infty$ and $L_\infty := C_\infty \cap [0, \frac{1}{2}]$. Then the set $H_2 := H_1 \cup \bigcup_{n=1}^\infty S_{J_{1,n}}(C_\infty)$ fulfills $\lambda(H_2) = \frac{3}{4}$ and we have $\lambda\big(\bigcup_{n=1}^\infty S_{J_{1,n}}(L_\infty)\big) = \frac{1}{8}$. Let $(J_{2,n})_{n\in\mathbb{N}}$ denote the closures of the countably many pairwise disjoint open intervals $(U_{2,n})_{n\in\mathbb{N}}$ constituting H_2^c. The set $H_3 := H_2 \cup \bigcup_{n=1}^\infty S_{J_{2,n}}(C_\infty)$ fulfills $\lambda(H_3) = \frac{2^3-1}{2^3}$ and we have $\lambda\big(\bigcup_{n=1}^\infty S_{J_{2,n}}(L_\infty)\big) = \frac{1}{4^2}$. We proceed inductively, i.e. for given H_k with $\lambda(H_k) = \frac{2^k-1}{2^k}$ let $(J_{k,n})_{n\in\mathbb{N}}$ denote the closures of the countably many pairwise disjoint open intervals $(U_{k,n})_{n\in\mathbb{N}}$ constituting H_k^c and set $H_{k+1} = H_k \cup \bigcup_{n=1}^\infty S_{J_{k,n}}(C_\infty)$. Then $\lambda(H_{k+1}) = \frac{2^{k+1}-1}{2^{k+1}}$ as well as $\lambda\big(\bigcup_{n=1}^\infty S_{J_{k,n}}(L_\infty)\big) = \frac{1}{4\,2^k}$ follows immediately. We will show now that Ω, defined by

$$\Omega := L_\infty \cup \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty S_{J_{k,n}}(L_\infty),$$

fulfills the properties stated in the lemma. Since both $\lambda(\Omega) = \frac{1}{2}$ and condition (i) follow immediately from the construction it suffices to prove condition (ii): Set $R_\infty := C_\infty \cap [\frac{1}{2}, 1]$ as well as $\Omega' := R_\infty \cup \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty S_{J_{k,n}}(R_\infty)$. Considering that for each pair $(k,n) \in \mathbb{N}^2$ both sets $R_\infty \cap S_{J_{k,n}}(L_\infty)$ and $L_\infty \cap S_{J_{k,n}}(R_\infty)$ contain at most one point $\lambda(\Omega' \cap \Omega) = 0$ follows, which in turn implies that $\lambda(\Omega' \Delta \Omega^c) = 0$. Since for every $x \in (0,1)$ we obviously have $\lambda([0,x] \cap L_\infty) > \lambda([0,x] \cap R_\infty)$ we get

$$\begin{aligned}
\lambda\big([0,x] \cap \Omega\big) &= \lambda\big([0,x] \cap L_\infty\big) + \sum_{k=1}^\infty \sum_{n=1}^\infty \lambda\big([0,x] \cap S_{J_{k,n}}(L_\infty)\big) \\
&\geq \lambda\big([0,x] \cap L_\infty\big) + \sum_{k=1}^\infty \sum_{n=1}^\infty \lambda\big([0,x] \cap S_{J_{k,n}}(R_\infty)\big) \\
&> \lambda\big([0,x] \cap R_\infty\big) + \sum_{k=1}^\infty \sum_{n=1}^\infty \lambda\big([0,x] \cap S_{J_{k,n}}(R_\infty)\big) = \lambda([0,x] \cap \Omega') \\
&= \lambda([0,x] \cap \Omega^c),
\end{aligned}$$

which completes the proof. ■

Theorem 3.2 *There exists a diagonal $\delta_0 \in \mathcal{D}$ such that $\hat{\delta}_0$ is not monotonic on any non-empty open interval.*

Proof: Consider a Borel set Ω fulfilling the properties of Lemma 3.1 and set

$$\delta_0(x) := 2 \int_{[0,x]} \mathbf{1}_{\Omega^c}(t) d\lambda(t) = 2\lambda([0,x] \cap \Omega^c). \tag{5}$$

for every $x \in [0,1]$. Then obviously δ_0 is monotonically non-decreasing, Lipschitz continuous with Lipschitz constant 2 and fulfills $\delta_0(0) = 0$, $\delta_0(1) = 1$. Furthermore, using condition (ii) in Lemma 3.1, it follows that $\delta_0(x) < x$ for every $x \in (0,1)$, so $\delta_0 \in \mathcal{D}$. According to Rudin (1987) δ_0 is as absolutely continuous function differentiable λ-a.e. and we have $\delta_0'(x) \in \{0, 2\}$ for λ-almost every $x \in [0,1]$. In terms of $\hat{\delta}_0$ this means $\hat{\delta}_0'(x) \in \{-1, 1\}$ for λ-almost every $x \in [0,1]$, from which, using condition (i) in Lemma 3.1, it follows immediately that $\hat{\delta}$ is not monotonic on any interval. ■

4 Markov kernels of diagonal copulas

For every diagonal δ define the so-called *diagonal copula* (see Nelsen et al., 2004, 2008, and Úbeda-Flores, 2008) E_δ by

$$E_\delta(x, y) = \min\left\{x, y, \frac{\delta(x) + \delta(y)}{2}\right\}. \tag{6}$$

(We will use the symbol E_δ instead of K_δ since the letter K will denote kernels throughout the whole paper.) It is well known that E_δ is singular and that for every symmetric copula $A \in \mathcal{C}_\delta$ we have $A \leq E_\delta$ (see again Nelsen et al., 2004, 2008, and Úbeda-Flores, 2008). The

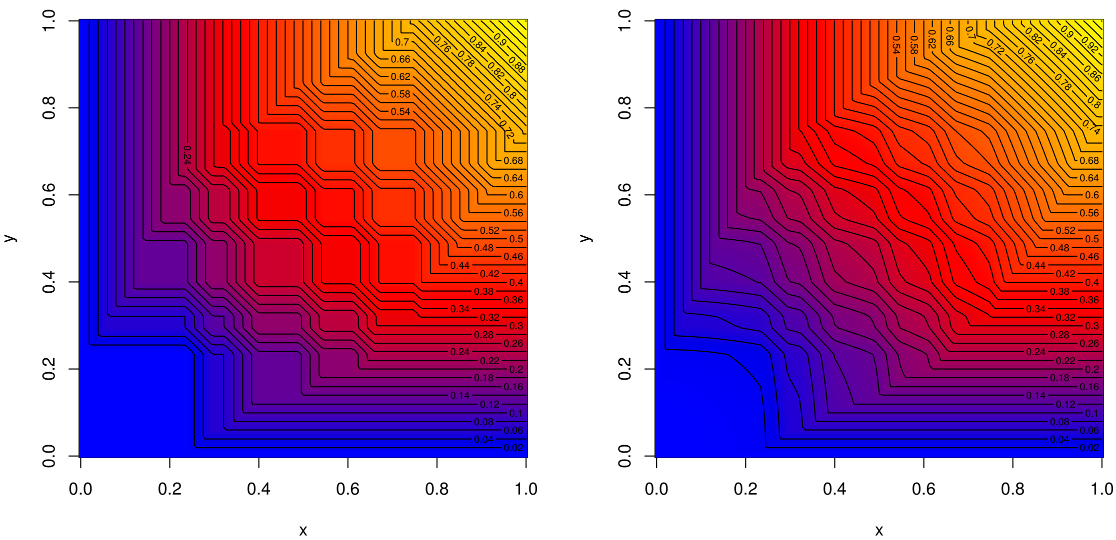

Figure 2: Image plots of the copulas E_{δ_1} E_{δ_2}, whereby δ_1, δ_2 are the diagonals from Figure 1.

following lemma will be useful for the calculation of the Markov kernel K_{E_δ} of E_δ:

Lemma 4.1 *Suppose that* $\delta \in \mathcal{D}$ *and set* $g(x) = 2x - \delta(x)$ *for* $x \in [0, 1]$. *Furthermore define two functions* $L, U : [0, 1] \to [0, 1]$ *by*

$$L(x) := \min\left\{z \in [0, 1] : g(z) \geq \delta(x)\right\}, \; U(x) := \min\left\{z \in [0, 1] : \delta(z) \geq g(x)\right\}.$$

Then the following assertions hold:

1. $L(x) \leq x$ *for all* $x \in [0, 1]$. *Furthermore* L *is non-decreasing and lower semicontinuous (hence left-continuous).*

2. $U(x) \geq x$ *for all* $x \in [0, 1]$. *Furthermore* U *is non-decreasing and upper semicontinuous (hence right-continuous).*

3. $\delta = \delta \circ U \circ L$ *and* $g = g \circ L \circ U$.

4. $\delta(x) < x$ *implies* $L(x) < x$ *and* $U(x) > x$.

Proof: Obviously the function g is Lipschitz-continuous with Lipschitz constant $L = 2$, non-decreasing. and fulfills $g(0) = 0$, $g(1) = 1$ as well as $g \geq \delta$. Hence, in particular, $L(x) \leq x$ and $U(x) \geq x$ for every $x \in [0, 1]$. The fact that L, U are non-decreasing is a direct consequence of the fact that δ, g are non-decreasing. If $L(x) > \alpha$ then $g(\alpha) < \delta(x)$ and continuity of δ implies the existence of $r > 0$ such that $g(\alpha) < \delta(z)$ for all $z \in B(x, r)$, which shows that the set $\{x \in [0, 1] : L(x) > \alpha\}$ is open. Since α was arbitrary lower semicontinuity of L follows. Using the fact that $L(x) \leq y$ if and only if $U(y) \geq x$ this shows upper semicontinuity of U, which completes the proof of the first two assertions. The third assertion is a direct consequence of continuity of δ and g. If $L(x) = x$ then, using $g(L(x)) = \delta(x)$ we have $2x - \delta(x) = \delta(x)$, so $\delta(x) = x$ follows. The fact that $U(x) = x$ implies $\delta(x) = x$ follows analogously. ■

E_δ lives on the graph of two functions - the following result holds:

Theorem 4.2 *Suppose that* $\delta \in \mathcal{D}$ *and let* $w_\delta : [0, 1] \to [0, 2]$ *be measurable with* $w_\delta(x) = \delta'(x)$ *for* λ*-almost every* $x \in [0, 1]$*. Then the Markov kernel* $K_{E_\delta}(\cdot, \cdot)$ *of* E_δ *is given by*

$$K_{E_\delta}(x, F) = \frac{w_\delta(x)}{2} \epsilon_{L(x)}(F) + \Big(1 - \frac{w_\delta(x)}{2}\Big) \epsilon_{U(x)}(F) \tag{7}$$

for λ*-almost every* $x \in [0, 1]$.

Proof: Obviously $K_{E_\delta} : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$ according to (7) is a Markov kernel. Using the fact that $L(x) \leq y$ if and only if $U(y) \geq x$ it follows that for every $y \in [0, 1]$ we have

$$L^{-1}\big([0, y]\big) = [0, U(y)] \quad \text{and} \quad U^{-1}\big([0, y)\big) = [0, L(y)).$$

Fix $y \in [0, 1]$. If $U^{-1}(\{y\})$ contains a compact interval of the form $[a, b]$ with $a < b$ then we have $U(a) = U(b)$ from which $\delta(b) - \delta(a) = 2(b - a)$, $\delta'(x) = 2$ for each $x \in (a, b)$, and $w_\delta(x) = 2$ for λ-almost every $x \in [a, b]$ follows. Having this it is straightforward to verify that K_{E_δ} corresponds to E_δ. In fact, taking into account

$$\begin{aligned}
\int_{[0,y]} K_{E_\delta}(t, [0, x]) \, d\lambda(t) &= \int_{[0,y]} \frac{w_\delta}{2} \mathbf{1}_{L^{-1}([0,x])} d\lambda + \int_{[0,y]} \Big(1 - \frac{w_\delta}{2}\Big) \mathbf{1}_{U^{-1}([0,x])} d\lambda \\
&= \int_{[0,y]} \frac{w_\delta}{2} \mathbf{1}_{[0,U(x)]} d\lambda + \int_{[0,y]} \Big(1 - \frac{w_\delta}{2}\Big) \mathbf{1}_{[0,L(x)]} d\lambda \\
&= \frac{1}{2} \delta\big(\min\{y, U(x)\}\big) + \frac{1}{2} g\big(\min\{y, L(x)\}\big),
\end{aligned}$$

using symmetry of E_δ, and considering the three cases (i) $y \leq L(x)$, (ii) $y \in [L(x), U(x)]$, (iii) $y \geq U(x)$ immediately yields the desired result. ■

The following result gives necessary and sufficient conditions for E_δ to be completely dependent.

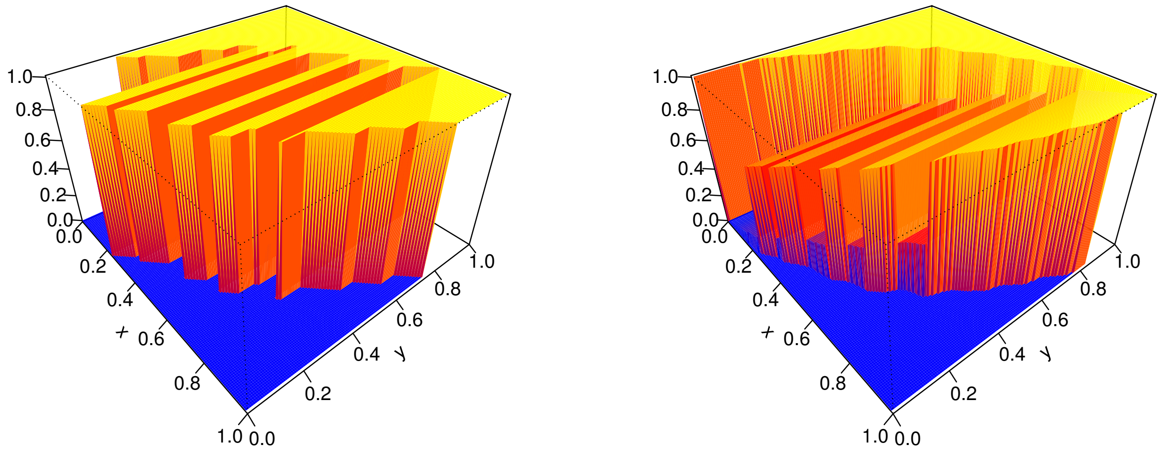

Figure 3: Image plots of the functions $(x,y) \mapsto K_{E_{\delta_1}}(x,[0,y])$ and $(x,y) \mapsto K_{E_{\delta_2}}(x,[0,y])$, whereby δ_1, δ_2 are as in Figure 1.

Theorem 4.3 *Suppose that δ is a diagonal. Then E_δ is a generalized shuffle of M living on the graph of a λ-preserving bijection $S : [0,1] \to [0,1]$ fulfilling $S \circ S = id_{[0,1]}$ if and only if for λ-almost every $x \in [0,1]$ either $\delta'(x) \in \{0,2\}$ or $\delta(x) = x$ holds.*

Proof: Suppose that E_δ lives on the graph of a λ-preserving transformation $S : [0,1] \to [0,1]$. Let $\Lambda \in \mathcal{B}([0,1])$ denote a Borel set such that $\delta'(x) = w_\delta(x)$ and $K_A(x,\cdot) = \epsilon_{Sx}(\cdot)$ for every $x \in \Lambda$. Fix $x \in \Lambda$. If $\delta(x) < x$ then, applying Lemma 4.1, yields $L(x) < x$ and $U(x) > x$, so, using (7) we have $w_\delta(x) \in \{0,2\}$, which proves one implication. On the other hand, if for λ-almost every $x \in [0,1]$ either $\delta'(x) \in \{0,2\}$ or $\delta(x) = x$ holds, then Theorem 4.2 and Lemma 4.1 imply complete dependence of E_δ, i.e. E_δ lives on the graph of a λ-preserving function $\tilde{S} : [0,1] \to [0,1]$. Symmetry implies

$$\lambda(F \times \hat{S}^{-1}(G)) = \lambda(G \times \hat{S}^{-1}(F))$$

for all $F, G \in \mathcal{B}([0,1])$, from which, setting $F = \hat{S}^{-1}(G)$, we immediately get that every Borel set G is $\hat{S}^2$-invariant (i.e. $\lambda(\hat{S}^{-2}(G)\Delta G) = 0$). Using the fact that $C_{\hat{S}} * C_{\hat{S}} = C_{\hat{S}^2}$ it follows that $C_{\hat{S}^2} = M$, so $\hat{S}^2 = id_{[0,1]}$ λ-almost everywhere. Set $\Psi := \{x \in [0,1] : \hat{S}^2(x) = x\}$, then the function $S : [0,1] \to [0,1]$, defined by $S(x) = \hat{S}(x)\mathbf{1}_\Psi(x) + x\,\mathbf{1}_{\Psi^c}(x)$ has the desired properties, E_δ lives on the graph of S and E_δ is a generalized shuffle of M. ■

The following direct consequence of Theorem 4.3 has already been proved in Nelsen and Fredricks (1997).

Proposition 4.4 *E_δ is a (straight) shuffle of M, if and only if, δ is piecewise linear and for λ-almost every $x \in [0,1]$ either $\delta'(x) \in \{0,2\}$ or $\delta(x) = x$ holds.*

Example 4.5 Consider E_{δ_0} for the diagonal δ_0 from Example 3.1. According to Proposition 4.3 E_{δ_0} is mutually completely dependent and lives on the graph of a λ-preserving bijection S. Furthermore, for λ-almost every x with $\delta_0'(x) = 2$ we have $S(x) = L(x)$ and for λ-almost every x with $\delta_0'(x) = 0$ we have $S(x) = U(x)$. It follows directly from the construction of δ_0 and Lemma 4.1 that both δ_0 and g are bijections and that $L(x) < x$ as well as $U(x) > x$ for every $x \in (0,1)$. Having this it is straightforward to verify that S is neither monotonic nor continuous on any open non-empty interval $(a,b) \subseteq [0,1]$: Choose $x_0 \in (a,b)$ such that $\delta_0'(x) = 0$ and $S(x_0) = U(x_0)$. Then there exists $R > 0$ such that $S(x_0) = U(x_0) = x_0 + R$ and for every integer $m \geq 2$ we can find points $z_1, z_2 \in (x_0 - \frac{R}{m}, x_0 + \frac{R}{m})$ with $z_1 < x_0 < z_2$, $\delta'(z_1) = \delta'(z_2) = 2$, and $S(z_1), S(z_2) < x_0 + \frac{R}{m} = S(x_0) - R\,\frac{m-1}{m}$. Obviously E_{δ_0} is a proper generalized shuffle of M.

5 Markov kernels of Bertino copulas

Given a diagonal δ the *Bertino copula* B_δ is defined by (see Fredricks and Nelsen, 2002)

$$B_\delta(x,y) = M(x,y) - \min\Big\{\hat{\delta}(t) : t \in \big[\min\{x,y\}, \max\{x,y\}\big]\Big\}. \tag{8}$$

It is well know that B_δ is the minimal element in $\mathcal{C}_\delta$ (see Fredricks and Nelsen, 2002, Nelsen at al., 2008). Analogous to the previous section we will now calculate the Markov kernel for B_δ and, as direct consequence of that, extend some results from Fredricks and Nelsen (2002). We start with the following two functions $l, u : [0,1] \to [0,1]$:

$$\begin{aligned} u(x) &:= \max\big\{y \geq x : \hat{\delta}(t) \geq \hat{\delta}(x) \text{ for all } t \in [x,y]\big\} \\ l(x) &:= \min\big\{y \leq x : \hat{\delta}(t) \geq \hat{\delta}(x) \text{ for all } t \in [y,x]\big\} \end{aligned} \tag{9}$$

The following lemma gathers some properties of l and u:

Lemma 5.1 *Suppose that δ is a diagonal and let u, l be defined according to (9). Then the following assertions hold:*

1. *u is upper semicontinuous, l lower semicontinuous.*
2. *$u(0) = u(1) = 1$, $u(x) \geq x$ and $\hat{\delta}(u(x)) = \hat{\delta}(x)$ for every $x \in [0,1]$.*
3. *$l(0) = l(1) = 0$, $l(x) \leq x$ and $\hat{\delta}(l(x)) = \hat{\delta}(x)$ for every $x \in [0,1]$.*
4. *$\hat{\delta}'(x) > 0$ implies $u(x) > x$ and $l(x) = x$, $\hat{\delta}'(x) < 0$ implies $l(x) < x$ and $u(x) = x$.*
5. *If $u(x) > x$ and $\hat{\delta}$ is differentiable at x then $\hat{\delta}'(x) \geq 0$ follows. If $l(x) < x$ and $\hat{\delta}$ is differentiable at x then $\hat{\delta}'(x) \leq 0$ follows.*
6. *Suppose that $x < y$; then we have $u(x) < y$ if and only if*

$$\hat{\delta}(x) > \min\big\{\hat{\delta}(t) : t \in [x,y]\big\}$$

7. *Suppose that $y < x$; then we $l(x) > y$ if and only if*

$$\hat{\delta}(x) > \min\left\{\hat{\delta}(t) : t \in [y, x]\right\}$$

Proof: We start with showing upper semicontinuity of u. Let $\alpha \in (0, 1]$ and suppose that $u(x) < \alpha$. Then, by definition, we can find $t_m \in (u(x), \alpha]$ such that

$$\hat{\delta}(t_m) = \min\left\{\hat{\delta}(t) : t \in [u(x), \alpha]\right\} < \hat{\delta}(x).$$

Continuity of $\hat{\delta}$ implies the existence of an interval $(x - r, x + r)$ with $r > 0$ such that $\hat{\delta}(z) > \hat{\delta}(t_m)$ and therefore $u(z) < t_m \leq \alpha$ for each $z \in B(x, r)$. This shows that the set $\{y \in [0, 1] : u(y) < \alpha\}$ is open proving upper semicontinuity of u since α was arbitrary. Lower semicontinuity of l can be proved in the same manner. Assertions two and three are direct consequences of continuity of $\hat{\delta}$, assertions four and five follow directly from the definition of the derivative. Assume that $x < y$. If $\hat{\delta}(x) > \min\left\{\hat{\delta}(t) : t \in [x, y]\right\}$, then there exists $t_0 \in (x, y]$ such that $\hat{\delta}(x) > \hat{\delta}(t_0)$, implying $u(x) < t_0 \leq y$. On the other hand, if $u(x) < y$ holds, then there exists $t_0 \in (x, y]$ with $\hat{\delta}(t_0) < \hat{\delta}(x)$. Assertion seven follows analogously. ■

Remark 5.2 Proposition 2.1 in Fredricks and Nelsen (2002) does not cover all possible cases. In fact, for the diagonal δ_0 from Theorem 3.2 $\hat{\delta}_0$ is not monotonic on any interval and, using Lemma 5.1 it is straightforward to see that the same is true for u and l. Furthermore, neither u nor l needs to be right- or leftcontinuous. Counterexamples are easily constructed: for the piecewise linear diagonal δ_1 fulfilling $\delta_1(1/4) = 0$, $\delta_1(1/2) = 3/8$ and $\delta_1(7/8) = 3/4$ obviously u is not right-continuous at $1/8$ and not left-continuous at $1/2$.

The following two lemmata help to calculate the Markov kernel $K_{B_\delta}(\cdot, \cdot)$ of the Bertino copula B_δ for every $\delta \in \mathcal{D}$.

Lemma 5.3 *Suppose that $\delta \in \mathcal{D}$, that the corresponding $\hat{\delta}$ is differentiable at $x_0 \in (0, 1)$ and that $x_0 < y_0$. Define a non-decreasing function g on $[0, y_0]$ by $g(z) = \min\left\{\hat{\delta}(t) : t \in [z, y_0]\right\}$. Then the following assertions hold:*

(a) If $y_0 > u(x_0)$ then g is differentiable at x_0 and we have $g'(x_0) = 0$.

(b) If $y_0 < u(x_0)$ then we have $D_-g(x_0) = D^-g(x_0) = \hat{\delta}'(x_0) \geq 0$, i.e. g is left-differentiable at x_0 with derivative $\hat{\delta}'(x_0) \geq 0$.

Proof: If(a) holds, then assertion five in Lemma 5.1 implies $\hat{\delta}(x_0) > \min\left\{\hat{\delta}(t) : t \in [x_0, y_0]\right\} := M$. Using continuity of $\hat{\delta}$ there exists an open ball $B(x_0, r)$ with $r > 0$ such that $\hat{\delta} > M$ on $B(x_0, r)$. Hence g is constant on $B(x_0, r)$ and $g'(x_0) = 0$ follows immediately. To prove assertion (b) we show that both the lower and upper left Dini derivative (see Hewitt and Stromberg, 1965) of g at x_0 coincide with $\hat{\delta}(x_0)$. Since, by assumption, $u(x_0) > y_0 > x_0$, Lemma 5.1 implies $\hat{\delta}'(x_0) \geq 0$ as well as $\hat{\delta}(x_0) = \min\left\{\hat{\delta}(t) : t \in [x_0, y_0]\right\}$. By definition of g we have

$$\frac{g(x_0) - g(x_0 - t)}{t} \geq \frac{g(x_0) - \hat{\delta}(x_0 - t)}{t} = \frac{\hat{\delta}(x_0) - \hat{\delta}(x_0 - t)}{t}$$

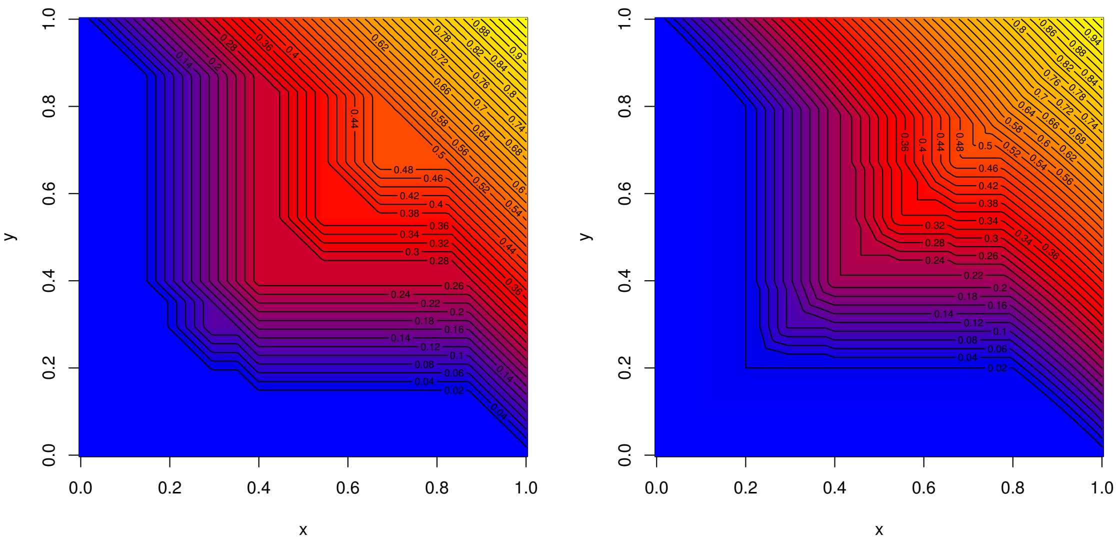

Figure 4: Image plots of the Bertino copulas B_{δ_1} B_{δ_2}, whereby δ_1, δ_2 are as in Figure 1.

from which it follows immediately that the lower left Dini derivative $D_{-}g(x_0)$ of g at x_0 fulfills $D_{-}g(x_0) \geq \hat{\delta}'(x_0)$. Furthermore, by definition of the upper left Dini derivative, for each $\varepsilon > 0$ there exists $\Delta_0 = \Delta_0(\varepsilon) > 0$ such that

$$\sup_{t \in (0,\Delta)} \frac{\hat{\delta}(x_0) - \hat{\delta}(x_0 - t)}{t} \leq \hat{\delta}'(x_0) + \varepsilon$$

for each $\Delta \in (0, \Delta_0]$. Hence, for each $t \in (0, \Delta)$ we have $\hat{\delta}(x_0 - t) \geq \hat{\delta}(x_0) - t(\hat{\delta}'(x_0) + \varepsilon)$, which, considering $\hat{\delta}(x_0) + \varepsilon > 0$, implies $g(x_0 - t) \geq \hat{\delta}(x_0) - t(\hat{\delta}'(x_0) + \varepsilon)$ for every $t \in (0, \Delta)$. Consequently the upper left Dini derivative $D^{-}g(x_0)$ of g at x_0 fulfills $D^{-}g(x_0) \leq \hat{\delta}'(x_0) + \varepsilon$, from which $D^{-}g(x_0) \leq \hat{\delta}'(x_0)$ follows since $\varepsilon > 0$ was arbitrary. ■

Lemma 5.4 *Suppose that $\delta \in \mathcal{D}$, that the corresponding $\hat{\delta}$ is differentiable at $x_0 \in (0, 1)$ and that $y_0 < x_0$. Define a non-increasing function g on $[y_0, 1]$ by $g(z) = \min\{\hat{\delta}(t) : t \in [y_0, z]\}$. Then the following two assertions hold:*

- *If $y_0 < l(x_0)$ then g is differentiable at x_0 and we have $g'(x_0) = 0$.*
- *If $y_0 > l(x_0)$ then we have $D_{+}g(x_0) = D^{+}g(x_0) = \hat{\delta}'(x_0) \leq 0$, i.e. g is right-differentiable at x_0 with derivative $\hat{\delta}'(x_0) \leq 0$.*

Proof: Analogous to the proof of Lemma 5.3. ■

Theorem 5.5 *Suppose that $\delta \in \mathcal{D}$ and let $\hat{w}_\delta : [0,1] \to [-1,1]$ be measurable with $\hat{w}_\delta(x) = \hat{\delta}'(x)$ for λ-almost every $x \in [0,1]$. Then the Markov kernel $K_{B_\delta}(\cdot,\cdot)$ of B_δ is given by*

$$K_{B_\delta}(x,E) = \begin{cases} (1-\hat{w}_\delta(x))\epsilon_x(E) + \hat{w}_\delta(x)\epsilon_{u(x)}(E) & \text{if } \hat{w}_\delta(x) > 0 \\ (1+\hat{w}_\delta(x))\epsilon_x(E) - \hat{w}_\delta(x)\epsilon_{l(x)}(E) & \text{if } \hat{w}_\delta(x) \leq 0, \end{cases} \tag{10}$$

for λ-almost every $x \in [0,1]$.

Proof: Fix $A \in \mathcal{C}$, (a version of) the corresponding Markov kernel $K_A \in \mathcal{K}$, $\delta \in \mathcal{D}$ and (a version of) the derivative $\hat{w}_\delta$ of $\hat{\delta}$. Then for all $x,y \in [0,1]$ we have $A(x,y) = \int_{[0,x]} K_A(t,[0,y])d\lambda(t)$. Hence (see Rudin, 1987) for every fixed $y \in [0,1]$ there exists a Borel set Λ_y with $\lambda(\Lambda_y) = 1$ such for every $x_0 \in \Lambda_y$ the function $f_y : x \mapsto A(x,y)$ is differentiable at x_0 and fulfills $f_y'(x_0) = K_A(x_0,[0,y])$. Use Lipschitz continuity of $\hat{\delta}$ to find a set $\Lambda' \in \mathcal{B}([0,1])$ with $\lambda(\Lambda') = 1$ and $\hat{\delta}'(x) = \hat{w}_\delta(x)$ for every $x \in \Lambda'$ and set $\Lambda_A^\delta := \Lambda' \cap \bigcap_{y \in \mathbb{Q}\cap[0,1]} \Lambda_y$. Then obviously $\Lambda_A^\delta \in \mathcal{B}([0,1])$ and $\lambda(\Lambda_A^\delta) = 1$ follows. Now consider the case $A = B_\delta$, set $\Lambda := \Lambda_{B_\delta}^\delta \in \mathcal{B}([0,1])$ and fix $x_0 \in \Lambda$ as well as $y_0 \in \mathbb{Q} \cap [0,1]$. (i) If $y_0 > u(x_0)$ then Lemma 5.1 implies $K_A(x_0,[0,y_0]) = 1$, (ii) if $x_0 < y_0 < u(x_0)$ then $w_\delta(x_0) \geq 0$ and $K_A(x_0,[0,y_0]) = 1 - \hat{w}_\delta(x_0)$ follows. (iii) If $x_0 > y_0 > l(x_0)$ then Lemma 5.3 implies $w_\delta(x_0) \leq 0$ as well as $K_A(x_0,[0,y_0]) = -w_\delta(x_0)$, if (iv) $y_0 < l(x_0)$ then $K_A(x_0,[0,y_0]) = 0$ follows. Having this and taking into account that (10) obviously is a Markov kernel the result follows immediately from right-continuity of $y \mapsto K(x,y)$ and the fact that $\mathbb{Q}$ is dense in $[0,1]$. ■

Proposition 5.6 *The support of the Bertino copula B_δ is contained in the union of the diagonal and the closure of the graph of the measurable function $S : [0,1] \to [0,1]$, defined by*

$$S(x) = \begin{cases} u(x) & \text{if } w_\delta(x) > 0 \\ l(x) & \text{if } w_\delta(x) \leq 0 \end{cases} \tag{11}$$

A result similar to Theorem 4.3 also holds for Bertino copulas:

Theorem 5.7 *Suppose that δ is a diagonal. If $\delta'(x) \in \{0,2\}$ holds for λ-almost every $x \in [0,1]$ then the Bertino copula B_δ is a generalized shuffles of M and lives on the graph of a λ-preserving bijection $S : [0,1] \to [0,1]$ fulfilling $S \circ S = id_{[0,1]}$. In case δ, in addition, is piecewise linear then B_δ is a (straight) shuffle of W.*

Proof: Analogous to the proof of Theorem 4.3. ■

Example 5.8 For δ_0 from Example 3.1 Proposition 5.7 implies that B_{δ_0} is a generalized shuffle of M living on the graph of a λ-preserving bijection $S : [0,1] \to [0,1]$ fulfilling $S \circ S = id_{[0,1]}$. Since for every $x \in (0,1)$ with $\hat{\delta}_0'(x) > 0$ we have $u(x) > x$ and for every $x \in (0,1)$ with $\hat{\delta}_0'(x) < 0$ we have $l(x) < x$ we can proceed analogously to Example 4.5 to show that S is neither monotonic nor continuous on any non-empty open interval $(a,b) \subseteq [0,1]$. As a consequence Theorem 2.2. in Fredricks and Nelsen (2002) does not cover all possible supports of Bertino copulas.

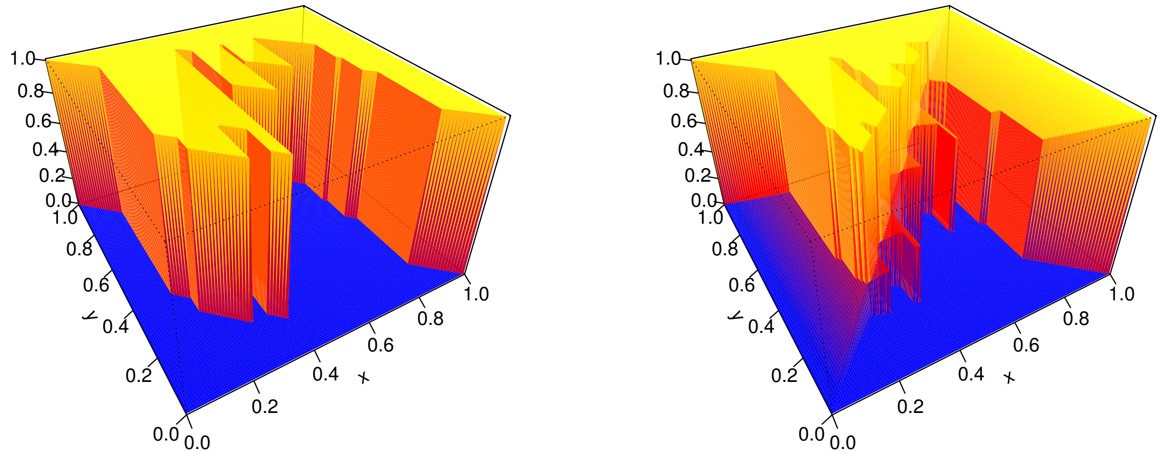

Figure 5: Image plots of the functions $(x,y) \mapsto K_{B_{\delta_1}}(x,[0,y])$ and $(x,y) \mapsto K_{B_{\delta_2}}(x,[0,y])$, whereby δ_1, δ_2 are as in Figure 1.

6 Signed Markov kernels for the maximum quasi-copulas with given diagonal

Given $\delta \in \mathcal{D}$ in the following A_δ will denote the quasi-copula introduced and studied in Nelsen et al. (2008) and Úbeda-Flores (2008), i.e.

$$A_\delta(x,y) := \min\left\{x, y, \max\{x,y\} - \max\left\{\hat{\delta}(t) : t \in \big[\min\{x,y\}, \max\{x,y\}\big]\right\}\right\} \tag{12}$$

for all $x, y \in [0,1]$. It is well known (see Nelsen et al., 2008, Úbeda-Flores, 2008) that A_δ is the maximal quasi-copula with given diagonal δ - in the sequel we will therefore refer to A_δ as the *MQC* with diagonal δ. Following a similar approach as in the last sections we will prove the conjecture stated in Nelsen et al. (2008), saying that A_δ is singular. Working with Markov kernels will also allow for a a very simple and short alternative proof of the characterization of diagonals for which A_δ is a copula given in Úbeda-Flores (2008). As in the previous two sections we start with the construction of some functions that will be useful in the sequel: For every $x \in [0,1]$ define two functions $\underline{g}_x : [0,x] \to [0,1]$ and $\overline{g}_x : [x,1] \to [0,1]$ by

$$\underline{g}_x(z) = z + \max\left\{\hat{\delta}(t) : t \in [z,x]\right\}, \quad \overline{g}_x(z) = z - \max\left\{\hat{\delta}(t) : t \in [x,z]\right\}.$$

It is straightforward to verify that both $\underline{g}_x$ and $\overline{g}_x$ are non-decreasing and Lipschitz continuous with Lipschitz constant $L = 1$. Furthermore we have $\underline{g}_x(0) \leq x$, $\underline{g}_x(x) \geq x$ as well as $\overline{g}_x(x) \leq x$

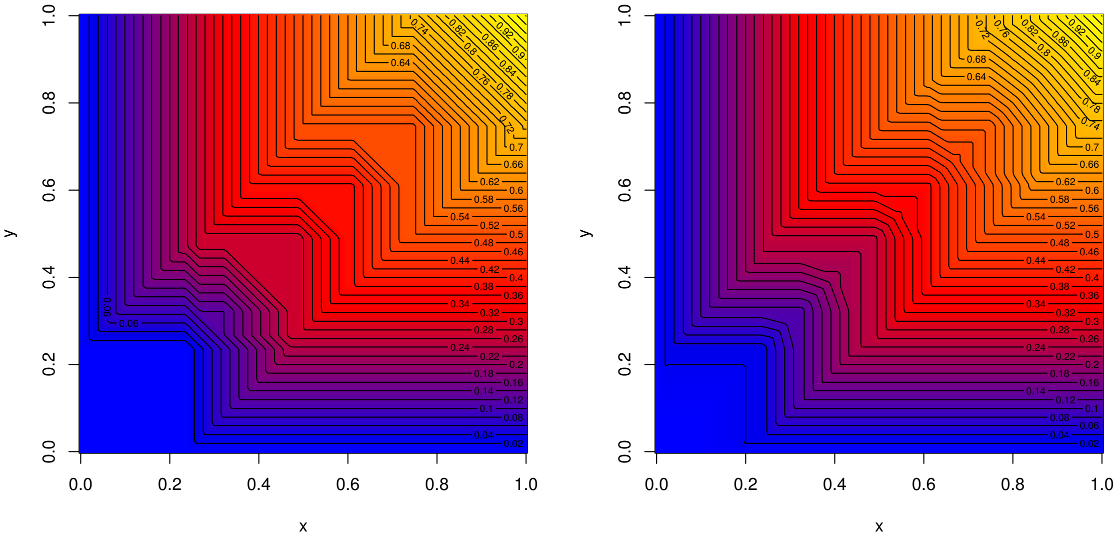

Figure 6: Image plots of the MQCs A_{δ_1} A_{δ_2}, whereby δ_1, δ_2 are as in Figure 1.

and $\overline{g}_x(1) \geq x$. Given the functions $\underline{g}_x, \overline{g}_x$ for every $x \in [0,1]$ define $\overline{f}, \underline{f}, u, l : [0,1] \to [0,1]$ by

$$\begin{aligned}
\overline{f}(x) &= \min\left\{z \in [x,1] : \overline{g}_x(z) \geq x\right\} \\
\underline{f}(x) &= \max\left\{z \in [0,x] : \underline{g}_x(z) \leq x\right\} \\
u(x) &= \max\left\{y \in [x,1] : \hat{\delta}(t) \leq \hat{\delta}(x) \text{ for all } t \in [x,y]\right\} \\
l(x) &= \min\left\{y \in [0,x] : \hat{\delta}(t) \leq \hat{\delta}(x) \text{ for all } t \in [y,x]\right\} \qquad (13)
\end{aligned}$$

The following lemma gathers some properties of the latter four functions:

Lemma 6.1 *Suppose that δ is a diagonal and let $\overline{f}, \underline{f} l, u$ be defined according to (13). Then the following assertions hold:*

1. *$\underline{f}(x) \leq x$ for all $x \in [0,1]$. Furthermore $\underline{f}$ is non-decreasing and upper semicontinuous (hence right-continuous).*

2. *$\overline{f}(x) \geq x$ for all $x \in [0,1]$. Furthermore $\overline{f}$ is non-decreasing and lower semicontinuous (hence left-continuous).*

3. *$\underline{f}(x) = \max\{z \in [0,x] : A_\delta(x,z) \geq z\}$, $\overline{f}(x) = \min\{z \in [x,1] : A_\delta(x,z) \geq x\}$.*

4. *For every $x \in [0,1]$ we have $\underline{f}(x) < x$ if and only if $\delta(x) < x$ if and only if $\overline{f}(x) > x$.*

5. *u is upper semicontinuous, l is lower semicontinuous.*

6. $\hat{\delta}'(x) < 0$ *implies* $u(x) > x$ *and* $l(x) = x$*,* $\hat{\delta}'(x) > 0$ *implies* $l(x) < x$ *and* $u(x) = x$*.*
7. *If* $u(x) > x$ *and* $\hat{\delta}$ *is differentiable at* x *then* $\hat{\delta}'(x) \leq 0$ *follows. If* $l(x) < x$ *and* $\hat{\delta}$ *is differentiable at* x *then* $\hat{\delta}'(x) \geq 0$ *follows.*
8. *Suppose that* $x < y$*; then we have* $u(x) < y$ *if and only if*
$$\hat{\delta}(x) < \max\left\{\hat{\delta}(t) : t \in [x, y]\right\}$$
9. *Suppose that* $y < x$*; then we* $l(x) > y$ *if and only if*
$$\hat{\delta}(x) < \max\left\{\hat{\delta}(t) : t \in [y, x]\right\}$$

Proof: First notice that the third assertion is a direct consequence of the definition of A_δ since, in case of $z \leq x$ we have $A_\delta(x, z) \geq z$ if and only if $\underline{g}_x(z) \leq x$ and in case of $z \geq x$ we have $A_\delta(x, z) \geq x$ if and only if $\overline{g}_x(z) \geq x$. In particular we get $A_\delta(x, \underline{f}(x)) = \underline{f}(x)$ and $A_\delta(x, \overline{f}(x)) = x$ for every $x \in [0, 1]$. Having this showing monotonicity and upper semicontinuity of $\underline{f}$ is straightforward. In fact, $x_1 < x_2$ implies $A_\delta(x_2, \underline{f}(x_1)) \geq \underline{f}(x_1)$, from which $\underline{f}(x_1) \leq \underline{f}(x_2)$ directly follows. Furthermore, considering a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0, 1]$ converging to x and fulfilling $\underline{f}(x_n) \geq \alpha$ for every $n \in \mathbb{N}$ it follows that $A_\delta(x_n, \alpha) = \alpha$, so, by continuity of A_δ, $A_\delta(x, \alpha) = \alpha$ and $\underline{f}(x) \geq \alpha$ follows. Hence the set $\{z \in [0, 1] : \underline{f}(z) \geq \alpha\}$ is closed and upper semicontinuity of $\underline{f}$ follows, which completes the proof of the first assertion. The second assertion can be proved analogously. If $\delta(x) = x$ then, using again assertion three, $\underline{f}(x) = \overline{f}(x) = x$ follows. On the other hand $\underline{f}(x) = x$ implies $\delta(x) = x$, which completes the proof of assertion four. Suppose now that $l(x) > \alpha$. Then there exists $t_0 \in [\alpha, l(x))$ such that $\hat{\delta}(t_0) = \max\{\hat{\delta}(t) : t \in [\alpha, x]\} > \hat{\delta}(x)$. Lipschitz continuity of $\hat{\delta}$ implies the existence of an interval $(x - r, x + r)$ with $r > 0$ such that $\hat{\delta}(z) < \hat{\delta}(t_0)$ holds for every $z \in B(x, r)$. This shows $l(z) > t_0 \geq \alpha$ for every $z \in B(x, r)$, so $\{y \in [0, 1] : l(y) > \alpha\}$ is open and l is lower semicontinuous since α was arbitrary. Upper semicontinuity of u can be proved in the same manner. Assertions six and seven follow directly from the definition of the derivative and assertions eight and nine are straightforward to verify. ■

Lemma 6.2 *Suppose that* δ *is a diagonal and let* $\hat{w}_\delta : [0, 1] \to [-1, 1]$ *denote the derivative of the corresponding* $\hat{\delta}$*. Then there exists a Borel set* $\Lambda \in \mathcal{B}([0, 1])$ *fulfilling* $\lambda(\Lambda) = 1$ *as well as* $\hat{\delta}'(x) = \hat{w}_\delta(x)$ *for every* $x \in \Lambda$*, such that for every* $y \in \mathbb{Q}$ *the derivative* s'_y *of the function* $s_y : x \mapsto A_\delta(x, y)$ *exists for every* $x \in \Lambda$ *and fulfills*
$$s'_y(x) = \begin{cases} 1 & \text{if } y \geq \overline{f}(x) \\ 0 & \text{if } y \in \big(u(x), \overline{f}(x)\big) \\ -\hat{w}_\delta(x) & \text{if } y \in \big(x, u(x)\big] \cap \big(x, \overline{f}(x)\big) \\ 1 - \hat{w}_\delta(x) & \text{if } y \in \big[l(x), x\big) \cap \big[\underline{f}(x), x\big) \\ 1 & \text{if } y \in \big[\underline{f}(x), l(x)\big) \\ 0 & \text{if } y < \underline{f}(x). \end{cases} \tag{14}$$

Proof: For every $y \in [0, 1]$ the function $s_y : x \mapsto A_\delta(x, y)$ is Lipschitz continuous with Lipschitz constant $L = 1$ and non-decreasing, so (see Rudin, 1987) there exists a Borel set

$\Lambda_y \subseteq (0,1)$ such that s_y is differentiable at every $x \in \Lambda_y$ and fulfills $s'_y(x) \in [0,1]$. Moreover, Lipschitz continuity of $\hat{\delta}$ implies the existence of another Borel set $\Gamma \in \mathcal{B}([0,1])$ with $\lambda(\Gamma) = 1$ such that $\hat{\delta}$ is differentiable at every $x \in \Gamma$ and fulfills $\hat{\delta}'(x) = \hat{w}_\delta(x)$. Finally, let $\mathcal{J}$ denote the (countable) set of all $x \in [0,1]$ such x being a discontinuity point of $\underline{f}$ or $\overline{f}$, and define $\Lambda = \Gamma \cap \mathcal{J}^c \cap \bigcap_{y \in \mathbb{Q} \cap [0,1]} \Lambda_y$. Then obviously $\Lambda \in \mathcal{B}([0,1])$ and $\lambda(\Lambda) = 1$. Suppose now that $x \in \Lambda$, $y \in \mathbb{Q} \cap [0,1]$ and distinguish the following two cases (Lemma 6.1 will be applied multiple times without reference):

Case I: $\mathbf{y < x}$: (i) If $y < \underline{f}(x)$ then there exists $r > 0$ such that for all $z \in (x-r, x+r)$ we have $A_\delta(z,y) = y$, from which $s'_y(x) = 0$ immediately follows. (ii) $y \in [\underline{f}(x), l(x))$ implies $\hat{\delta}(x) < \max\{\hat{\delta}(t) : t \in [y,x]\}$. Hence, taking into account that $\underline{f}$ is non-decreasing and $\hat{\delta}$ is Lipschitz continuous, we can find $r > 0$ such that $A_\delta(z,y) = z - \max_{t \in [y,z]} \hat{\delta}(z)$ for every $z \in (x-r, x)$ and the function $g : z \mapsto \max_{t \in [y,z]} \hat{\delta}(t)$ is constant on $(x-r, x]$. Since $s_y(x)$ exists $s'_y(x) = 1$ follows. (iii) If $y \geq l(x)$ and $y > \underline{f}(x)$, then $\hat{\delta}'(x) \geq 0$ as well as $\hat{\delta}(x) = \max_{t \in [y,x]} \hat{\delta}(t)$ follows. Furthermore we can find $r > 0$ such that $A_\delta(z,y) = z - \max_{t \in [y,z]} \hat{\delta}(t)$ for every $z \in (x-r, x+r)$. Setting $g(z) := \max_{t \in [y,z]} \hat{\delta}(t)$ for $z \in (x-r, x+r)$ and considering that

$$\begin{aligned}
\frac{g(x+t) - g(x)}{t} &\geq \frac{\hat{\delta}(x+t) - g(x)}{t} = \frac{\hat{\delta}(x+t) - \hat{\delta}(x)}{t} \\
\frac{g(x) - g(x-t)}{t} &\leq \frac{g(x) - \hat{\delta}(x-t)}{t} = \frac{\hat{\delta}(x) - \hat{\delta}(x-t)}{t}
\end{aligned} \tag{15}$$

holds for every $t \in (0,r)$, $s'_y(x) = 1 - \hat{\delta}'(x)$ follows immediately. (iv) If $y \geq l(x)$ and $y = \underline{f}(x)$, then, as in (iii), $\hat{\delta}'(x) \geq 0$ as well as $\hat{\delta}(x) = \max_{t \in [y,x]} \hat{\delta}(t)$ follows. It suffices to consider the case that $\underline{f}(z) > y$ for every $z > x$ (otherwise the arguments in (iii) may be applied). In this case $A_\delta(z,y) = y$ for all $z > x$ follows, which implies $s'_y(x) = 0$. Furthermore, we can find $r > 0$ such that $A_\delta(z,y) = z - g(z)$ for every $z \in (x-r, x]$, whereby g is defined as in (iii). Applying (15) it follows that $g'(x) \leq \hat{\delta}'(x)$, from which, using $s'_y(x) = 0$, $\hat{\delta}'(x) = 1$ follows, i.e. $s'_y(x) = 1 - \hat{\delta}'(x)$ as in (iii). This completes the proof of the case $y < x$.

Case II: $\mathbf{y > x}$ (i) If $y \geq \overline{f}(x)$ then we have $A_\delta(z,y) = z$ for all $z < x$ from which $s'_y(x) = 1$ directly follows. (ii) If $y > u(x)$ and $y < \overline{f}(x)$ then we have $\hat{\delta}(x) < \max_{t \in [x,y]} \hat{\delta}(t)$. Hence the function $g : z \mapsto \max_{t \in [z,y]} \hat{\delta}(t)$ is constant on an interval $(x-r, x+r)$ with $r > 0$, implying $s'_y(x) = -g'(z) = 0$. (iii) If $y \leq u(x)$ and $y < \overline{f}(x)$ then $\hat{\delta}'(x) \leq 0$ as well as $\hat{\delta}(x) = \max_{t \in [x,y]} \hat{\delta}(t)$ follows. Furthermore there exists $r > 0$ such that for every $z \in (x-r, x+r)$ we have $A_\delta(z,y) = y - g(z)$ whereby $g(z) = \max_{t \in [z,y]} \hat{\delta}(t)$. Since g obviously fulfills (15) it follows that $g'(x) = \hat{\delta}'(x)$ and $s'_y(x) = -\hat{\delta}'(x)$. ■

It is well known (see Fernández Sánchez, 2010, Nelsen et al., 2010) that, given a quasi-copula Q there need not exist a doubly stochastic signed measure $\mu_Q : \mathcal{B}([0,1]^2) \to \mathbb{R}$ fulfilling

$$\mu([x_1,x_2] \times [y_1,y_2]) = Q(x_2,y_2) - Q(x_1,y_2) - Q(x_2,y_1) + Q(x_1,y_1) =: V_Q([x_1,x_2] \times [y_1,y_2]) \tag{16}$$

for all intervals $[x_1,x_2], [y_1,y_2] \subseteq [0,1]$. Nevertheless, we will show now that in case of the MQC A_δ a signed measure μ with the afore-mentioned properties can be constructed.

For every $x \in \Lambda$ the function $y \mapsto s_y'(x) \in [0,1]$ is a step-function that is right-continuous at all $y \in \mathbb{Q} \setminus \{x, u(x)\}$. Additionally, for given $y \in \mathbb{Q}$ and arbitrary $x \in [0,1]$ we obviously have

$$A_\delta(x,y) = \int_{[0,x]} s_y'(t) d\lambda(t). \tag{17}$$

For every $x \in \Lambda$ let $y \mapsto K(x,[0,y])$ denote the right-continuous extension of $y \mapsto s_y'(x)$ to full $[0,1]$, for every $x \in \Lambda^c$ set $K(x,[0,y]) = 1$, i.e.

$$K(x,[0,y]) = \begin{cases} 1 & \text{if } x \in \Lambda \text{ and } y \geq \overline{f}(x) \\ 0 & \text{if } x \in \Lambda \text{ and } y \in \big[u(x), \overline{f}(x)\big) \\ -\hat{w}_\delta(x) & \text{if } x \in \Lambda \text{ and } y \in \big[x, u(x)\big) \cap \big[x, \overline{f}(x)\big) \\ 1 - \hat{w}_\delta(x) & \text{if } x \in \Lambda \text{ and } y \in \big[l(x), x\big) \cap \big[\underline{f}(x), x\big) \\ 1 & \text{if } x \in \Lambda \text{ and } y \in \big[\underline{f}(x), l(x)\big) \\ 0 & \text{if } x \in \Lambda \text{ and } y < \underline{f}(x) \\ 1 & \text{if } x \in \Lambda^c. \end{cases} \tag{18}$$

Then $y \mapsto K(x,[0,y])$ is a step-function too and for every $x \in \Lambda$ we have $s_y(x) = K_A(x,[0,y])$ for all $y \in \mathbb{Q} \setminus \{x, u(x)\}$. Fix $y \in \mathbb{Q}$ and consider $\{z \in \Lambda : u(z) = y < \overline{f}(z)\}$. If the latter

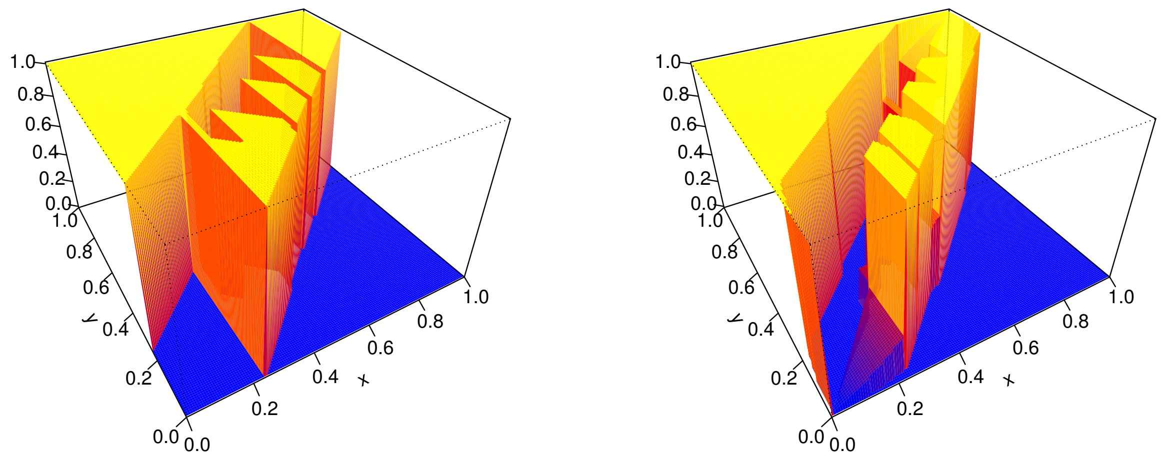

Figure 7: Image plots of the functions $(x,y) \mapsto K_{A_{\delta_1}}(x,[0,y])$ and $(x,y) \mapsto K_{A_{\delta_2}}(x,[0,y])$, whereby δ_1, δ_2 are as in Figure 1.

contains two points $x_1 < x_2$ then we have $\hat{\delta}(x_1) = \hat{\delta}(x_2)$ as well as $A_\delta(x_1,y) = A_\delta(x_2,y)$, from which $s_y'(x_1) = s_y'(x_2) = 0$ immediately follows. Since $s_y'(x) = 0$ is exactly the case where

$y \mapsto s_y(x)$ has no jump at $u(x)$ it follows that for every $y \in \mathbb{Q}$ we have $s'_y(x) = K(x, [0, y])$ for λ-almost every $x \in \Lambda$, so, in particular

$$A_\delta(x, y) = \int_{[0,x]} K(t, [0, y]) d\lambda(t) \tag{19}$$

for every $x \in [0, 1]$ and every $y \in \mathbb{Q}$. Altogether we have constructed a function $K : [0, 1] \times [0, 1] \to [0, 1]$ such that $x \mapsto K(x, [0, y])$ is measurable for every $y \in [0, 1]$ and $y \mapsto K(x, [0, y])$ is a right-continuous step-function with values in $[0, 1]$ for every $x \in [0, 1]$. For $x \in \Lambda^c$ the function $y \mapsto K(x, [0, y])$ corresponds to the Dirac measure ϵ_x in x, set $\vartheta_x^+ = \epsilon_x$ and $\vartheta_x^- = 0$. For $x \in \Lambda$ the function $y \mapsto K(x, [0, y])$ induces a signed measure $\vartheta_x : \mathcal{B}([0, 1]) \to [-1, 2]$ whose Hahn decomposition $\vartheta_x = \vartheta_x^+ - \vartheta_x^-$ consists of two finite discrete measures $\vartheta_x^+, \vartheta_x^-$, whereby the support of ϑ_x^+ is contained in the set $\{\underline{f}(x), \overline{f}(x)\}$ and the support of ϑ_x^- in the set $\{l(x), x, u(x)\}$. It is straightforward to verify that $K^+ : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 2]$ and $K^- : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$, defined by

$$K^+(x, E) := \vartheta_x^+(E), \quad \text{and} \quad K^-(x, E) := \vartheta_x^-(E)$$

are finite discrete kernels fulfilling $K(x, [0, y]) = K^+(x, [0, y]) - K^-(x, [0, y])$. In the sequel we will therefore write $K(x, F) := K^+(x, F) - K^-(x, F)$ for every $x \in [0, 1]$ and $F \in \mathcal{B}([0, 1])$. Moreover, using disintegration (see Kallenberg, 1997), we get that $\mu^+ : \mathcal{B}([0, 1]^2) \to [0, 2]$ and $\mu^- : \mathcal{B}([0, 1]^2) \to [0, 1]$, defined by

$$\mu^+(E) = \int_{[0,1]} K^+(t, E_x) d\lambda(t), \quad \mu^-(E) = \int_{[0,1]} K^-(t, E_x) d\lambda(t) \tag{20}$$

for every $E \in \mathcal{B}([0, 1]^2)$ are measures with $\mu^+([0, 1]^2) - \mu^-([0, 1]^2) = 1$. Hence, setting $\mu(E) := \mu^+(E) - \mu^-(E) = \int_{[0,1]} K(x, E_x) d\lambda(x)$ for every $E \in \mathcal{B}([0, 1]^2)$, defines a finite signed measure μ on $\mathcal{B}([0, 1]^2)$. Considering that both μ^+ and μ^- live on the graph of finitely many measurable functions by construction, it follows that μ is a singular finite signed measure by definition. We will show now that μ fulfills (16) and start with showing that (19) holds for every $y \in [0, 1]$. Fix $y \in [0, 1]$, $x \in [0, 1]$, and let $(y_n)_{n \in \mathbb{N}}$ denote a decreasing sequence in $\mathbb{Q}$ with limit y. Then Lebesgue's theorem on Dominated Convergence and continuity of A_δ imply

$$A_\delta(x, y) = \lim_{n \to \infty} A_\delta(x, y_n) = \lim_{n \to \infty} \int_{[0,x]} K(t, [0, y_n]) d\lambda(t) = \int_{[0,x]} K(t, [0, y]) d\lambda(t).$$

Suppose now that $x_1 < x_2$, that $y \in [0, 1]$, and that $(y_n)_{n \in \mathbb{N}}$ is a monotonically increasing sequence in $[0, y)$ with limit y, then we have

$$\begin{aligned} \int_{[x_1,x_2]} K(t, [0, y)) d\lambda(t) &= \lim_{n \to \infty} \int_{(x_1,x_2]} K(t, [0, y_n]) d\lambda(t) = \lim_{n \to \infty} \big(A_\delta(x_2, y_n) - A_\delta(x_1, y_n)\big) \\ &= A_\delta(x_2, y) - A_\delta(x_1, y) = \int_{[x_1,x_2]} K(t, [0, y]) d\lambda(t), \end{aligned}$$

so, in particular $\int_{[x_1,x_2]} K(t,\{y\})d\lambda(t) = 0$. Having this, equation (16) follows from

$$\begin{aligned} V_{A_\delta}([x_1,x_2]\times[y_1,y_2]) &= \int_{[x_1,x_2]} K(t,[0,y_2])d\lambda(t) - \int_{[x_1,x_2]} K(t,[0,y_1])d\lambda(t) \\ &= \int_{[x_1,x_2]} K(t,(y_1,y_2])d\lambda(t) = \int_{[x_1,x_2]} K(t,[y_1,y_2])d\lambda(t) \\ &= \mu([x_1,x_2]\times[y_1,y_2]). \end{aligned}$$

Altogether we have the following result confirming singularity of A_δ as conjectured in Nelsen et al. (2008) (note that we used the standard definition of singularity of a signed measure and not $\frac{\partial^2 A_\delta}{\partial x \partial y} = 0$ λ-almost everywhere):

Theorem 6.3 *Suppose that δ is a diagonal. Then there exists a doubly stochastic (finite) signed measure $\mu : \mathcal{B}([0,1]^2) \to [-1,2]$ such that*

$$A_\delta(x_2,y_2) - A_\delta(x_1,y_2) - A_\delta(x_2,y_1) + A_\delta(x_1,y_1) = \mu([x_1,x_2]\times[y_1,y_2])$$

holds for all intervals $[x_1,x_2],[y_1,y_2] \subseteq [0,1]$. Additionally, both measures μ^+,μ^- of the Hahn decomposition $\mu = \mu^+ - \mu^-$ of μ live on the graph of at most three measurable functions, i.e. μ is singular.

We conclude the paper by showing that the chosen approach with kernels also allows for a very simply and short proof of the main result in Úbeda-Flores (2008) characterizing all diagonals for which A_δ is a copula.

Lemma 6.4 *Suppose that δ is a diagonal for which A_δ is a copula. Then for almost every $x \in [0,1]$ we have either $\hat{\delta}'(x) \in \{-1,1\}$ or $\delta(x) = x$ and A_δ is completely dependent.*

Proof: Consider Λ according to Lemma 6.2 and $K(\cdot,\cdot)$ as in equation (18). Since, by assumption, A_δ is a copula, there exists a subset $\Lambda' \subseteq \Lambda$ with $\lambda(\Lambda') = 1$ such that $K(x,\cdot)$ is a probability measure for every $x \in \Lambda'$. (i) If $x \in \Lambda'$ and $\hat{\delta}'(x) > 0$ then $l(x) < x = u(x)$ follows. In this case $\underline{f}(x) < l(x)$ can not hold since for every $y \in [\underline{f}(x), l(x))$ we have $K(x,[0,y]) = 1$, for every $y \in [l(x), x)$ we have $K(x,[0,y]) = 1 - \hat{\delta}'(x)$, so monotonicity would imply $\hat{\delta}'(x) = 0$. Hence $\underline{f}(x) \geq l(x)$ follows. Since, additionally, $\underline{f}(x) = x$ would imply $\underline{f}(x) = \overline{f}(x) = \delta(x) = x$, $\hat{\delta}(x) = 0$ and therefore $\hat{\delta}'(x) = 0$, it suffices to consider the case $l(x) \leq \underline{f}(x) < x$. Since $K(x,[0,y]) = 1 - \hat{\delta}'(x)$ for $y \in [\underline{f}(x), x)$ and $K(x,[0,y]) = 0$ for $y \in [u(x), \overline{f}(x)) = [x, \overline{f}(x))$ it follows immediately that $\hat{\delta}'(x) = 1$ and that $K(x,E) = \epsilon_{\overline{f}(x)}(E)$ for every $E \in \mathcal{B}([0,1])$. (ii) If $x \in \Lambda'$ and $\hat{\delta}'(x) < 0$ then $\hat{\delta}'(x) = -1$ and $K(x,E) = \epsilon_{\underline{f}(x)}(E)$ can be shown analogously. (iii) Finally, suppose that $x \in \Lambda'$ and $\hat{\delta}'(x) = 0$. Since $\delta(x) < x$ would imply $\underline{f}(x) < x < \overline{f}(x)$, $K(x,[0,y]) = 1$ for $y \in [\underline{f}(x), 1)$ and $K(x,[0,y]) = 0$ for $y \in [x, \overline{f}(x))$, both $\delta(x) = x$ and $K(x,E) = \epsilon_x(E)$ follow. ■

Theorem 6.5 (Úbeda-Flores, 2008) *Suppose that δ is a diagonal. Then A_δ is a copula if and only if for λ-almost every $x \in [0,1]$ one of the following three conditions holds:*

(a) $\delta(x) = x$

(b) $\delta(x) < x$, $\hat{\delta}'(x) = 1$ *and* δ *is constant on the interval* $[\underline{f}(x), x]$

(c) $\delta(x) < x$, $\hat{\delta}'(x) = -1$ *and* δ *has slope two on the interval* $[x, \overline{f}(x)]$

Proof: Suppose that A_δ is a copula. Then $A_\delta = E_\delta$ and, using Lemma 4.1 and Lemma 6.1, $L = \underline{f}$ as well as $U = \overline{f}$ follows. Consider Λ' from the proof of Lemma 6.4 and suppose that $x \in \Lambda'$. (i) If $\delta(x) < x$ and $\hat{\delta}'(x) = 1$, then we have $l(x) \leq \underline{f}(x) < x$ and $\hat{\delta}(x) = \max_{t\in[y,x]} \hat{\delta}(t)$ for every $y \in [\underline{f}(x), x)$. Hence $x - \hat{\delta}(x) = A_\delta(x, y) = E_\delta(x, y) = \frac{\delta(x)+\delta(y)}{2}$ and $\delta(y) = \delta(x)$ for every $y \in [\underline{f}(x), x]$. (ii) If $\delta(x) < x$ and $\hat{\delta}'(x) = -1$ then we have $u(x) \geq \overline{f}(x) > x$ and $\hat{\delta}(x) = \max_{t\in[x,y]} \hat{\delta}(t)$ for every $y \in [x, \overline{f}(x)]$. Hence $y - \hat{\delta}(x) = A_\delta(x, y) = E_\delta(x, y) = \frac{\delta(x)+\delta(y)}{2}$ and $\delta(y) = \delta(x)+2(y-x)$ for every $y \in [x, \overline{f}(x)]$. This completes the proof of one implication. Suppose now that $\Gamma \in \mathcal{B}([0, 1])$ fulfills $\lambda(\Gamma) = 1$ and for every $x \in \Gamma$ (a), (b) or (c) holds. Let Λ as in Lemma 6.2 and consider $x \in \Gamma \cap \Lambda$. (i) If $\delta(x) = x$ then $\hat{\delta}'(x) = 0$ follows and equation (18) implies $K(x, E) = \epsilon_x(E)$. (ii) If (b) holds then $l(x) < x = u(x) < \overline{f}(x)$ and $\hat{\delta}$ has slope 1 on $[\underline{f}(x), x]$. Hence $l(x) \leq \underline{f}(x)$ and it follows immediately that $K(x, E) = \epsilon_{\overline{f}(x)}(E)$ for every $E \in \mathcal{B}([0, 1])$. (iii) If (c) holds then $\underline{f}(x) < l(x) = x < u(x)$ and $\hat{\delta}$ has slope -1 on $[x, \overline{f}(x)]$. Hence $u(x) \geq \overline{f}(x)$ and $K(x, E) = \epsilon_{\underline{f}(x)}(E)$ for every $E \in \mathcal{B}([0, 1])$ follows. Altogether we have shown that $K(x, \cdot)$ is a probability measure for λ-almost every $x \in [0, 1]$, which implies that μ is a doubly stochastic measure. Applying Theorem 6.3 completes the proof. ■

Theorem 6.5 may be reformulated as follows (for the definition of ordinal sums see Nelsen, 2006):

Proposition 6.6 *A_δ is a copula if and only if it is an ordinal sum of W.*

Remark 6.7 Considering, for instance, the set C_∞ and the family $(J_{1,n})_{n\in\mathbb{N}}$ used in the proof of Lemma 3.1 we can easily construct a diagonal δ_3 for which A_{δ_3} is an ordinal sum of W although δ_3 is not piecewise linear (compare with Corollary 10 in Úbeda-Flores, 2008). In fact, setting $\delta_3(t) := t$ for every $t \in C_\infty$ and filling the holes $(J_{1,n})_{n\in\mathbb{N}}$ with affine copies of the diagonal δ_W of W yields a diagonal δ_3 with the desired property.

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