

# On the size of the class of bivariate extreme-value copulas with a fixed value of Spearman's rho or Kendall's tau

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## Abstract

A bivariate extreme-value copula is characterized by a function of one variable, called a Pickands dependence function, which is convex and comprised between two bounds. The authors identify the smallest possible compact set containing the graph of all Pickands dependence functions whose corresponding bivariate extreme-value copula has a fixed value of Spearman's rho or Kendall's tau. The consequences of this result for statistical modeling are outlined.

*Keywords:* Extreme-value copula, Kendall's tau, Pickands dependence function, Spearman's rho

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## 1. Introduction

A copula is the restriction to the unit  $d$ -cube  $[0, 1]^d$  of the cumulative distribution function of a vector  $(U_1, \dots, U_d)$  of standard uniform random variables, i.e., for all  $u_1, \dots, u_d \in [0, 1]$ ,  $C(u_1, \dots, u_d) = \Pr(U_1 \leq u_1, \dots, U_d \leq u_d)$  and for all  $i \in \{1, \dots, d\}$ ,  $\Pr(U_i \leq u_i) = u_i$ . In statistics, copulas are commonly used to model the dependence between the components of a random vector; see, e.g., [14] for an introduction to copulas and [2, 9, 10] for reviews of statistical modeling techniques using copulas.

To understand the nature of copula models, let  $H$  denote the joint distribution of the vector  $X = (X_1, \dots, X_d)$  and let  $F_1, \dots, F_d$  be its univariate margins, i.e., for all  $x_1, \dots, x_d \in \mathbb{R}$ ,  $H(x_1, \dots, x_d) = \Pr(X_1 \leq x_1, \dots, X_d \leq x_d)$  and for all  $i \in \{1, \dots, d\}$ ,  $F_i(x_i) = \Pr(X_i \leq x_i)$ . Assume that the functions  $F_1, \dots, F_d$  are continuous, as is often the case when the variables  $X_1, \dots, X_d$  are measurements. Sklar [18] showed that in this case, there exists a unique copula  $C : [0, 1]^d \rightarrow [0, 1]$  such that, for all  $x_1, \dots, x_d \in \mathbb{R}$ ,

$$H(x_1, \dots, x_d) = C\{F_1(x_1), \dots, F_d(x_d)\}. \quad (1)$$

Thus if  $H$  is known, its margins  $F_1, \dots, F_d$  can be deduced from it and the copula  $C$  induces the dependence between them. In fact,  $C$  is simply the restriction to  $[0, 1]^d$  of the joint distribution function of the vector  $(U_1, \dots, U_d) = (F_1(X_1), \dots, F_d(X_d))$ . In practice, however,  $H$  is often unknown and a model for it can be constructed by selecting  $F_1, \dots, F_d$  and  $C$  of specific forms in Eq. (1).

As an example, suppose that  $X_1$  and  $X_2$  are exponential random variables, i.e., for all  $x_1, x_2 \in [0, \infty)$ ,  $\Pr(X_1 \leq x_1) = 1 - e^{-\lambda_1 x_1}$  and  $\Pr(X_2 \leq x_2) = 1 - e^{-\lambda_2 x_2}$  for some  $\lambda_1, \lambda_2 \in (0, \infty)$ . A joint distribution for the pair  $(X_1, X_2)$  is then obtained if  $C$  is taken to be a Farlie–Gumbel–Morgenstern (FGM) copula, defined for all  $u_1, u_2 \in [0, 1]$  by  $C(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2)$  in terms of a third parameter  $\theta \in [-1, 1]$ . Upon substitution into (1), one finds

$$\Pr(X_1 \leq x_1, X_2 \leq x_2) = F_1(x_1)F_2(x_2)[1 + \theta(1 - F_1(x_1))(1 - F_2(x_2))] = (1 - e^{-\lambda_1 x_1})(1 - e^{-\lambda_2 x_2})(1 + \theta e^{-\lambda_1 x_1} e^{-\lambda_2 x_2})$$

for all  $x_1, x_2 \in [0, \infty)$ . If  $\theta = 0$ , the variables  $X_1$  and  $X_2$  are stochastically independent; otherwise, they are dependent.

In statistical practice, it is often assumed that an unknown copula  $C$  belongs to a given parametric class  $\{C_\theta : \theta \in \Theta\}$ . There is then an interest in estimating the parameter  $\theta$  from data. A standard approach, useful when the parameter space  $\Theta$  is an interval in  $\mathbb{R}$ , is to compute the rank correlation, say  $\rho_n$ , in a random sample of size  $n$  from  $(X_1, X_2)$ . The theoretical analog, called Spearman's rho, is given by

$$\text{corr}\{F_1(X_1), F_2(X_2)\} = \rho(C) = -3 + 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2.$$

An estimate of  $\theta$  is then obtained by solving the equation  $\rho(C) = \rho_n$ . When  $C = C_\theta$  is an FGM copula with unknown parameter  $\theta \in [-1, 1]$ , for instance, one finds  $\rho(C_\theta) = \theta/3$ ; if  $\rho_n \in [-1/3, 1/3]$ , one can then estimate  $\theta$  by  $\hat{\theta}_n = 3\rho_n$ .

An alternative approach consists of computing Kendall's coefficient of concordance,  $\tau_n$ , whose population value is

$$\tau(C) = -1 + 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2).$$

Thus when  $C$  is an FGM copula, one finds  $\tau(C_\theta) = 2\theta/9$  and so another estimate of  $\theta$  is  $\check{\theta}_n = 9\tau_n/2$  if  $\tau_n \in [-2/9, 2/9]$ . In general, the inversion of Spearman's rho and Kendall's tau both lead to consistent estimators of the dependence parameter  $\theta$  if  $C \in \{C_\theta : \theta \in \Theta\}$  and  $\Theta \subset \mathbb{R}$  as  $\rho_n \rightarrow \rho$  and  $\tau_n \rightarrow \tau$  almost surely as  $n \rightarrow \infty$ . For details, see, e.g., [2].

Copula modeling is of particular interest in assessing the joint probability of occurrence of rare events with potentially catastrophic consequences; see, e.g., [13]. It is then known from multivariate extreme-value theory that the copula is max-stable, i.e., such that for all  $u_1, \dots, u_d \in [0, 1]$  and  $m \in \mathbb{N}$ ,  $C(u_1, \dots, u_d) = C^m(u_1^{1/m}, \dots, u_d^{1/m})$ ; see, e.g., [1]. In the bivariate case, Pickands [15] showed that every max-stable (or extreme-value) copula can be written in terms of a convex function  $A : [0, 1] \rightarrow [0, 1/2]$  such that, for all  $t \in [0, 1]$ ,  $A(t) \geq \max(t, 1-t)$ . Specifically, one has, for all  $u_1, u_2 \in (0, 1)$ ,

$$C(u_1, u_2) = \exp[\ln(u_1 u_2) A(\ln(u_2)/\ln(u_1 u_2))]. \quad (2)$$

The mapping  $A$  is called a Pickands dependence function and in modeling the dependence between  $d = 2$  extreme risks, the choice of copula  $C$  in Eq. (1) amounts to selecting an appropriate function  $A$ . For example, Tawn's mixed model is obtained by setting  $A(t) = \theta t^2 - \theta t + 1$  for all  $t \in [0, 1]$  and some  $\theta \in [0, 1]$ . Several other examples are given in Table 1, excerpted from [4]. When  $C$  is of the form (2) for some  $A$ , it was shown by Ghoudi et al. [8] that

$$\rho(C) = -3 + 12 \int_0^1 \frac{1}{\{1 + A(t)\}^2} dt, \quad \tau(C) = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t),$$

where  $A'$  denotes the right-hand derivative of  $A$  on  $[0, 1]$ , which always exists because  $A$  is convex, and  $A'(1)$  is defined as the supremum of  $A'(t)$  on  $(0, 1)$ . It is easily seen that  $\rho(C)$  and  $\tau(C)$  are both elements of  $[0, 1]$ .

Suppose, for example, that one has observed  $\rho(C) = 0.25$  or  $\tau(C) = 0.25$  in a sample. It is then of interest to know how broad is the choice of Pickands dependence functions with this specific value of Spearman's rho or Kendall's tau. To explore this issue, the four functions listed in Table 1 were plotted in the left panel of Figure 1 for parameter values corresponding to  $\tau = 1/4$ . As one can see, there is barely any difference between them. The other two panels of Figure 1 show that the same observation holds for  $\tau = 1/2$  and  $\tau = 3/4$ .

Repeating the exercise for other values of Kendall's tau or Spearman's rho, and for many other common parametric models, leads to the same conclusion, in part because in many of these models, the Pickands dependence function  $A$  is symmetric with respect to  $1/2$ , i.e., one has  $A(t) = A(1-t)$  for all  $t \in [0, 1]$ . When this condition is relaxed, somewhat larger differences between models can be observed, as in Figure 2, which displays five Pickands dependence

Table 1: Common Pickands dependence functions.

| Copula Family   | Pickands Dependence Function $A(t)$ , $t \in (0, 1)$  | Parameter(s)                           |
|-----------------|---|--|
| Galambos        | $1 - \{t^{-\theta} + (1-t)^{-\theta}\}^{-1/\theta}$   | $\theta \in (0, \infty)$               |
| Gumbel–Hougaard | $\{t^\theta + (1-t)^\theta\}^{1/\theta}$  | $\theta \in [1, \infty)$               |
| Hüsler–Reiß     | $t \Phi[\theta + (2\theta)^{-1} \ln\{t/(1-t)\}] + (1-t) \Phi[\theta + 2\theta^{-1} \ln\{(1-t)/t\}]$   | $\theta \in (0, \infty)$               |
| $t$ -EV         | $t T_{\nu+1} \left[ \frac{\{t/(1-t)\}^{1/\nu} - \rho}{\sqrt{(1-\rho^2)/(\nu+1)}} \right] + (1-t) T_{\nu+1} \left[ \frac{\{(1-t)/t\}^{1/\nu} - \rho}{\sqrt{(1-\rho^2)/(\nu+1)}} \right]$ | $\nu \in (0, \infty), \rho \in (0, 1)$ |

$\Phi$  and  $T_\nu$  denote the cumulative distribution functions of the standard Normal and Student  $t$  with  $\nu$  degrees of freedom, respectively.

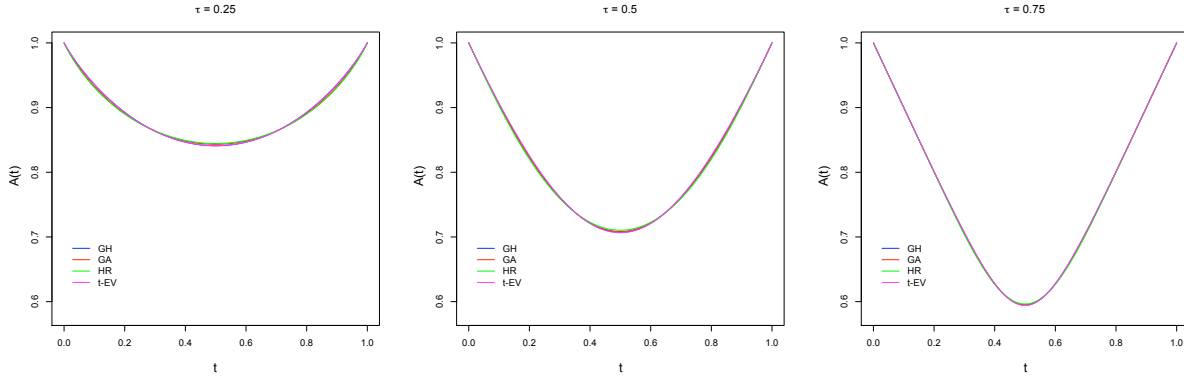


Figure 1: Graphs of the Pickands dependence function for the Gumbel–Hougaard, Galambos, Hüsler–Reiß and t-EV copulas with 4 degrees of freedom when  $\tau = 0.25$  (left),  $\tau = 0.50$  (middle) and  $\tau = 0.75$  (right).

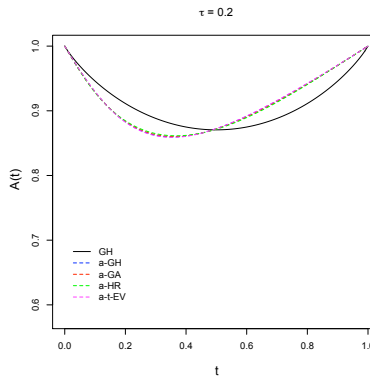


Figure 2: Graphs of the Pickands dependence function for the Gumbel–Hougaard copula and four asymmetric extreme-value copulas with  $\tau = 0.20$ : the asymmetric Gumbel–Hougaard (a-GH), Galambos (a-GA), Hüsler–Reiß (a-HR) and t-EV (a-t-EV) with 4 degrees of freedom.

functions with  $\tau = 1/5$ : the symmetric Gumbel–Hougaard copula and asymmetric versions of the functions in Table 1 obtained by an application of Khoudraji’s device [4]. Still, the difference is not that great.

This paper investigates analytically the size of the class of Pickands dependence functions with a fixed value of Spearman’s rho or Kendall’s tau. Notation and preliminaries are reviewed in Section 2. Derivations presented in Sections 3 and 4 lead to the smallest possible compact set containing the graph of all Pickands dependence functions with a fixed value of Spearman’s rho and Kendall’s tau, respectively. Bounds on the size of these sets are then derived in Section 5. Section 6 contains a short discussion and some technical details are deferred to the Appendix.

## 2. Notation and preliminaries

Let  $\mathcal{A}$  be the class of Pickands dependence functions, i.e., the collection of convex functions  $A : [0, 1] \rightarrow [0, 1/2]$  such that, for all  $t \in [0, 1]$ ,  $A(t) \geq \max(t, 1 - t)$ . For every  $A \in \mathcal{A}$ , the copula  $C$  induced by Eq. (2) is denoted  $C_A$ . We write  $A \in \mathcal{A}_S(\rho)$  if and only if  $\rho(C_A) = \rho$ ; similarly,  $A \in \mathcal{A}_K(\tau)$  if and only if  $\tau(C_A) = \tau$ . For arbitrary  $A, B \in \mathcal{A}$ , we further write  $A \leq B$  if  $A(t) \leq B(t)$  for all  $t \in [0, 1]$  and  $A < B$  if the inequality is strict in at least one point and hence, by continuity, on an interval.

The following three types of piece-wise linear Pickands dependence functions will be used extensively in subsequent derivations. See Figure 3 for archetypical examples of these functions.

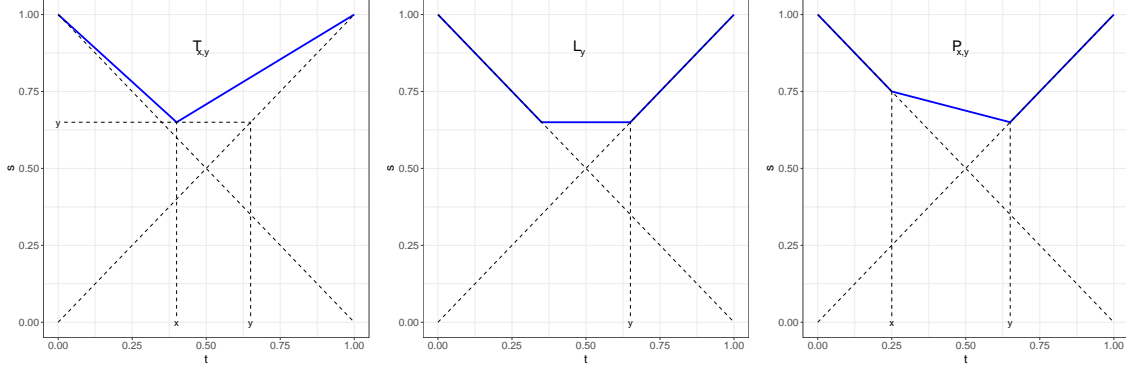


Figure 3: Graphs illustrating the three types of auxiliary functions defined in Section 2.

(i) For all  $y \in [1/2, 1]$  and  $x \in [1 - y, y]$ , define  $T_{x,y} \in \mathcal{A}$  at every  $t \in [0, 1]$  by

$$T_{x,y}(t) = \begin{cases} 1 - (1 - y)t/x & \text{if } 0 \leq t \leq x, \\ 1 - (1 - y)(1 - t)/(1 - x) & \text{if } x < t \leq 1. \end{cases}$$

(ii) For all  $y \in [1/2, 1]$ , define  $L_y \in \mathcal{A}$  at every  $t \in [0, 1]$  by

$$L_y(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq 1 - y, \\ y & \text{if } 1 - y < t \leq y, \\ t & \text{if } y < t \leq 1. \end{cases}$$

(iii) For all  $y \in [1/2, 1]$  and  $x \in [0, 1/2]$ , define  $P_{x,y} \in \mathcal{A}$  at every  $t \in [0, 1]$  by

$$P_{x,y}(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq x, \\ (1 - x - y)(y - t)/(y - x) + y & \text{if } x < t \leq y, \\ t & \text{if } y < t \leq 1. \end{cases}$$

Note that for all  $y \in [1/2, 1]$  and all  $t \in [0, 1]$ ,  $L_y(t) = P_{1-y,y}(t)$ .

### 3. Bounds for the Pickands dependence function when Spearman's $\rho$ is fixed

Straightforward calculations yield

$$\int_0^1 \frac{1}{\{1 + T_{x,y}(t)\}^2} dt = \frac{1}{2(1 + y)}, \quad \int_0^1 \frac{1}{\{1 + L_y(t)\}^2} dt = \frac{y(2 - y)}{(1 + y)^2}, \quad \int_0^1 \frac{1}{\{1 + P_{x,y}(t)\}^2} dt = \frac{1 - x + xy}{(2 - x)(1 + y)}$$

for all values of  $x$  and  $y$  where these functions are defined. Accordingly,

$$\rho(C_{T_{x,y}}) = -3 + \frac{6}{1 + y}, \quad \rho(C_{L_y}) = -3 + 12 \frac{y(2 - y)}{(1 + y)^2}, \quad \rho(C_{P_{x,y}}) = -3 + 12 \frac{1 - x + xy}{(2 - x)(1 + y)}.$$

Let  $\varphi_1 : [1/2, 1] \rightarrow [0, 1]$  and  $\psi_1 : [1/2, 1] \rightarrow [0, 1]$  be strictly decreasing homeomorphisms defined, for all  $y \in [1/2, 1]$ , by  $\varphi_1(y) = \rho(C_{T_{x,y}})$  and  $\psi_1(y) = \rho(C_{L_y})$ . Let  $\varphi_1^{-1}$  and  $\psi_1^{-1}$  denote their inverses. Then, for all  $\rho \in [0, 1]$ ,  $L_{\psi_1^{-1}(\rho)} \in \mathcal{A}_S(\rho)$  and  $T_{x, \varphi_1^{-1}(\rho)} \in \mathcal{A}_S(\rho)$  whatever  $x \in [1 - \varphi_1^{-1}(\rho), \varphi_1^{-1}(\rho)]$ , where  $\varphi_1^{-1}(\rho) = (3 - \rho)/(3 + \rho)$ .

Next, for any fixed  $y \in (1/2, 1]$  and  $x_1, x_2 \in [0, 1]$  with  $x_1 < x_2$ , one has  $P_{x_1, y} > P_{x_2, y}$ , implying that the map  $x \mapsto \rho(C_{P_{x, y}})$  is a strictly increasing homeomorphism which maps  $[0, 1/2]$  onto  $[\varphi_1(y), 1]$ . Fix  $\rho \in [0, 1)$ . Then, for every  $y \geq \varphi_1^{-1}(\rho) > 1/2$ , there is exactly one  $x \in [0, 1/2]$  with  $\rho(C_{P_{x, y}}) = \rho$ . It is easily checked that  $x = h_\rho(y)$ , where

$$h_\rho(y) = \frac{2(-3 + \rho + 3y + \rho y)}{-9 + \rho + 15y + \rho y}$$

and  $h_\rho\{\varphi_1^{-1}(\rho)\} = 0$ . Furthermore,  $h_\rho(1) = 2\rho/(3 + \rho) = 1 - \varphi_1^{-1}(\rho)$  holds. The derivative  $h'_\rho$  of  $h_\rho$  is given, for all  $y \in (\varphi_1^{-1}(\rho), 1)$ , by  $h'_\rho(y) = 36(1 - \rho)/(-9 + \rho + 15y + \rho y)^2$ . Therefore,  $h'_\rho$  is decreasing and hence

$$\min_{y \in [\varphi_1^{-1}(\rho), 1]} h'_\rho(y) = h'_\rho(1) = \frac{36(1 - \rho)}{(6 + 2\rho)^2} > 0.$$

Consequently,  $h_\rho$  is strictly increasing and concave. Thus for arbitrary  $\underline{y}$  and  $\bar{y}$  such that  $\varphi_1^{-1}(\rho) \leq \underline{y} < \bar{y} \leq 1$ , one has

$$h_\rho(\bar{y}) - h_\rho(\underline{y}) \geq h'_\rho(1)(\bar{y} - \underline{y}) = \frac{36(1 - \rho)}{(6 + 2\rho)^2} (\bar{y} - \underline{y}). \quad (3)$$

This non-contractivity property will be key to the proof of Lemma 2.

Tedious but straightforward calculations reported in Appendix A show that the upper envelope  $U_\rho$  of the class  $(P_{h_\rho(y), y})_{y \in [\varphi_1^{-1}(\rho), 1]}$  is given by

$$U_\rho(t) = \begin{cases} 1 - \frac{2\rho}{3 - \rho} t & \text{if } 0 \leq t < (3 - \rho)/(6 + \rho), \\ 1 - \frac{6 + 2\rho - 2\sqrt{9(1 - \rho) + (15 + \rho)(2t - 1)^2}}{15 + \rho} & \text{if } (3 - \rho)/(6 + \rho) \leq t \leq (3 + 2\rho)/(6 + \rho), \\ 1 - \frac{2\rho}{3 - \rho} (1 - t) & \text{if } (3 + 2\rho)/(6 + \rho) < t \leq 1. \end{cases} \quad (4)$$

The function  $U_\rho \in \mathcal{A}$  is symmetric with respect to  $1/2$ , convex and continuously differentiable on  $(0, 1)$ .

As shown below,  $L_{\varphi_1^{-1}(\rho)} \leq A \leq U_\rho$  holds for all  $A \in \mathcal{A}_S(\rho)$  and  $\rho \in [0, 1]$ . The lower bound is derived first.

**Lemma 1.** *Suppose that  $\rho \in [0, 1]$ . Then every  $A \in \mathcal{A}_S(\rho)$  fulfills  $A \geq L_{\varphi_1^{-1}(\rho)}$ .*

*Proof.* The assertion is trivial for  $\rho \in \{0, 1\}$ , so assume  $\rho \in (0, 1)$  and argue by contradiction, i.e., suppose that there exist  $A \in \mathcal{A}_S(\rho)$  and  $t_0 \in (1 - \varphi_1^{-1}(\rho), \varphi_1^{-1}(\rho))$  such that  $A(t_0) < L_{\varphi_1^{-1}(\rho)}(t_0) = \varphi_1^{-1}(\rho)$ . The convexity of  $A$  and the boundary conditions  $A(0) = A(1) = 1$  together imply that  $A \leq T_{t_0, A(t_0)}$ , from which one can immediately conclude that

$$\rho(C_A) \geq \rho(C_{T_{t_0, A(t_0)}}) = \varphi_1 \circ A(t_0) > \varphi_1 \circ \varphi_1^{-1}(\rho) = \rho,$$

which contradicts the fact that  $A \in \mathcal{A}_S(\rho)$ . Thus the argument is complete.  $\square$

The upper bound is established next.

**Lemma 2.** *Suppose that  $\rho \in [0, 1]$ . Then every  $A \in \mathcal{A}_S(\rho)$  fulfills  $A \leq U_\rho$ .*

*Proof.* When  $\rho = 0$  or  $\rho = 1$ , one finds  $U_0 = 1$  and  $U_1 = P_{0, 1/2}$ , respectively. Therefore, the result trivially holds in these two cases. For the rest of the proof, assume that  $\rho \in (0, 1)$ .

Suppose that  $A \in \mathcal{A}_S(\rho)$  and that  $A(t_0) > U_\rho(t_0)$  for some  $t_0 \in [0, 1]$ . Write  $s_0 = A(t_0)$ . The case  $s_0 = 1$  yields an immediate contradiction, so one can assume  $s_0 < 1$ . Because  $U_\rho$  is symmetric, it suffices to consider  $t_0 \in (0, 1/2]$ , and the continuity of  $U_\rho$  implies that

$$r = \min_{t \in [0, 1]} \sqrt{(t - t_0)^2 + \{U_\rho(t) - s_0\}^2} > 0.$$

Define, for all  $t \in (0, 1)$ ,  $f^*(t) = (s_0 - 1)(t - t_0)/(t_0 - 1) + s_0$ ; this increasing line is depicted in green in Figure 4. Because  $A$  is convex, one has  $f^*(t) \leq A(t)$  for all  $t \leq t_0$  and  $f^*(t) \geq A(t)$  for all  $t \geq t_0$ . Let  $x^*$  denote the unique point in the interval  $(0, 1 - \varphi_1^{-1}(\rho))$  such that  $f^*(x^*) = 1 - x^*$ . The following observation is crucial for completing the proof.

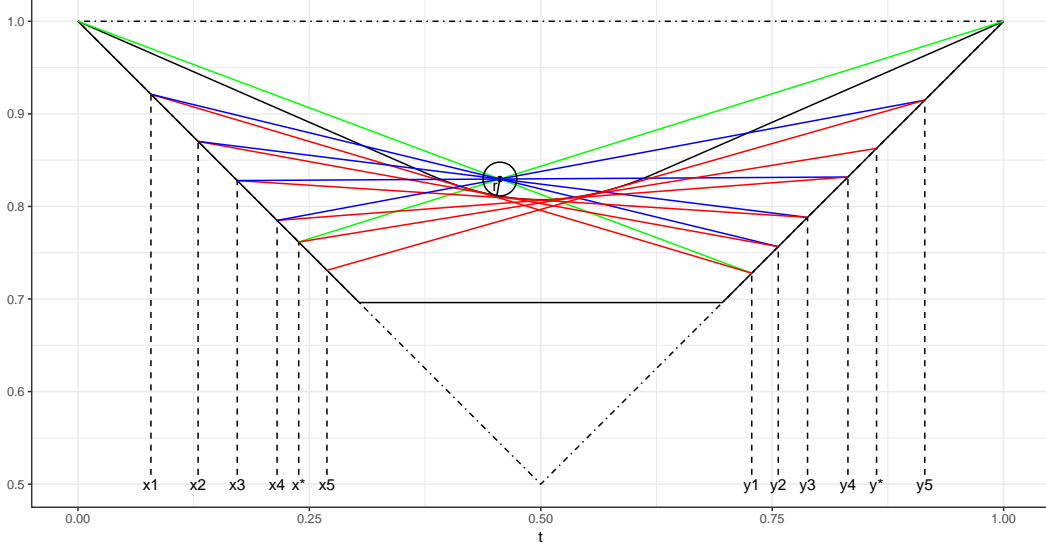


Figure 4: Illustration of the construction used in the proof of Lemma 2.

For every  $x \in [0, x^*]$ , let  $y' = h_\rho^{-1}(x) \in [\varphi_1^{-1}(\rho), 1]$  and define the map  $f : [x, 1] \rightarrow [\varphi_1^{-1}(\rho), 1]$  by setting, for each  $t \in [x, 1]$ ,  $f(t) = (s_0 - 1 + x)(t - t_0)/(t_0 - x) + s_0$ . Moreover, let  $y''$  denote the unique point in  $[\varphi_1^{-1}(\rho), 1]$  such that  $f(y'') = y''$  and  $y'' - y' \geq r/\sqrt{2}$  holds.

*Step 1:* For each  $t \in [0, 1]$ , let  $f_1(t) = (s_0 - 1)(t - t_0)/t_0 + s_0$ ; this decreasing line is also depicted in green in Figure 4. Because  $A$  is convex, one has  $A(t) \leq f_1(t)$  for all  $t \leq t_0$  and  $A(t) \geq f_1(t)$  for all  $t \geq t_0$ . Let  $y_1 \in [\varphi_1^{-1}(\rho), 1]$  denote the unique point such that  $f_1(y_1) = y_1$  and set  $x_1 = h_\rho(y_1)$ . Then  $P_{x_1, y_1} \in \mathcal{A}_S(\rho)$  and  $f_1(t) \geq P_{x_1, y_1}(t)$  for all  $t \leq y_1$ . Three cases must be distinguished as follows.

Case 1: If  $x_1 \geq x^*$ , then  $A(t) \geq P_{x_1, y_1}(t)$ , so  $\rho(C_A) < \rho(C_{P_{x_1, y_1}}) = \rho$ , which is a contradiction.

Case 2: If  $x_1 < x^*$  and  $A(t) \geq P_{x_1, y_1}(t)$  for all  $t \in (0, t_0)$ , then  $A(t) \geq P_{x_1, y_1}(t)$ , which is a contradiction.

Case 3: If  $x_1 < x^*$  and  $A(t) < P_{x_1, y_1}(t)$  holds for some  $t < t_0$ , proceed to Step 2.

*Step 2:* For each  $t \in [0, 1]$ , let  $f_2(t) = (s_0 - 1 + x_1)(t - t_0)/(t_0 - x_1) + s_0$ ; this is the blue line starting at  $(x_1, 1 - x_1)$  in Figure 4. Then  $f_2(t) \geq P_{x_1, y_1}(t)$  for  $t \in [x_1, y_1]$  and the convexity of  $A$  implies that  $A(t) \geq f_2(t)$  for all  $t \geq t_0$ . Let  $y_2$  denote the unique point such that  $f_2(y_2) = y_2$  and set  $x_2 = h_\rho(y_2)$ . Considering that  $y_2 - y_1 > r/\sqrt{2}$  and using Inequality (3), one finds  $x_2 - x_1 > r h'_\rho(1)/\sqrt{2} > 0$ . If  $x_2 \geq x^*$ , then proceed to the final step. Otherwise, continue in the same manner to construct  $x_3, \dots, x_\ell$ , where  $\ell$  denotes the first integer such that  $x_{\ell+1} \geq x^*$  and  $x_\ell < x^*$ . Note that  $x^*$  can be reached in finitely many steps because  $x_{j+1} - x_j > r h'_\rho(1)/\sqrt{2}$  holds. Figure 4 depicts the case  $\ell = 4$ .

*Final step:* Given that  $A \geq P_{x_\ell, y_\ell}$  implies  $A > P_{x_\ell, y_\ell}$ , which directly yields a contradiction, assume that there exists  $t \in (0, t_0)$  with  $A(t) < P_{x_\ell, y_\ell}(t)$  and set  $y^* = h_\rho^{-1}(x^*) \in [y_\ell, y_{\ell+1}]$ . The convexity of  $A$  implies that  $A(t) \geq f_{\ell+1}(t) \geq P_{x^*, y^*}(t)$  for all  $t \in [t_0, 1]$  as well as  $A(t) \geq f^*(t) \geq P_{x^*, y^*}(t)$  for all  $t \in [0, t_0]$ . Because  $P_{x^*, y^*}(t_0) < A(t_0)$ , we find  $A > P_{x^*, y^*}$  and  $\rho(C_A) < \rho(C_{P_{x^*, y^*}}) = \rho$ , which yields a contradiction.  $\square$

Summing up, we have the following result.

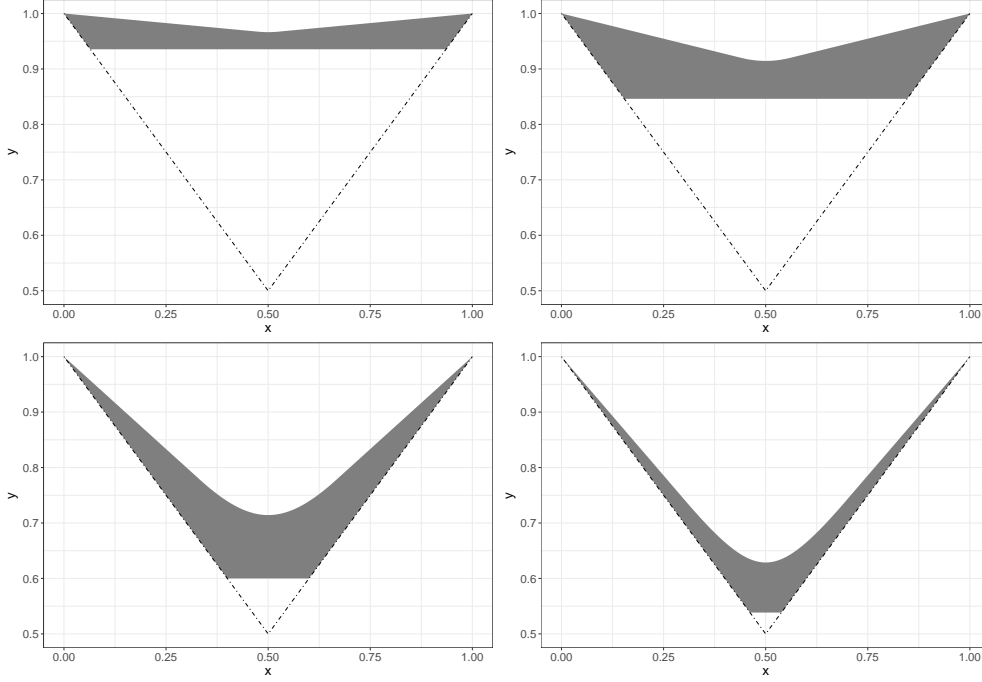


Figure 5: Graphs depicting the shape of  $\Omega_S(\rho)$  when  $\rho = 0.1$  (upper left panel),  $\rho = 0.25$  (upper right panel),  $\rho = 0.75$  (lower left panel), and  $\rho = 0.9$  (lower right panel). In each case, all  $A \in \mathcal{A}_S(\rho)$  lie in the shaded region.

**Proposition 1.** *Let  $\rho \in [0, 1]$  be arbitrary but fixed. Then every  $A \in \mathcal{A}_S(\rho)$  fulfills  $L_{\varphi_1^{-1}(\rho)} \leq A \leq U_\rho$ . Moreover, the set*

$$\Omega_S(\rho) = \{(t, y) \in [0, 1] \times [1/2, 1] : L_{\varphi_1^{-1}(\rho)}(t) \leq y \leq U_\rho(t)\}$$

*is best possible in the sense that for each point  $(t, y) \in \Omega_S(\rho)$  there exists  $A \in \mathcal{A}_S(\rho)$  with  $A(t) = y$ .*

*Proof.* The first assertion has already been proved. The second assertion is a direct consequence of the construction via the functions  $T_{x,y}$  and  $P_{x,y}$ .  $\square$

Note that the bounds  $L_{\varphi_1^{-1}(\rho)}$  and  $U_\rho$  given in Proposition 1 are themselves members of the class  $\mathcal{A}$ . An explicit formula for  $U_\rho$  was already given in Eq. (4). The lower bound is even simpler and reads as follows:

$$L_{\varphi_1^{-1}(\rho)}(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq 2\rho/(3 + \rho), \\ (3 - \rho)/(3 + \rho) & \text{if } 2\rho/(3 + \rho) < t \leq (3 - \rho)/(3 + \rho), \\ t & \text{if } (3 - \rho)/(3 + \rho) < t \leq 1. \end{cases}$$

Figure 5 depicts the set  $\Omega_S(\rho)$  for four different choices of  $\rho \in [0, 1]$ . Note in passing that  $L_{\varphi_1^{-1}(\rho)} \in \mathcal{A}_S(2\rho - \rho^2)$  while  $U_\rho \in \mathcal{A}_S(\rho + 0.5 \times (1 - \rho) \ln(1 - \rho))$ .

#### 4. Bounds for the Pickands dependence function when Kendall's $\tau$ is fixed

As the argument developed here is similar to the approach used in Section 3, the text below is designed to emphasize the parallel. Before proceeding, note that if  $A \geq B$ , then  $\tau(C_A) \leq \tau(C_B)$ ; see, e.g., [5, 12]. The following slight generalization of this fact will be needed below; its proof is deferred to Appendix B.

**Lemma 3.** *If  $A, B \in \mathcal{A}$  and  $A > B$ , then  $\tau(C_A) < \tau(C_B)$ .*

Straightforward calculations yield

$$\tau(C_{T_{x,y}}) = -1 + \frac{1}{y}, \quad \tau(C_{L_y}) = 2(1 - y), \quad \tau(C_{P_{x,y}}) = \frac{1 - 3x - y + 4xy}{y - x}$$

for all values of  $x$  and  $y$  where these functions are defined. Furthermore,  $\tau(C_{P_{x,y}}) = \tau(C_{P_{y,y}}) = 1$  when  $x = y = 1/2$ .

Let  $\varphi_2 : [1/2, 1] \rightarrow [0, 1]$  and  $\psi_2 : [1/2, 1] \rightarrow [0, 1]$  be strictly decreasing homeomorphisms defined, for all  $y \in [1/2, 1]$ , by  $\varphi_2(y) = \tau(C_{T_{x,y}})$  and  $\psi_2(y) = \tau(C_{L_y})$ . Let  $\varphi_2^{-1}$  and  $\psi_2^{-1}$  denote their inverses. Then, for all  $\tau \in [0, 1]$ ,  $L_{\varphi_2^{-1}(\tau)} \in \mathcal{A}_K(\tau)$  and  $T_{x, \varphi_2^{-1}(\tau)} \in \mathcal{A}_K(\tau)$  whatever  $x \in [1 - \varphi_2^{-1}(\tau), \varphi_2^{-1}(\tau)]$ , where  $\varphi_2^{-1}(\tau) = 1/(1 + \tau)$ .

Next, for any fixed  $y \in (1/2, 1]$  and  $x_1, x_2 \in [0, 1]$  with  $x_1 < x_2$ , one has  $P_{x_1, y} > P_{x_2, y}$ , implying that the map  $x \mapsto \tau(C_{P_{x,y}})$  is a strictly increasing homeomorphism which maps  $[0, 1/2]$  onto  $[\varphi_2(y), 1]$ . Fix  $\tau \in [0, 1)$ . Then, for every  $y \geq \varphi_2^{-1}(\tau) > 1/2$ , there is exactly one  $x \in [0, 1/2)$  with  $\tau(C_{P_{x,y}}) = \tau$ . It is easily checked that  $x = h_\tau(y)$ , where

$$h_\tau(y) = \frac{-1 + y + \tau y}{-3 + \tau + 4y}$$

and  $h_\tau\{\varphi_2^{-1}(\tau)\} = 0$ . Furthermore,  $h_\tau(1) = \tau/(1 + \tau) = 1 - \varphi_2^{-1}(\tau)$  holds. The derivative  $h'_\tau$  of  $h_\tau$  is given, for all  $y \in (\varphi_2^{-1}(\tau), 1)$ , by  $h'_\tau(y) = (1 - \tau)^2/(-3 + \tau + 4y)^2$ . Therefore,  $h'_\tau$  is decreasing and hence

$$\min_{y \in \{\varphi_2^{-1}(\tau), 1\}} h'_\tau(y) = h'_\tau(1) = \left(\frac{1 - \tau}{1 + \tau}\right)^2 > 0.$$

Consequently,  $h_\tau$  is strictly increasing and for arbitrary  $\underline{y}$  and  $\bar{y}$  such that  $\varphi_2^{-1}(\tau) \leq \underline{y} < \bar{y} \leq 1$ , one has

$$h_\tau(\bar{y}) - h_\tau(\underline{y}) \geq h'_\tau(1)(\bar{y} - \underline{y}) = \left(\frac{1 - \tau}{1 + \tau}\right)^2 (\bar{y} - \underline{y}). \quad (5)$$

This non-contractivity property will be key to the proof of Lemma 5.

For every  $y \in [\varphi_2^{-1}(\tau), 1]$ , one has  $P_{h_\tau(y), y}(1/2) = 1 - \tau/2 = \psi_2^{-1}(\tau)$ , so the upper envelope  $V_\tau$  of the class  $(P_{h_\tau(y), y})_{y \in [\varphi_2^{-1}(\tau), 1]}$  is given by

$$V_\tau(t) = T_{1/2, \psi_2^{-1}(\tau)}(t) = T_{1/2, 1 - \tau/2}(t) = \begin{cases} P_{h_\tau(\varphi_2^{-1}(\tau)), \varphi_2^{-1}(\tau)}(t) = 1 - \tau t & \text{if } t \in [0, 1/2], \\ P_{h_\tau(1), 1}(t) = 1 - \tau(1 - t) & \text{if } t \in (1/2, 1]. \end{cases} \quad (6)$$

The function  $V_\tau$  is symmetric with respect to  $t = 1/2$  and convex on  $[0, 1]$ . It is in fact the Pickands dependence function of the Cuadras–Augé copula with parameter  $\tau$ , which is a special case of extreme-value copula from the Marshall–Olkin family; see, e.g., Section 3.1 of [14].

As shown below,  $L_{\varphi_2^{-1}(\tau)} \leq A \leq V_\tau$  holds for all  $A \in \mathcal{A}_K(\tau)$  and  $\tau \in [0, 1]$ . The lower bound is derived first.

**Lemma 4.** *Suppose that  $\tau \in [0, 1]$ . Then every  $A \in \mathcal{A}_K(\tau)$  fulfills  $A \geq L_{\varphi_2^{-1}(\tau)}$ .*

*Proof.* The assertion is trivial for  $\tau \in \{0, 1\}$ , so assume  $\tau \in (0, 1)$  and argue by contradiction, i.e., suppose that there exist  $A \in \mathcal{A}_K(\tau)$  and  $t_0 \in (1 - \varphi_2^{-1}(\tau), \varphi_2^{-1}(\tau))$  such that  $A(t_0) < L_{\varphi_2^{-1}(\tau)}(t_0) = \varphi_2^{-1}(\tau)$ . The convexity of  $A$  and the boundary conditions  $A(0) = A(1) = 1$  together imply that  $A \leq T_{t_0, A(t_0)}$ , from which one can immediately conclude that

$$\tau(C_A) \geq \tau(C_{T_{t_0, A(t_0)}}) = \varphi_2 \circ A(t_0) > \varphi_2 \circ \varphi_2^{-1}(\tau) = \tau,$$

which contradicts the fact that  $A \in \mathcal{A}_K(\tau)$ . Thus the argument is complete.  $\square$

The upper bound is established next.

**Lemma 5.** *Suppose that  $\tau \in [0, 1]$ . Then every  $A \in \mathcal{A}_K(\tau)$  fulfills  $A \leq V_\tau$ .*

*Proof.* When  $\tau = 0$  or  $\tau = 1$ , one finds  $V_0 = 1$  and  $V_1 = T_{1/2,1/2}$ , respectively. Therefore, the result trivially holds in these two cases. For the rest of the proof, assume that  $\tau \in (0, 1)$ .

Suppose that  $A \in \mathcal{A}_K(\tau)$  and that  $A(t_0) > V_\tau(t_0)$  for some  $t_0 \in [0, 1]$ . Write  $s_0 = A(t_0)$ . The case  $s_0 = 1$  yields an immediate contradiction, so one can assume  $s_0 < 1$ . Because  $V_\tau$  is symmetric, it suffices to consider  $t_0 \in (0, 1/2]$ , and the continuity of  $V_\tau$  implies that

$$r = \min_{t \in [0,1]} \sqrt{(t - t_0)^2 + \{V_\tau(t) - s_0\}^2} > 0.$$

Define, for all  $t \in (0, 1)$ ,  $f^*(t) = (s_0 - 1)(t - t_0)/(t_0 - 1) + s_0$ ; this increasing line is depicted in green in Figure 6. Because  $A$  is convex, one has  $f^*(t) \leq A(t)$  for all  $t \leq t_0$  and  $f^*(t) \geq A(t)$  for all  $t \geq t_0$ . Let  $x^*$  denote the unique point in the interval  $(0, 1 - \varphi_2^{-1}(\tau))$  such that  $f^*(x^*) = 1 - x^*$ . The following observation is crucial for completing the proof.

For every  $x \in [0, x^*]$ , let  $y' = h_\tau^{-1}(x) \in [\varphi_2^{-1}(\tau), 1]$  and define the map  $f : [x, 1] \rightarrow [\varphi_2^{-1}(\tau), 1]$  by setting, for each  $t \in [x, 1]$ ,  $f(t) = (s_0 - 1 + x)(t - t_0)/(t_0 - x) + s_0$ . Moreover, let  $y''$  denote the unique point in  $[\varphi_2^{-1}(\tau), 1]$  such that  $f(y'') = y''$  and  $y'' - y' \geq r/\sqrt{2}$  holds.

*Step 1:* For each  $t \in [0, 1]$ , let  $f_1(t) = (s_0 - 1)(t - t_0)/t_0 + s_0$ ; this decreasing line is also depicted in green in Figure 6. Because  $A$  is convex, one has  $A(t) \leq f_1(t)$  for all  $t \leq t_0$  and  $A(t) \geq f_1(t)$  for all  $t \geq t_0$ . Let  $y_1 \in [\varphi_2^{-1}(\tau), 1]$  denote the unique point such that  $f_1(y_1) = y_1$  and set  $x_1 = h_\tau(y_1)$ . Then  $P_{x_1, y_1} \in \mathcal{A}_K(\tau)$  and  $f_1(t) \geq P_{x_1, y_1}(t)$  for all  $t \leq y_1$ . Three cases must be distinguished as follows.

Case 1: If  $x_1 \geq x^*$ , then  $A(t) \geq P_{x_1, y_1}(t)$ , so  $\tau(C_A) < \tau(C_{P_{x_1, y_1}}) = \tau$ , which is a contradiction.

Case 2: If  $x_1 < x^*$  and  $A(t) \geq P_{x_1, y_1}(t)$  for all  $t \in (0, t_0)$ , so  $A(t) \geq P_{x_1, y_1}(t)$ , which is a contradiction.

Case 3: If  $x_1 < x^*$  and  $A(t) < P_{x_1, y_1}(t)$  holds for some  $t < t_0$ , proceed to Step 2.

*Step 2:* For each  $t \in [0, 1]$ , let  $f_2(t) = (s_0 - 1 + x_1)(t - t_0)/(t_0 - x_1) + s_0$ ; this is the blue line starting starting at  $(x_1, 1 - x_1)$  in Figure 6. Then  $f_2(t) \geq P_{x_1, y_1}(t)$  for  $t \in [x_1, y_1]$  and the convexity of  $A$  implies that  $A(t) \geq f_2(t)$  for all  $t \geq t_0$ . Let  $y_2$  denote the unique point such that  $f_2(y_2) = y_2$  and set  $x_2 = h_\tau(y_2)$ . Considering that  $y_2 - y_1 > r/\sqrt{2}$  and using Inequality (5), one finds  $x_2 - x_1 > r h'_\tau(1)/\sqrt{2} > 0$ . If  $x_2 \geq x^*$ , then proceed to the final step. Otherwise, continue in the same manner to construct  $x_3, \dots, x_\ell$ , where  $\ell$  denotes the first integer such that  $x_{\ell+1} \geq x^*$  and  $x_\ell < x^*$ . Note that  $x^*$  can be reached in finitely many steps because  $x_{j+1} - x_j > r h'_\tau(1)/\sqrt{2}$  holds. Figure 6 depicts the case  $\ell = 3$ .

*Final step:* Given that  $A \geq P_{x_\ell, y_\ell}$  implies  $A > P_{x_\ell, y_\ell}$ , which directly yields a contradiction, assume that there exists  $t \in (0, t_0)$  with  $A(t) < P_{x_\ell, y_\ell}(t)$  and set  $y^* = h_\tau^{-1}(x^*) \in [y_\ell, y_{\ell+1}]$ . The convexity of  $A$  implies that  $A(t) \geq f_{\ell+1}(t) \geq P_{x^*, y^*}(t)$  for all  $t \in [t_0, 1]$  as well as  $A(t) \geq f^*(t) \geq P_{x^*, y^*}(t)$  for all  $t \in [0, t_0]$ . Because  $P_{x^*, y^*}(t_0) < A(t_0)$ , we find  $A > P_{x^*, y^*}$  and hence  $\tau(C_A) < \tau(C_{P_{x^*, y^*}}) = \tau$  by Lemma 3, which yields a contradiction.  $\square$

Summing up, we have the following result.

**Proposition 2.** *Let  $\tau \in [0, 1]$  be arbitrary but fixed. Then every  $A \in \mathcal{A}_K(\tau)$  fulfills  $L_{\varphi_2^{-1}(\tau)} \leq A \leq V_\tau$ . Moreover, the set*

$$\Omega_K(\tau) = \{(t, y) \in [0, 1] \times [1/2, 1] : L_{\varphi_2^{-1}(\tau)}(t) \leq y \leq V_\tau(t)\}$$

*is best possible in the sense that for each point  $(t, y) \in \Omega_\tau$ , there exists  $A \in \mathcal{A}_K(\tau)$  with  $A(t) = y$ .*

*Proof.* The first assertion has already been proved. The second assertion is a direct consequence of the construction via the functions  $T_{x, y}$  and  $P_{x, y}$ .  $\square$

Note that the bounds  $L_{\varphi_2^{-1}(\tau)}$  and  $V_\tau$  given in Proposition 2 are themselves members of the class  $\mathcal{A}$ . An explicit formula for  $V_\tau$  was already given in Eq. (6). The lower bound is just as simple and reads as follows:

$$L_{\varphi_2^{-1}(\tau)}(t) = \begin{cases} 1 - t & \text{if } 0 \leq t \leq \tau/(1 + \tau), \\ 1/(1 + \tau) & \text{if } \tau/(1 + \tau) < t \leq 1/(1 + \tau), \\ t & \text{if } 1/(1 + \tau) < t \leq 1. \end{cases}$$

Figure 7 depicts the set  $\Omega_K(\tau)$  for four different choices of  $\tau \in [0, 1]$ . Note in passing that  $L_{\varphi_2^{-1}(\tau)} \in \mathcal{A}_K\{2\tau/(1 + \tau)\}$  while  $V_\tau \in \mathcal{A}_K\{\tau/(2 - \tau)\}$ .

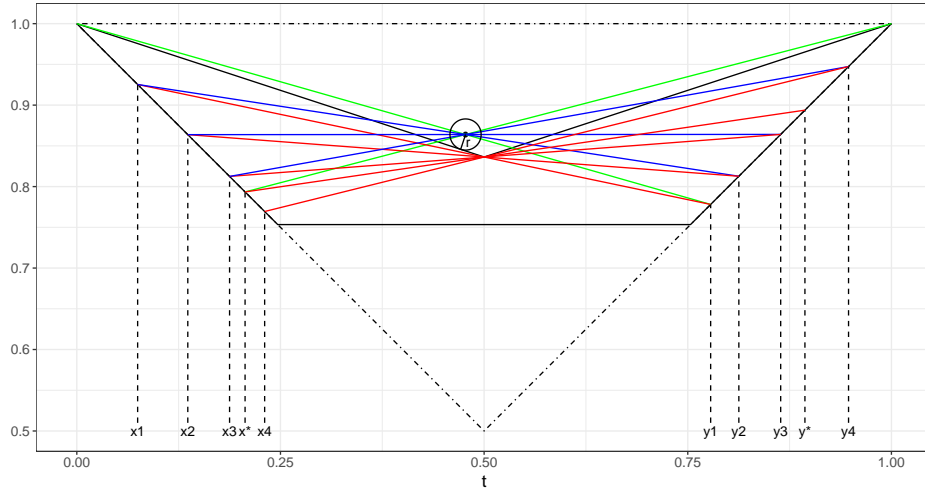


Figure 6: Illustration of the construction used in the proof of Lemma 5.

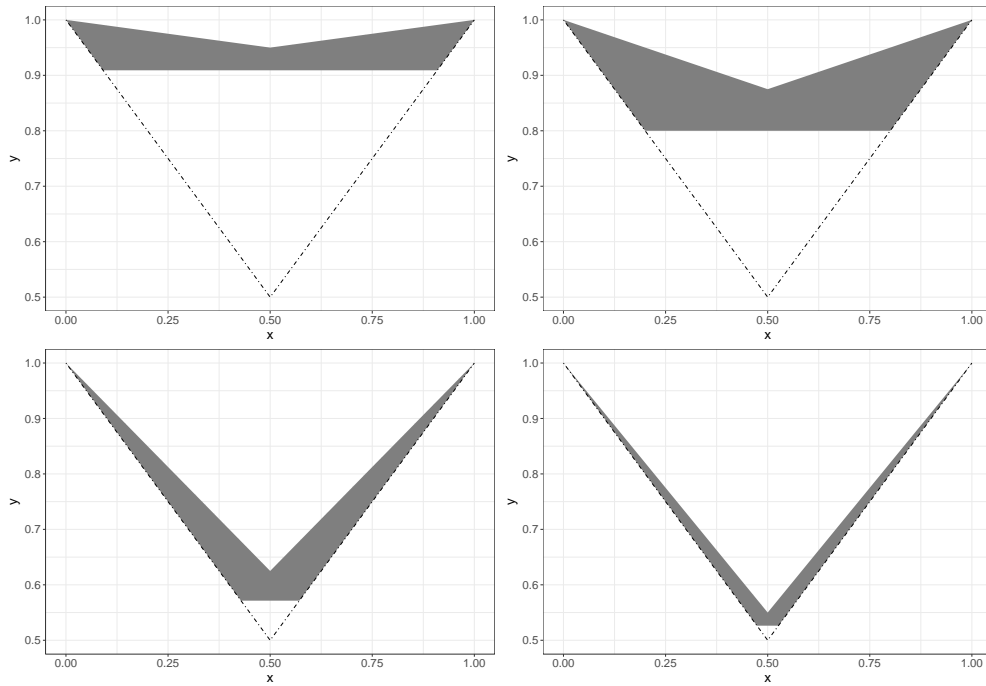


Figure 7: Graphs depicting the shape of  $\Omega_K(\tau)$  when  $\tau = 0.1$  (upper left panel),  $\tau = 0.25$  (upper right panel),  $\tau = 0.75$  (lower left panel), and  $\tau = 0.9$  (lower right panel). In each case, all  $A \in \mathcal{A}_K(\tau)$  lie in the shaded region.

## 5. Dimension of the classes of extreme-value copulas with fixed Spearman's rho or Kendall's tau

Propositions 1 and 2 shed some light on the size of the class of bivariate max-stable (or extreme-value) copulas, i.e., functions  $C : [0, 1]^2 \rightarrow [0, 1]$  of the form (2) for some  $A \in \mathcal{A}$  and given value of Spearman's rho or Kendall's tau.

A natural way to characterize this size is in terms of the supremum norm defined, for arbitrary copulas  $C_1$  and  $C_2$ , by

$$\|C_1 - C_2\|_\infty = \sup\{|C_1(u_1, u_2) - C_2(u_1, u_2)| : u_1, u_2 \in [0, 1]\}.$$

Given that extreme-value copulas are completely determined by their Pickands dependence function, this depends in turn on the supremum norm on  $\mathcal{A}$  defined, for arbitrary  $A_1, A_2 \in \mathcal{A}$ , by  $\|A_1 - A_2\| = \sup\{|A_1(t) - A_2(t)| : t \in [0, 1]\}$ .

**Proposition 3.** *Let  $C_1$  and  $C_2$  be copulas of the form (2) with corresponding Pickands dependence functions  $A_1$  and  $A_2$ , respectively. Then  $\|C_1 - C_2\| \leq 2\gamma/(1 + 2\gamma)^{1+1/(2\gamma)}$ , where  $\gamma = \|A_1 - A_2\|$ .*

*Proof.* Fix  $u_1, u_2 \in (0, 1)$  and set  $t = \ln(u_2)/\ln(u_1 u_2)$ . Assume without loss of generality that  $C_1(u_1, u_2) \leq C_2(u_1, u_2)$ . Then  $A_1(t) \geq A_2(t)$  and hence

$$|C_1(u_1, u_2) - C_2(u_1, u_2)| \leq C_2(u_1, u_2) - C_1(u_1, u_2) = C_2(u_1, u_2) \left\{ 1 - \frac{C_1(u_1, u_2)}{C_2(u_1, u_2)} \right\} = C_2(u_1, u_2) \{1 - (u_1 u_2)^{A_1(t) - A_2(t)}\}, \quad (7)$$

where the fact that  $C_1$  and  $C_2$  are extreme-value copulas was used. Now  $A_1(t) - A_2(t) \leq \gamma$  and hence  $1 - (u_1 u_2)^{A_1(t) - A_2(t)} \leq 1 - (u_1 u_2)^\gamma$ . Furthermore,  $C_2(u_1, u_2) \leq \min(u_1, u_2)$ , which holds true for any copula; see Section 2.5 in [14]. Therefore, the right-hand side of Eq. (7) can be bounded above by

$$\min(u_1, u_2) \{1 - (u_1 u_2)^\gamma\} \leq \min(u_1, u_2) \{1 - \min(u_1, u_2)^{2\gamma}\} \leq 2\gamma/(1 + 2\gamma)^{1+1/(2\gamma)}$$

because the map  $t \mapsto t - t^{2\gamma+1}$  reaches its maximum at  $t = 1/(1 + 2\gamma)^{1/(2\gamma)}$  on  $[0, 1]$ . This completes the argument.  $\square$

Note that in general, one has  $\gamma \leq 1/2$  and that the upper bound is reached when  $A_1(t) = 1$  and  $A_2(t) = \max(t, 1 - t)$  for all  $t \in [0, 1]$ . These choices correspond to  $C_1 = \Pi$  and  $C_2 = M$ , respectively defined, for all  $u_1, u_2 \in [0, 1]$ , by  $\Pi(u_1, u_2) = u_1 u_2$  and  $M(u_1, u_2) = u_1 \wedge u_2$ . In this case, one can easily see that  $\|C_1 - C_2\| = 1/4 = 2\gamma/(1 + 2\gamma)^{1+1/(2\gamma)}$ . In other words, the upper bound given in Proposition 3 is best possible when  $C_1$  is the product copula and  $C_2$  is the so-called upper Fréchet–Hoeffding bound, which are the two limiting cases of max-stable copulas: the first corresponds to independence between the variables  $X_1$  and  $X_2$ ; the other implies a monotonic functional dependence between them.

The following result gives stricter bounds for  $\gamma$  when the copulas are known to have the same value of Spearman's rho or Kendall's tau.

**Proposition 4.** *If  $A_1, A_2 \in \mathcal{A}_K(\tau)$ , then  $\|A_1 - A_2\| \leq B_K(\tau) = \tau(1 - \tau)/(1 + \tau)$ . If  $A_1, A_2 \in \mathcal{A}_S(\rho)$ , then*

$$\|A_1 - A_2\| \leq B_S(\rho) = \begin{cases} 6 \frac{(1 - \rho)\rho}{(3 - \rho)(3 + \rho)} & \text{if } \rho \leq \sqrt{7} - 2, \\ \frac{6}{(3 + \rho)(15 + \rho)} \{2\sqrt{2(3 - \rho)(1 - \rho)} - 3(1 - \rho)\} & \text{otherwise.} \end{cases}$$

*Proof.* Given that, for all  $t \in [0, 1]$ ,

$$V_\tau(t) - L_{\varphi_2^{-1}(\tau)}(t) = \begin{cases} (1 - \tau)t & \text{if } 0 \leq t \leq \tau/(1 + \tau), \\ \tau/(1 + \tau) - \tau t & \text{if } \tau/(1 + \tau) < t \leq 1/2, \\ \tau/(1 + \tau) - \tau(1 - t) & \text{if } 1/2 < t \leq 1/(1 + \tau), \\ (1 - \tau)(1 - t) & \text{if } 1/(1 + \tau) < t \leq 1, \end{cases}$$

a simple calculation shows that the maximum difference is  $\tau(1 - \tau)/(1 + \tau)$ ; it occurs at  $t = \tau/(1 + \tau)$  and  $t = 1/(1 + \tau)$  given that the map  $V_\tau - L_{\varphi_2^{-1}(\tau)}$  is symmetric about  $1/2$ . Turning to the second statement, note that the map  $U_\rho - L_{\varphi_1^{-1}(\rho)}$  is also symmetric about  $1/2$ . Focusing on the difference over the interval  $[0, 1/2]$ , one must then distinguish between the cases where  $(3 - \rho)/(6 + \rho)$  is smaller or larger than  $2\rho/(3 + \rho)$ . Values of  $\rho$  that are smaller or larger than  $\rho_0 = \sqrt{7} - 2$  must then be discussed separately. When  $\rho \leq \rho_0$ , one has  $2\rho/(3 + \rho) \leq (3 - \rho)/(6 + \rho) \leq 1/2$  and the difference  $U_\rho(t) - L_{\varphi_1^{-1}(\rho)}(t)$  is largest at  $t = 2\rho/(3 + \rho)$ . This difference is the reported value. When  $\rho > \rho_0$ , the largest difference also occurs at  $t = 2\rho/(3 + \rho)$ . The value of this difference is what is reported above.  $\square$

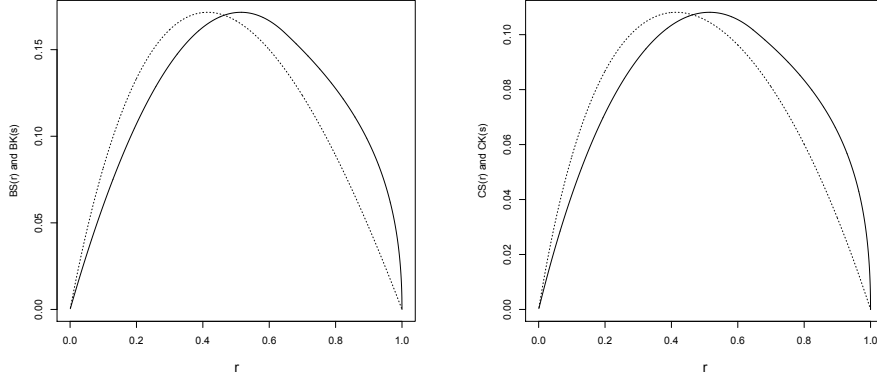


Figure 8: Left: Graphs of the upper bounds  $B_S$  (solid) and  $B_K$  (dashed) for  $\|A_1 - A_2\|$  when  $A_1, A_2$  are arbitrary Pickands dependence functions with the same value of Spearman's rho or Kendall's tau. Right: Graphs of the corresponding bounds for  $\|C_1 - C_2\|$  based on Proposition 3.

The upper bounds  $B_K$  and  $B_S$  are plotted in the left panel of Figure 8 as a function of  $r = \tau$  or  $\rho \in [0, 1]$ . The right panel displays the corresponding bounds on  $\|C_1 - C_2\|$  derived from Proposition 3. The bounds  $B_K$  and  $B_S$  are tight when  $\tau, \rho \in \{0, 1\}$ . Indeed, if the dependence structure in a pair  $(X_1, X_2)$  of continuous random variables is characterized by an extreme-value copula  $C$ , then  $\tau(X_1, X_2)$  or  $\rho(X_1, X_2)$  equal to 0 or 1 can occur only when  $C = \Pi$  or  $M$ , respectively. Furthermore, one can see that the maximum of the curve  $B_K$  occurs at  $\tau = \sqrt{2} - 1 \approx 0.4142$ ; this is in a sense the least informative value of  $\tau$ . Similarly, the least informative value of  $\rho$  is found to be  $9 - 6\sqrt{2} \approx 0.5147$ .

## 6. Discussion

The derivations reported here show that except in limiting cases, the classes  $\mathcal{A}_S(\rho)$  and  $\mathcal{A}_K(\tau)$  of Pickands dependence functions with a given value of Spearman's rho or Kendall's tau are not as small as initially suggested by Figure 1. As already mentioned in the Introduction, the apparent narrowness of these classes was induced in part by the required symmetry of the parametric classes of copulas drawn there. In that sense, Figure 2 is more illustrative of the broader reality. Still, it is not that large.

It often makes sense in statistical practice to assume that the underlying dependence structure is exchangeable. In the bivariate case, exchangeability means that  $C(u_1, u_2) = C(u_2, u_1)$  for all  $u_1, u_2 \in [0, 1]$ , and hence  $A(t) = A(1-t)$  for all  $t \in [0, 1]$ . This assumption can actually be validated from data; see, e.g., [6, 11]. It would be of interest, therefore, to refine the bounds given herein by taking into account this additional constraint.

Also, Spearman's rho and Kendall's tau are not the only copula-based dependence measures used in statistical practice. Other examples include Blomqvist's beta [14] and Blest's coefficient [7]. Particularly relevant in extreme-value analysis is Joe's coefficient of upper-tail tail dependence [10, 16] defined, in the notation of the Introduction, by

$$\lambda_C = \lim_{u \uparrow 1} \Pr\{X_2 > F_2^{-1}(u) | X_1 > F_1^{-1}(u)\} = \lim_{u \uparrow 1} \frac{1 - 2u + C(u, u)}{1 - u}.$$

When  $C$  is a max-stable (or extreme-value) copula, one finds simply  $\lambda = 2 - 2A(1/2) \in [0, 1]$ . If desired, it would be a simply matter to extend the above analysis to the class  $\mathcal{A}_J(\lambda)$  of Pickands dependence functions  $A \in \mathcal{A}$  such that  $\lambda(C_A) = \lambda$ . However, it would be of greater interest still to examine the relation between constraints on  $\rho$ ,  $\tau$ , and  $\lambda$ , in the spirit of [3, 17]. In view of Propositions 1–2, it is obvious that

$$A \in \mathcal{A}_S(\rho) \Rightarrow 2 \frac{6 + 2\rho - 6\sqrt{1-\rho}}{15 + \rho} \leq \lambda \leq \frac{4\rho}{3 + \rho}, \quad A \in \mathcal{A}_K(\tau) \Rightarrow \tau \leq \lambda \leq \frac{2\tau}{1 + \tau}.$$

However, what would be the best possible bounds on  $\rho$  and  $\tau$  when it is known that  $A \in \mathcal{A}_J(\lambda)$ ?

### Appendix A. Complementary calculations for Section 3

Fix  $\rho \in (0, 1)$ . To compute the upper envelope  $U_\rho(t)$  of the class  $(P_{h_\rho(y), y})_{y \in [\varphi_1^{-1}(\rho), 1]}$ , we begin by defining an auxiliary family of functions  $g_{\rho, t} : \mathbb{R} \rightarrow \mathbb{R}$  by letting, for all  $y \in \mathbb{R}$  and  $t \in [0, 1]$ ,

$$g_{\rho, t}(y) = \frac{3t + \rho t + 3y - 3\rho y - 18ty + 2\rho ty + 3y^2 - 3\rho y^2 + 15ty^2 + \rho ty^2}{6 - 2\rho - 15y - \rho y + 15y^2 + \rho y^2}.$$

Note that for all  $y \in [\varphi_1^{-1}(\rho), 1]$  and  $t \in (h_\rho(y), y)$ ,  $g_{\rho, t}(y) = P_{h_\rho(y), y}(t)$ . Hence, for all  $\rho \in (0, 1)$  and  $t \in [0, 1]$ ,

$$g_{\rho, t}(t) = t, \quad g_{\rho, t}(1) = 1 - \frac{2\rho}{3 - \rho}(1 - t), \quad g_{\rho, t}\{\varphi_1^{-1}(\rho)\} = 1 - \frac{2\rho}{3 - \rho}t.$$

Furthermore, the first derivative of  $g_{\rho, t}$  is given, for all  $y \in \mathbb{R}$ , by

$$g'_{\rho, t}(y) = \frac{3 - 3\rho - 18t + 2\rho t + 6y - 6\rho y + 30ty + 2\rho ty}{6 - 2\rho - 15y - \rho y + 15y^2 + \rho y^2} - \frac{(-15 - \rho + 30y + 2\rho y)(3t + \rho t + 3y - 3\rho y - 18ty + 2\rho ty + 3y^2 - 3\rho y^2 + 15ty^2 + \rho ty^2)}{(6 - 2\rho - 15y - \rho y + 15y^2 + \rho y^2)^2}.$$

The only root of the equation  $g'_{\rho, t}(y) = 0$  which is larger than  $1/2$  is

$$y_\rho(t) = \frac{-6 + 2\rho - 15t - \rho t - 6\sqrt{6 - 2\rho - 15t(1 - t) - \rho t(1 - t)}}{-30 - 2\rho + 15t + \rho t}.$$

Moreover

$$g'_{\rho, t}(1) = \frac{12(1 - \rho)}{(15 + \rho)^2} \{(6 + \rho)t - (3 + 2\rho)\}.$$

Finally, denote by  $t_\rho = (3 + 2\rho)/(6 + \rho)$  the value of  $t$  for which  $g'_{\rho, t}(1) = 0$ .

**Lemma A.** For all  $t \in [0, 1]$ ,

$$U_\rho(t) = \begin{cases} g_{\rho, t}\{\varphi_1^{-1}(\rho)\} & \text{if } 0 \leq t \leq 1 - t_\rho, \\ g_{\rho, t}\{y_\rho(t)\} & \text{if } 1 - t_\rho < t < t_\rho, \\ g_{\rho, t}(1) & \text{if } t_\rho \leq t \leq 1. \end{cases}$$

*Proof.* Because the envelope  $U_\rho$  is symmetric, we restrict ourselves to  $t \in [1/2, 1]$ . Then

$$U_\rho(t) = \max\{g_{\rho, t}(y) : y \in [\max\{t, \varphi_1^{-1}(\rho)\}, 1]\}.$$

Observe that for all  $t \in [1/2, 1]$ , one has  $g_{\rho, t}[\max\{t, \varphi_1^{-1}(\rho)\}] \leq g_{\rho, t}(1)$ . Therefore, the maximum of  $g_{\rho, t}$  is either attained at the boundary point  $y = 1$  or at an inner point  $y^*$ . In the second case,  $y^*$  is in fact the only critical point  $y_\rho(t)$  and must be a local maximum point, i.e., the derivative  $g'_{\rho, t}$  is changing sign from  $+$  to  $-$  at  $y_\rho(t)$ .

If  $t > t_\rho$ ,  $g'_{\rho, t}(1)$  is positive and a change of sign from  $+$  to  $-$  at the first critical point left from 1 is not possible. Thus the maximum is attained at 1. Next if  $t = t_\rho$ , then  $g'_{\rho, t}(1) = 0$  and 1 is a critical point,  $y_\rho(t) = 1$ . Finally if  $1/2 \leq t < t_\rho$ ,  $g'_{\rho, t}(1)$  is negative and  $g_{\rho, t}$  is decreasing left of the right boundary point 1; consequently, 1 is not a point where the maximum is attained. Thus the maximum is attained at the inner point.  $\square$

### Appendix B. Proof of Lemma 3

If  $A \geq B$ , then  $C_A \leq C_B$  holds and

$$\tau(C_B) - \tau(C_A) = 4 \left\{ \int_0^1 \int_0^1 (C_B - C_A) dC_B + \int_0^1 \int_0^1 (C_B - C_A) dC_A \right\} \geq 0, \quad (\text{A.1})$$

owing to the fact that

$$\int_0^1 \int_0^1 C_B dC_A = \int_0^1 \int_0^1 C_A dC_B$$

by a simple application of Fubini's Theorem. Following Trutschnig et al. [19], set, for every  $D \in \mathcal{A}$ ,

$$L_D = \max\{x \in [0, 1] : D(x) = 1 - x\}, \quad R_D = \min\{x \in [0, 1] : D(x) = x\}.$$

For each  $t \in (0, 1)$  and  $x \in [0, 1]$ , let  $f_t(x) = x^{1/t-1}$ ; let also  $f_0 = 0$  and  $f_1 = 1$ . Then the support of  $C_D$  can be written as  $\text{supp}(C_D) = \{(x, y) \in [0, 1]^2 : f_{L_D}(x) \leq y \leq f_{R_D}(x)\}$ .

Now suppose that  $A \geq B$ . If  $B$  is the map  $t \mapsto \max(1 - t, t)$ , then  $C_B$  is the upper Fréchet–Hoeffding bound and  $\tau(C_A) < 1 = \tau(C_B)$  is trivial. Therefore, assume that  $B(t) \neq \max(1 - t, t)$  for some  $t \in (0, 1)$ . Obviously  $L_A \leq L_B < 1/2$  as well as  $R_A \geq R_B > 1/2$  and one can find  $t_0 \in (L_B, R_B)$  such that  $A(t_0) > B(t_0)$ . By continuity there exists  $\delta > 0$  such that  $A(t) > B(t)$  holds for every  $t$  in the interval  $[t, \bar{t}] = [t_0 - \delta, t_0 + \delta] \subseteq (L_B, R_B) \subseteq (L_A, R_A)$ . Considering that

$$\{(u, v) \in (0, 1)^2 : f_{\bar{t}}(u) < v < f_t(u)\} \subseteq \{(u, v) \in (0, 1)^2 : C_B(u, v) > C_A(u, v)\} = \{C_B > C_A\}$$

and

$$\{(u, v) \in (0, 1)^2 : f_{\bar{t}}(u) < v < f_t(u)\} \subseteq \text{supp}(C_B) \subseteq \text{supp}(C_A),$$

one can see that the event  $\{C_B > C_A\}$  has non-zero probability with respect to the distributions  $C_A$  and  $C_B$ . Hence

$$\int_0^1 \int_0^1 (C_B - C_A) dC_B > 0 \quad \text{and} \quad \int_0^1 \int_0^1 (C_B - C_A) dC_A > 0,$$

and the conclusion follows from Eq. (A.1). □

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