

On some properties of reflected maxmin copulas

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Abstract

Aiming at a the simplest possible representation of maxmin copulas in terms of two generators Košir and Omladič recently introduced the class of so-called reflected maxmin (RMM) copulas in [10]. All results for RMM copulas can easily be translated to results on maxmin copulas. Building upon the afore-mentioned article we provide alternative simplified (mainly Markov kernel based) proofs for some of their theorems and, more importantly, derive numerous new results. In particular, we show that the family of all RMM copulas is a compact subset of the metric space of all copulas (with the standard uniform metric), characterize absolutely continuous RMM copulas, and show that the limit of absolutely continuous RMM copulas is absolutely continuous too. Finally, we determine Kendall's τ and Spearman's ρ of RMM copulas, prove some first inequalities for τ and ρ and state a conjecture on sharp inequalities. Some examples illustrating the main results round off the paper.

Keywords: Copula, Maxmin copulas, Markov kernel, Kendall's tau, Spearman's rho

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1. Introduction

Let X_1, X_2, Z be independent random variables with continuous distribution functions and consider the random variables $X = \max\{X_1, Z\}$ and $Y = \min\{X_2, Z\}$. Following [3] and interpreting X_1 and X_2 as lifetimes of components and Z as endogenous shock affecting both components, the first one in a positive, and the second one in a negative way, X and Y are the resulting lifetimes of the components. According to [3, 12] the joint distribution of (X, Y) has a so-called *maxmin* copula as underlying copula. Every maxmin copula C can be expressed in terms of functions $\phi, \psi : [0, 1] \rightarrow [0, 1]$ as

$$C(x, y) = \min\{x, \phi(x)y - \phi(x)\psi(y) + x\psi(y)\}, \quad x, y \in [0, 1] \quad (1)$$

whereby the functions ϕ, ψ fulfill the following three properties (again see [3, 12]):

(M1) $\phi(0) = \psi(0) = 0, \phi(1) = \psi(1) = 1.$

(M2) ϕ, ψ are non-decreasing.

(M3) $x \mapsto \frac{\phi(x)}{x}$ and $x \mapsto \frac{1-\psi(x)}{x-\psi(x)}$ are non-increasing on $(0, 1)$. Thereby we set $\frac{1-\psi(x)}{x-\psi(x)} := \infty$ if $\psi(x) = x$.

Vice versa, if functions $\phi, \psi : [0, 1] \rightarrow [0, 1]$ fulfill (M1)-(M3) then C , defined according to equation (1) is a maxmin copula.

Aiming at a simplified representation of maxmin copulas, in [10] Košir and Omladič considered a reflection of maxmin copulas in the second variable and assigned the resulting copulas the name *reflected maxmin copulas* (RMM copulas, for short). As direct consequence of the afore-mentioned characterization RMM copulas correspond to pairs (f, g) of so-called *generators*, i.e. functions satisfying the following three properties:

(R1) $f(0) = 0$ and $f(1) = 0$

(R2) $x \mapsto f(x) + x$ is non-decreasing on $[0, 1]$

(R3) $x \mapsto \frac{f(x)}{x}$ is non-increasing on $(0, 1]$.

In fact, for every RMM copula C there exist generators f and g such that

$$C(x, y) = C_{f,g}(x, y) := \max\{0, xy - f(x)g(y)\} \quad (2)$$

for all $x, y \in [0, 1]$. For further results on RMM copula we refer to [1, 2, 10].

The main objective of the current paper is twofold: On the one hand we provide alternative simplified (mainly Markov kernel based) proofs for some of the theorems contained in [10]. And on the other hand we derive various new results on RMM copulas: In particular we show that the family of all RMM copulas is compact but not convex, we characterize absolutely continuous RMM copulas, and show that the limit of absolutely continuous RMM copulas is absolutely continuous too. Additionally, we derive simple formulas for Kendall's τ and Spearman's ρ of RMM copulas, prove some first inequalities for the latter and then state a conjecture on sharp inequalities between Kendall's τ and Spearman's ρ of RMM copulas. Each of the new results can easily be translated to maxmin copulas.

The rest of the paper is organized as follows: Section 2 gathers preliminaries and notations that will be used throughout the paper. Section 3 contains various results on the mass distributions of RMM copulas and, as main theorem, states and proves compactness of the family of all two-dimensional RMM copulas. After calculating the Markov kernel of RMM copulas at the beginning of Section 4 we derive necessary and sufficient conditions for absolute continuity and prove that the limit of absolutely continuous RMM copulas is absolutely continuous too. The afore-mentioned results on Kendall's τ and Spearman's ρ are gathered in Section 5, in order to facilitate reading of the paper some of the corresponding tedious proofs are moved to the Appendix.

50 **2. Notations and preliminaries**

In the sequel \mathcal{C} will denote the family of all two-dimensional *copulas*, $\mathcal{P}_{\mathcal{C}}$ the family of all *doubly stochastic measures*, i.e. the family of all probability measures on $[0, 1]^2$ whose marginals are uniformly distributed on $[0, 1]$. For every $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C , C^t will denote the *transpose* of C , i.e. the copula defined by $C^t(x, y) = C(y, x)$. Letting d_{∞} denote the uniform metric on \mathcal{C} it is well known that $(\mathcal{C}, d_{\infty})$ is a compact metric space. For further background on copulas we refer to [4] and [11].

In the sequel $\mathcal{B}(\mathbb{R})$ denotes the Borel σ -field in \mathbb{R} , λ the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. If X, Y are real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ then we will write $\mathcal{P}^{X \otimes Y}$ for their joint distribution and \mathcal{P}^X for the distribution of X . $\mathbb{E}(Y|X)$ will denote the *conditional expectation of Y given X* . Since by definition $\mathbb{E}(Y|X)$ is $\mathcal{A}_{\sigma}(x) := X^{-1}(\mathcal{B}(\mathbb{R}))$ -measurable there exists a measurable function $g : \mathbb{R} \mapsto \mathbb{R}$ such that $\mathbb{E}(Y|X) = g \circ X$ holds \mathcal{P} -almost surely. A *Markov kernel* from \mathbb{R} to $\mathcal{B}(\mathbb{R})$ is a mapping $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ such that $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}(\mathbb{R})$ and $B \mapsto K(x, B)$ is a probability measure for every fixed $x \in \mathbb{R}$. A Markov kernel $K : \mathbb{R} \times \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ is called a *regular conditional distribution of Y given X* if for every $B \in \mathcal{B}(\mathbb{R})$

$$K(X(\omega), B) = \mathbb{E}(\mathbf{1}_B \circ Y|X)(\omega) \tag{3}$$

holds \mathcal{P} -a.e. It is well known that for each pair (X, Y) of real-valued random variables a regular conditional distribution $K(\cdot, \cdot)$ of Y given X exists, that $K(\cdot, \cdot)$ is unique \mathcal{P}^X -a.s. (i.e. unique for \mathcal{P}^X -almost all $x \in \mathbb{R}$) and that $K(\cdot, \cdot)$ only depends on $\mathcal{P}^{(X, Y)}$. Hence, given $C \in \mathcal{C}$ we will denote (a version of) the regular conditional distribution of Y given X by $K_C(\cdot, \cdot)$, directly view it as Markov kernel from $[0, 1]$ to $\mathcal{B}([0, 1])$, and refer to $K_C(\cdot, \cdot)$ simply as *regular conditional distribution of C* or as *Markov kernel of C* . Note that for every $C \in \mathcal{C}$, its regular conditional distribution $K_C(\cdot, \cdot)$, and every Borel set $G \in \mathcal{B}([0, 1]^2)$ we have the following *disintegration* (here $G_x := \{y \in [0, 1] : (x, y) \in G\}$ denotes

the x -section of G for every $x \in [0, 1]$)

$$\int_{[0,1]} K_C(x, G_x) d\lambda(x) = \mu_C(G), \quad (4)$$

so in particular

$$\int_{[0,1]} K_C(x, F) d\lambda(x) = \lambda(F) \quad (5)$$

80 for every $F \in \mathcal{B}([0, 1])$. On the other hand, every Markov kernel $K : [0, 1] \times \mathcal{B}([0, 1]) \rightarrow [0, 1]$ fulfilling (5) induces a unique element $\mu \in \mathcal{P}_C([0, 1]^2)$ via (4). For every $C \in \mathcal{C}$ and $x \in [0, 1]$ the function $y \mapsto F_x^C(y) := K_C(x, [0, y])$ will be called *conditional distribution function of C at x* . For more details and properties of conditional expectation, regular conditional distributions, and disintegration see [8] and [9]. For examples underlining the usefulness of Markov
85 kernels in the copula setting we refer, for instance, to [5, 6, 16, 18] and at this point only remark that the two main advantages of working with the Markov kernel over working with partial derivatives are that, firstly, $K_A(x, [0, y])$ can be interpreted as the conditional probability $\mathbb{P}(Y \in [0, y] | X = x)$ and, secondly,
90 that one does not have to worry about measurability issues.

For every pair of generators f, g (i.e. for every pair f, g of functions fulfilling (R1)-(R3) in the Introduction) we will refer to

$$C_{f,g}(x, y) = \max\{0, xy - f(x)g(y)\}. \quad (6)$$

as *RMM copula induced by (f, g)* . In the sequel we will let \mathcal{C}_{RMM} the family of all RMM copulas. Notice that the product copula $\Pi(x, y) = xy$ is an RMM copula
95 for $f = g = 0$ and the Fréchet Hoeffding lower bound $W(x, y) = \max\{0, x+y-1\}$ is an RMM copula for $f(t) = g(t) = (1-t)$ for every $t \in (0, 1]$. As direct consequence of equation (6) we obviously have $W \leq C_{f,g} \leq \Pi$ for every RMM copula $C_{f,g}$. It is straightforward to verify (and already stated in [10]) that different pairs $(f_1, g_1), (f_2, g_2)$ of generators may induce the same RMM copula,
100 we will therefore avoid expressions like ‘the generators corresponding to the RMM copula C ’ etc. Figure 1 depicts a scatterplot of a sample of an RMM copula $C_{f,g}$ (fulfilling $f = g$) as well as the marginal histograms of the sample.

Property (R2) of generators yields that $f(x) + x \leq f(1) + 1 = 1$ holds for all $x \in [0, 1]$, implying, firstly, that $f(x) \leq 1 - x$ for every $x \in [0, 1]$, secondly
105 that the function f is differentiable λ -almost everywhere on $[0, 1]$, and, thirdly, for every point x at which f is differentiable we have $f'(x) \geq -1$. Moreover, property (R3) allows for a simple geometric interpretation: the angle between the vectors $(x, f(x))$ and $(x, 0)$ is non-increasing in x .

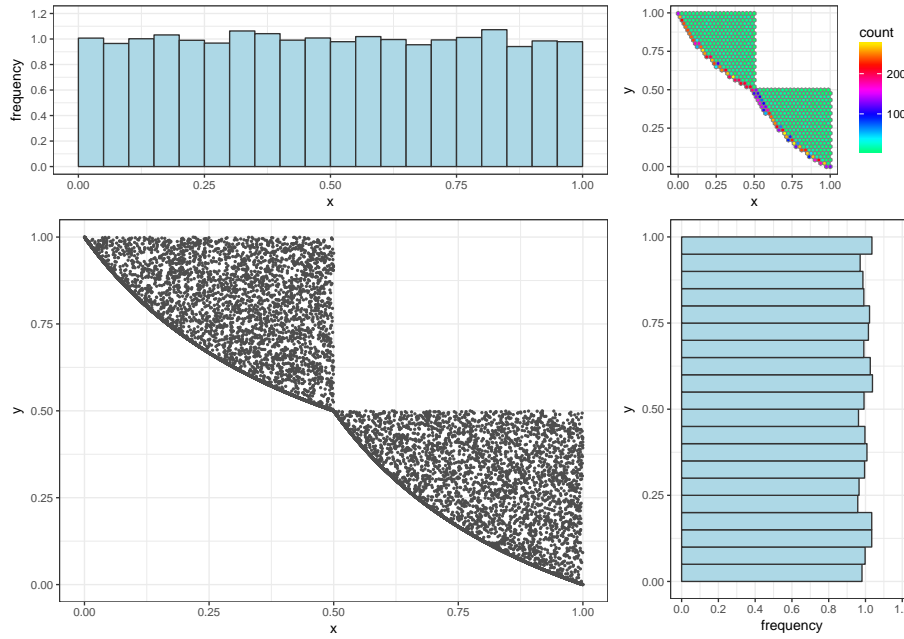


Figure 1: Sample of size 20.000 of $C_{f,g}$ with $f(x) = g(x) = \min\{\frac{1}{2}, 1 - x\}$ for all $x \in (0, 1]$

3. Compactness of $(\mathcal{C}_{RMM}, d_\infty)$ and some properties of the function b carrying the singular mass

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It has already been shown in Example 8(a) of [10] that for every $s \in [0, 1]$ there exists an RMM copula C such that the mass $\mu_C^{sing}([0, 1]^2)$ of the singular component (of the Lebesgue decomposition of μ_C with respect to the two-dimensional Lebesgue measure λ_2) fulfills $\mu_C^{sing}([0, 1]^2) = s$. In this section
115 we explicitly derive the function b carrying all singular mass of $C_{f,g}$ (if any)

and derive some simple properties of b that will prove handy in the next two sections. Before doing so we show that the family of all generators is equicontinuous on every interval of the form $[\delta, 1]$ with $\delta \in (0, 1)$ - this result will be essential for proving compactness of $(\mathcal{C}_{RMM}, d_\infty)$ at the end of this section. A
120 similar property was shown in Lemma 10 of [10] in a different way.

Lemma 3.1. *Suppose that $\delta \in (0, 1)$ and set $M = \max\{\frac{1-\delta}{\delta}, 1\}$. Then on the interval $[\delta, 1]$ every generator is Lipschitz continuous with Lipschitz constant M .*

Proof. Suppose that $\delta \in (0, 1)$ and consider $\delta \leq u \leq v \leq 1$. Then property (R3) implies

$$f(v) \leq \frac{v}{u}f(u) = f(u) + \frac{v-u}{u}f(u) \leq f(u) + (v-u)\frac{f(\delta)}{\delta}$$

and using (R2) we get

$$-M(v-u) \leq (-1)(v-u) \leq f(v) - f(u) \leq M(v-u),$$

from which the assertion follows immediately. \square

Suppose that (f, g) is a pair of generators. Using (R1)-(R3) it is straightforward to see that for every $x \in [0, 1]$ the set

$$\{y \in [0, 1] : xy - f(x)g(y) \leq 0\}$$

is compact and non-empty. Setting

$$b_{f,g}(x) = \max\{y \in [0, 1] : xy - f(x)g(y) \leq 0\} \tag{7}$$

125 therefore defines a function $b_{f,g}$ from $[0, 1]$ to $[0, 1]$ (also compare with the ‘zero-set’ and the ‘stand’ mentioned in [10]). In the sequel we will simply write b instead of $b_{f,g}$ whenever no confusion can arise. Obviously the function b has the following properties: $C_{f,g}(x, b(x)) = 0$ for all $x \in [0, 1]$; for every $x \in (0, 1]$ we have $C_{f,g}(x, y) > 0$ for every $y > b(x)$ as well as $xy - f(x)g(y) \leq 0$ for every
130 $y \leq b(x)$.

The subsequent lemmata gather some properties of the function b .

Lemma 3.2. *b is non-increasing, left continuous, and fulfills $b(0) = 1, b(1) = 0$. Furthermore b fulfills $b(x) \leq 1 - x$ for every $x \in [0, 1]$ and we have $b(x) = 1 - x$ for all $x \in [0, 1]$ if, and only if, $C_{f,g} = W$.*

Proof. The first assertion is a direct consequence of the fact that if $0 < x_1 < x_2 \leq 1$ and $b(x_1) < b(x_2)$ holds, then, using property (R3), we would get

$$b(x_2) > \frac{f(x_1)}{x_1}g(b(x_2)) \geq \frac{f(x_2)}{x_2}g(b(x_2)),$$

i.e. $x_2b(x_2) - f(x_2)g(b(x_2)) > 0$, which is a contradiction. Hence b is non-increasing. To show left continuity suppose that $x \in (0, 1]$ and that $(x_n)_{n \in \mathbb{N}}$ is an increasing sequence in $[0, x)$ converging to x . Being monotonic and bounded, the sequence $(b(x_n))_{n \in \mathbb{N}}$ converges to some y' fulfilling $y' \in [b(x), b(x_n)]$ for every $n \in \mathbb{N}$. Continuity of f at the point x implies

$$xy' - f(x)g(y') = \lim_{n \rightarrow \infty} (x_n y' - f(x_n)g(y')) \leq 0,$$

135 hence $y' \leq b(x)$, from which we altogether get $y' = b(x)$. Since $b(0) = 1, b(1) = 0$ are immediate consequences of the definition of b it remains to show the assertions concerning the interrelation between b and the function $1 - id_{[0,1]}$, which can be done as follows: If we had $b(x) > 1 - x$ for some $x \in (0, 1)$ then we would get $0 = C_{f,g}(x, b(x)) \geq W(x, b(x)) > 0$. On the other hand, if
140 $b(x) = 1 - x$ holds for every $x \in [0, 1]$, then we have $C_{f,g}(x, 1 - x) = 0$ for all $x \in [0, 1]$ from which, using Lipschitz continuity of copulas in each coordinate (see [4, 11]) $C_{f,g} = W$ follows immediately. Since the converse is trivial this completes the proof. \square

As the next step we characterize flats and jumps of the function b in terms
145 of (f, g) . Despite the simplicity of the proofs of the subsequent three results the observations will turn out very useful in the sequel.

Lemma 3.3. *Suppose that $[\alpha, \beta] \subset (0, 1]$ and that $b(\alpha) > 0$. Then the following two conditions are equivalent:*

1. b is constant on $[\alpha, \beta]$,

- 150 2. the map $x \mapsto \frac{f(x)}{x}$ is constant on $[\alpha, \beta]$ and there exists some $y \in (0, 1)$
fulfilling $\frac{y}{g(y)} = \frac{f(x)}{x}$.

Proof. Assume that $b(x) = c$ for all $x \in [\alpha, \beta]$ and for some $c \in (0, 1)$. Since in case of $g(c) = 0$ we would get $xc = f(x)g(c) = 0$, whence $x = 0 \notin [\alpha, \beta]$, we may assume that $g(c) > 0$. In this case we have $\frac{f(x)}{x} = \frac{c}{g(c)}$ for all $x \in [\alpha, \beta]$,
155 i.e. the second assertion of the lemma holds.

Conversely, assume that the second assertion holds and that x_1, x_2 fulfill $\alpha \leq x_1 < x_2 \leq \beta$. For $y \in [0, 1]$ we then have $y \leq \frac{f(x_1)}{x_1}g(y)$ if and only if $y \leq \frac{f(x_2)}{x_2}g(y)$, so $b(x_1) = b(x_2)$ follows, which completes the proof. \square

Lemma 3.4. *Suppose that $x \in (0, 1)$ fulfills $b(x) > 0$ and that $\epsilon \in (0, 1 - x]$
160 holds. Then $b(x) > b(x + \epsilon)$ if and only if $\frac{f(x)}{x} > \frac{f(x+\epsilon)}{x+\epsilon}$.*

Proof. If $\frac{f(x)}{x} = \frac{f(x+\epsilon)}{x+\epsilon}$ holds, then $g(b(x)) > 0$ follows and we get

$$\frac{f(x + \epsilon)}{x + \epsilon} = \frac{f(x)}{x} = \frac{b(x)}{g(b(x))},$$

which in turn implies that $b(x + \epsilon) \geq b(x)$. Starting with $\frac{f(x)}{x} > \frac{f(x+\epsilon)}{x+\epsilon}$ and proceeding analogously shows the other implications. \square

Theorem 3.5. *Suppose that $x \in (0, 1)$. Then the following two conditions are equivalent:*

- 165 1. x is a discontinuity point of b with jump height $h = b(x) - b(x+) > 0$.
2. $\frac{f(x+\epsilon)}{x+\epsilon} < \frac{f(x)}{x}$ for every $\epsilon \in (0, 1-x)$ and $E := \{y \in (0, 1] : xy - f(x)g(y) = 0\}$ is a (closed or left-open and right-closed) interval with $\lambda(E) = h > 0$.

Proof. Assume that x is a discontinuity point of b with jump height $h = b(x) - b(x+) > 0$. Then $b(x) > 0$ and $b(x) > b(x + \epsilon)$ for every $\epsilon \in (0, 1 - x)$, so according to Lemma 3.4 we have $\frac{f(x+\epsilon)}{x+\epsilon} < \frac{f(x)}{x}$ for every $\epsilon \in (0, 1 - x)$. We will show that

$$(b(x) - h, b(x)) \subseteq E \subseteq [b(x) - h, b(x)]$$

holds and proceed as follows. If $y \in (b(x) - h, b(x))$ then $C_{f,g}(x + \frac{1}{n}, y) > 0$ for all $n \in \mathbb{N}$ fulfilling $x + \frac{1}{n} \leq 1$, which yields

$$0 \leq \lim_{n \rightarrow \infty} ((x + \frac{1}{n})y - f(x + \frac{1}{n})g(y)) = xy - f(x)g(y) \leq 0.$$

It follows that $xy - f(x)g(y) = 0$ and therefore $y \in E$. Since $y \in (b(x) - h, b(x)]$ was arbitrary this proves the first inclusion. Now suppose that $y \in E$. The very
170 construction of b implies $y \leq b(x)$, applying Lemma 3.4 yields $\frac{f(x+\frac{1}{n})}{x+\frac{1}{n}} < \frac{f(x)}{x} = \frac{y}{g(y)}$ and $b(x + \frac{1}{n}) < y$ for all $n \in \mathbb{N}$ fulfilling $x + \frac{1}{n} \leq 1$. As direct consequences we get $b(x) - h = \lim_{n \rightarrow \infty} b(x + \frac{1}{n}) \leq y$, so $y \in [b(x) - h, b(x)]$, which proves the second inclusion. Altogether we have shown $\lambda(E) = h = b(x) - b(x+)$.

Conversely, assume now that the second assertion of the theorem holds. Then E is either of the form $(b(x) - h, b(x)]$ or $[b(x) - h, b(x)]$. For every $y \in E$ and every $n \in \mathbb{N}$ with $x + \frac{1}{n} \leq 1$ we have

$$\frac{f(x + \frac{1}{n})}{x + \frac{1}{n}} < \frac{f(x)}{x} = \frac{y}{g(y)},$$

from which $b(x+) \leq y \leq b(x)$ follows, i.e. b has a discontinuity at x with
175 jump height of at least h . If the jump height would be greater than h then the already proved implication would yield that $\lambda(E) > h$, so the desired equality $h = b(x) - b(x+)$ follows. \square

As shown in Example 3.8, Lemma 3.3 and Theorem 3.5 open the door for the construction of RMM copulas for which the function b has infinitely many
180 jumps and flats. We start with some simple, more regular examples, and then turn to the more irregular setting.

Example 3.6. Let f denote the piecewise linear generator fulfilling $f(\frac{1}{8}) = f(\frac{1}{4}) = \frac{1}{4}$, $f(\frac{3}{8}) = \frac{3}{8}$, $f(\frac{1}{2}) = \frac{1}{4}$, and $f(\frac{2}{3}) = \frac{1}{3}$ as depicted in the left panel of Figure 2. According to Theorem 3.3, b is constant on $(0, \frac{1}{8}]$, $[\frac{1}{4}, \frac{3}{8}]$, $[\frac{1}{2}, \frac{2}{3}]$ and
185 $0, \frac{1}{8}, \frac{3}{8}, \frac{2}{3}$ are discontinuities of b . Figure 3 depicts a sample of the corresponding RMM copula $C_{f,f}$ as well as the two marginal histograms and a two-dimensional histogram of the sample in terms of a heatmap with hexagonal bins.

Example 3.7. Let $f(t) = 1 - t$ for every $t \in (0, 1]$ and let g denote the piecewise linear generator fulfilling $g(0) = g(\frac{1}{3}) = \frac{1}{3}$, $g(\frac{1}{6}) = \frac{1}{6}$, $g(\frac{7}{12}) = \frac{1}{12}$, $g(\frac{7}{8}) = \frac{1}{8}$ as
190 depicted in Figure 4. Then b is strictly decreasing and $\frac{1}{8}$ and $\frac{1}{2}$ are discontinuity points of b . Figure 5 depicts a sample of the corresponding RMM copula $C_{f,g}$.

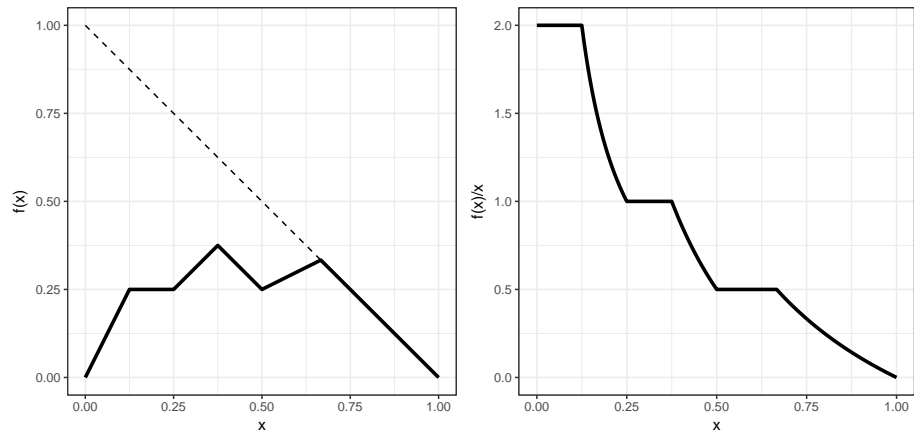


Figure 2: Piecewise linear generator f from Example 3.6 (left panel) and the corresponding function $f(x)/x$ on $(0, 1]$ (right panel)

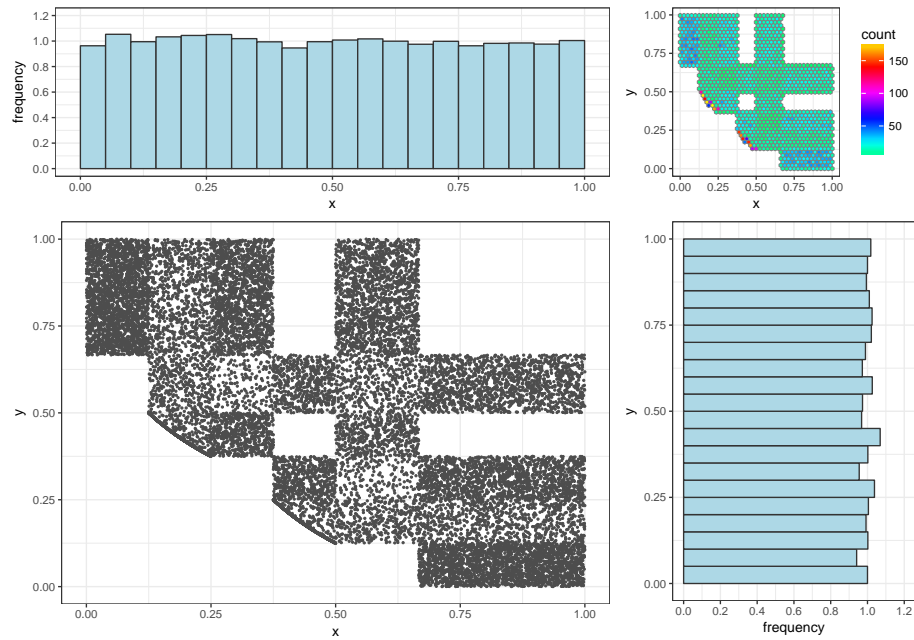


Figure 3: Sample of size 20.000 from the RMM copula $C_{f,f}$ with f being the piecewise linear generator depicted in Figure 2

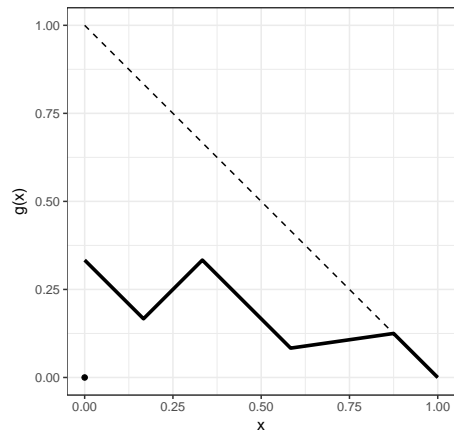


Figure 4: Piecewise linear generator g from Example 3.7

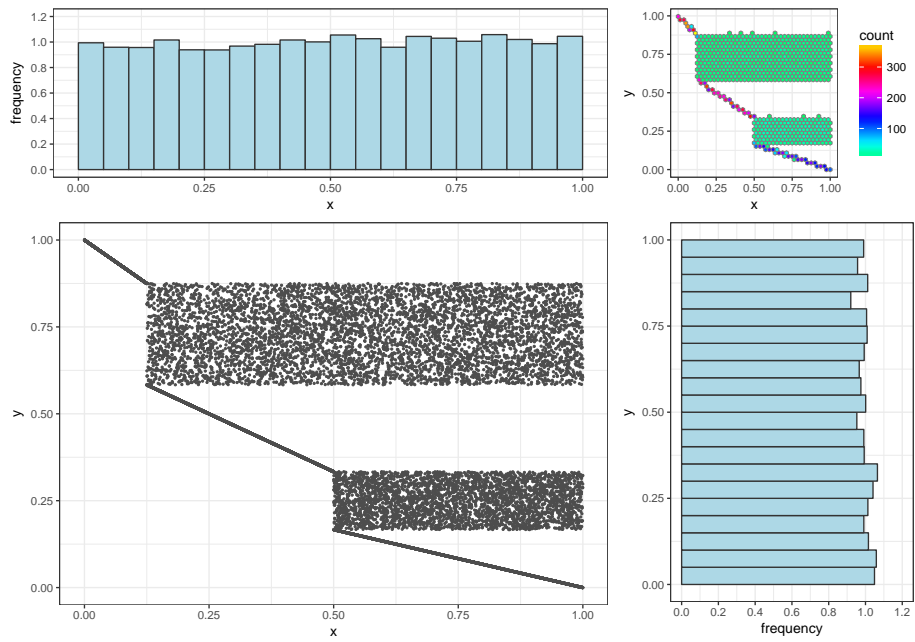


Figure 5: Sample of size 20.000 from the RMM copula $C_{f,g}$ with $f(t) = 1 - t$ and g being the piecewise linear generator depicted in Figure 4

The next example shows the afore-mentioned existence of an RMM copula $C_{f,f}$ whose corresponding function b has infinitely many discontinuities and infinitely many flats.

Example 3.8. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence in $(0, 1]$ with limit $x = 0$ and $x_1 = 1$. Set $y_1 = 0$ and define $(y_n)_{n \geq 2}$ inductively by

$$y_i = \begin{cases} x_{i-1} + y_{i-1} - x_i & \text{if } i \text{ is even,} \\ \frac{y_{i-1}}{x_{i-1}} x_i & \text{if } i \text{ is odd} \end{cases}$$

and let f denote the unique function fulfilling $f(0) = 0$ and $f(x_i) = y_i$ for all $i \in \mathbb{N}$, being continuous on $(0, 1]$ and linear on each of the segments $[x_{i+1}, x_i]$. It is straightforward to verify that f is a generator, Figure 6 depicts the analogous construction for finite sequences $1 = x_1 > x_2 > \dots > x_n > 0$ for $n = 20, 100, 500, 1000$. As direct application of Lemma 3.3 and Theorem 3.5 the resulting symmetric RMM copula $C_{f,f}(x, y) = \max\{0, xy - f(x)f(y)\}$ has the property that the corresponding function b has infinitely many flats (b is constant on all intervals of the form $[x_{2i+1}, x_{2i}]$) and infinitely many discontinuities (each point x_{2i} for $i \in \mathbb{N}$ is a discontinuity point).

We conclude this section with proving that the family \mathcal{C}_{RMM} of all RMM copulas is compact and proceed in two steps formulated as separate lemmata. Notice that according to Lemma 3.1 the family of all generators is equi-continuous on $[\delta, 1]$ for every $\delta \in (0, 1)$. Since this does, however, not imply equicontinuity on the full interval $(0, 1]$, an additional idea of proof complementing Arzelà-Ascoli theorem (see [14]) is needed.

Lemma 3.9. *Suppose that $(C_{f_n, g_n})_{n \in \mathbb{N}}$ is a sequence of RMM copulas that converges to a copula $C \in \mathcal{C}$. If $C(x, y) > 0$ for all $(x, y) \in (0, 1)^2$ then there exist generators f and g such that $C = C_{f,g}$.*

Proof. Since the case $C = \Pi$ is trivial we may assume that $C \neq \Pi$. Then there exists a point $(x_0, y_0) \in (0, 1)^2$ with

$$\Pi(x_0, y_0) > C(x_0, y_0) = \lim_{n \rightarrow \infty} C_{f_n, g_n}(x_0, y_0) = x_0 y_0 - \lim_{n \rightarrow \infty} f_n(x_0) g_n(y_0).$$

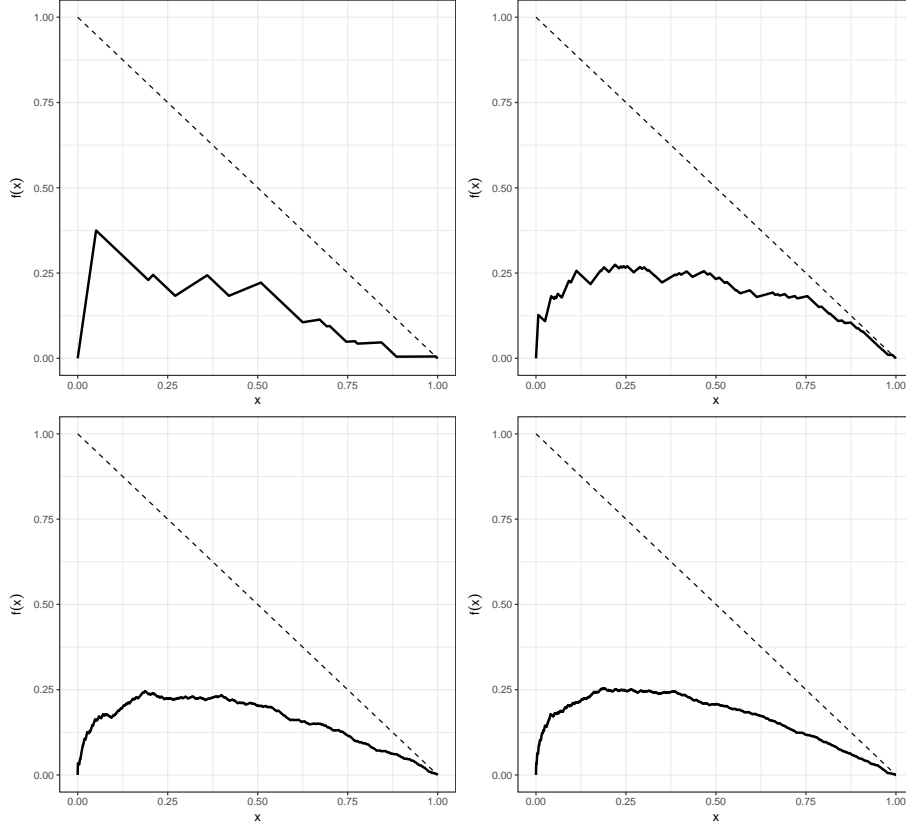


Figure 6: Examples of function f with $n = 20, 100, 500, 1000$

Using Lemma 3.1 and applying Arzelà-Ascoli theorem (see [14]) yields the existence of a strictly increasing sequence $(n_j)_{j \in \mathbb{N}}$ such that $(f_{n_j})_{j \in \mathbb{N}}$ and $(g_{n_j})_{j \in \mathbb{N}}$ converge uniformly on $[\min(x_0, y_0), 1]$ to functions \tilde{f} and \tilde{g} respectively.

Since $\tilde{g}(y_0) = 0$ would imply $C(x_0, y_0) = x_0 y_0 = \Pi(x_0, y_0)$ it follows that $\tilde{g}(y_0) > 0$. Define a function f by

$$f(x) = \frac{1}{\tilde{g}(y_0)} \lim_{n \rightarrow \infty} f_n(x) g_n(y_0)$$

for all $x \in [0, 1]$. Considering $\tilde{g}(y_0) = \lim_{j \rightarrow \infty} g_{n_j}(y_0)$ we get

$$f(x) = \lim_{j \rightarrow \infty} f_{n_j}(x) \frac{g_{n_j}(y_0)}{\tilde{g}(y_0)} = \lim_{j \rightarrow \infty} f_{n_j}(x)$$

for every $x \in [0, 1]$. As direct consequence, f is the pointwise limit of generators

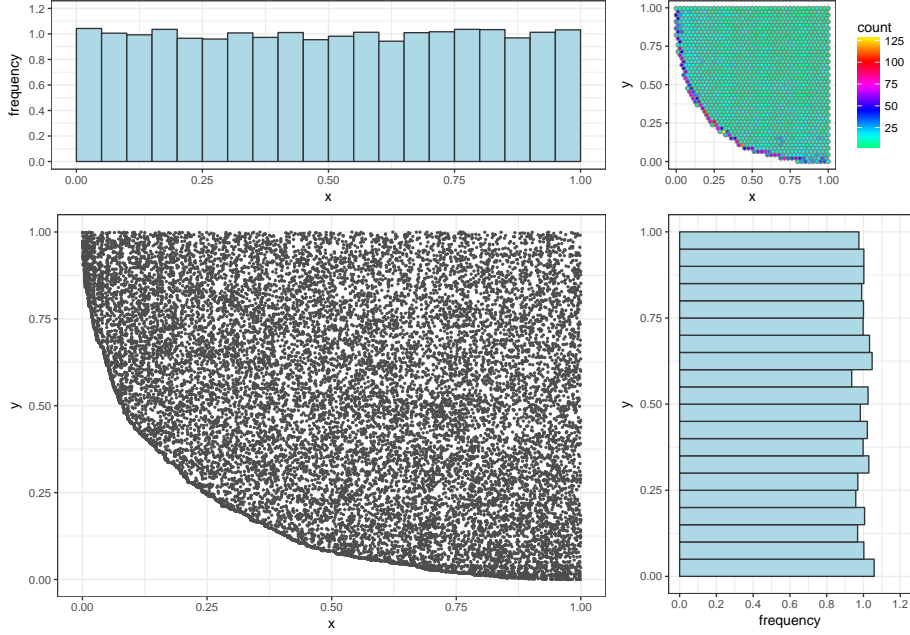


Figure 7: Sample of size 20.000 from the RMM copula with generator f from the right, lower panel in Figure 6

and as such a generator too. Proceeding analogously shows the existence of a generator g with $\lim_{j \rightarrow \infty} g_{n_j}(x) = g(x)$ for all $x \in [0, 1]$. Since for all $x, y \in (0, 1]$ by assumption we have $C(x, y) > 0$ the desired assertion now follows from

$$\begin{aligned} 0 < C(x, y) &= \lim_{j \rightarrow \infty} C_{f_{n_j}, g_{n_j}}(x, y) = xy - \lim_{j \rightarrow \infty} f_{n_j}(x)g_{n_j}(y) \\ &= xy - f(x)g(y) = C_{f, g}(x, y) \end{aligned}$$

□

Lemma 3.10. *Let $(C_{f_n, g_n})_{n \in \mathbb{N}}$ be a sequence of RMM copulas that converges to a copula $C \in \mathcal{C}$. If $C(x, y) = 0$ holds for some $(x, y) \in (0, 1)^2$ then there exist generators f and g such that $C = C_{f, g}$.*

220 *Proof.* Define a non-increasing function $b_0 : [0, 1] \rightarrow [0, 1]$ by $b_0(x) = \max\{y \in [0, 1] : C(x, y) = 0\}$. By assumption there exists some $x \in (0, 1)$ with $b_0(x) > 0$.

We set $\delta = \min\{x, b_0(x)\} > 0$ and proceed as follows: Arzelà-Ascoli theorem implies the existence of subsequences $(f_{n_j})_{j \in \mathbb{N}}, (g_{n_j})_{j \in \mathbb{N}}$ and functions $\tilde{f}, \tilde{g} : [\delta, 1] \rightarrow [0, 1]$ such that $(f_{n_j})_{j \in \mathbb{N}}$ and $(g_{n_j})_{j \in \mathbb{N}}$ converge to \tilde{f} and \tilde{g} uniformly on $[\delta, 1]$.

For $t \in (0, x]$ and $y > b_0(t)$ we have

$$0 < C(t, y) = \lim_{j \rightarrow \infty} C_{f_{n_j}, g_{n_j}}(t, y) = ty - \lim_{j \rightarrow \infty} f_{n_j}(t)g_{n_j}(y)$$

Suppose that $\tilde{g}(b_0(t)) = 0$. Then we get $\tilde{g}(y) = 0$ and $C(t, y) = ty$ for every $y > b_0(t)$, which, using (Lipschitz-) continuity of C , implies

$$0 = C(t, b_0(t)) = tb_0(t) > 0.$$

Assume therefore that $\tilde{g}(b_0(t)) > 0$. Proceeding analogously to the proof of Lemma 3.9 and defining $f : [0, 1] \rightarrow [0, 1]$ by

$$f(t) = \frac{1}{\tilde{g}(b_0(t))} \lim_{n \rightarrow \infty} f_n(t)g_n(b_0(t))$$

it follows that $\lim_{j \rightarrow \infty} f_{n_j}(t) = f(t)$ holds for all $t \in [0, 1]$, so f is a generator too. Finally, considering $C_{g_{n_j}, f_{n_j}} = C_{f_{n_j}, g_{n_j}}^t \rightarrow C^t$ we can repeat the previous steps to prove the existence of a generator g such that $\lim_{k \rightarrow \infty} g_{n_{j_k}}(x) = g(x)$ for some subsubsequence $(n_{j_k})_{k \in \mathbb{N}}$. Altogether we get $C(x, y) = \lim_{k \rightarrow \infty} C_{f_{n_{j_k}}, g_{n_{j_k}}}(x, y) = C_{f, g}(x, y)$ for all $x, y \in [0, 1]$, which completes the proof. \square

Summing up we have proved the following result.

Theorem 3.11. *The class \mathcal{C}_{RMM} of all RMM copulas is a compact subset of (\mathcal{C}, d_∞) .*

As illustrated by the subsequent example, uniform convergence of a sequence $(C_{f_n, g_n})_{n \in \mathbb{N}}$ of RMM copulas to an RMM copula $C_{f, g}$ does not imply convergence of the corresponding sequence $(b_n)_{n \in \mathbb{N}}$ to the function b .

Example 3.12. For every $n \in \mathbb{N}$ define the generator f_n by

$$f_n(x) = (1 - \frac{1}{n}) \min\{x, 1 - x\}$$

and set $f(x) = \min\{x, 1 - x\}$ for every $x \in [0, 1]$. Then $C_{f_n, f_n}(x, y)$ converges to $C_{f, f}(x, y)$, but $b_n(x)$ does not converge to $b(x)$ for any $x \in (0, \frac{1}{2}]$. In fact, we have $b_n(x) = 0$ for every $n \in \mathbb{N}$ whereas b is given by $b(x) = \frac{1}{2}\mathbf{1}_{(0, \frac{1}{2}]}(x)$ for all $x \in (0, 1]$.

4. Markov kernel and mass distributions of RMM copulas

For arbitrary generators f, g and $x, y \in [0, 1]$ let $K_{f, g}(x, [0, y])$ be defined by

$$K_{f, g}(x, [0, y]) = \begin{cases} 1 & \text{if } x \in D_f^c, \\ 0 & \text{if } x \in D_f, y < b(x), \\ y - f'(x)g(y) & \text{if } x \in D_f, y \geq b(x), \end{cases} \quad (8)$$

where $D_f = \{t \in [0, 1] : f'(t) \text{ exists}\}$. The following theorem holds:

Theorem 4.1. *Suppose that f, g are generators and let $K_{f, g}(\cdot, \cdot)$ be defined according to equation (8). Then $K_{f, g}(\cdot, \cdot)$ is a Markov kernel of the RMM copula $C_{f, g}$.*

Proof. Obviously $y \mapsto K_{f, g}(x, [0, y])$ is a distribution function for $x \in D_f^c$. To show right-continuity of $y \mapsto K_{f, g}(x, [0, y])$ for $x \in D_f$ we proceed as follows: In case of $b(x) > 0$ considering the fact that g is continuous on $(0, 1]$ shows right-continuity (the only possible point at which $y \mapsto K_{f, g}(x, [0, y])$ is not left-continuous is $b(x)$). For $b(x) = 1$ (again by continuity of g on $(0, 1]$) it suffices to show right-continuity at $y = 0$: If $f(x) = 0$ then we get $f(z) = 0$ for every $z \in [x, 1]$, so $f'(x) = 0$ and $y \mapsto K_{f, g}(x, [0, y]) = y$ is obviously right-continuous. If $f(x) > 0$ using $0 < C_{f, g}(x, y) = xy - f(x)g(y)$ for every $y \in (0, 1]$ we get $\limsup_{y \rightarrow 0^+} \frac{f(x)}{x}g(y) \leq \limsup_{y \rightarrow 0^+} y = 0$, so $y \mapsto K_{f, g}(x, [0, y])$ is right-continuous at $y = 0$. Since $K_{f, g}(x, [0, 1]) = 1$ obviously holds for every $x \in D_f$ it remains to show that $y \mapsto K_{f, g}(x, [0, y])$ is non-decreasing on $[b(x), 1]$, which, however is a direct consequence of Theorem 2.2.7 in [11]. Extending $K_{f, g}(x, \cdot)$ from $\{[0, y] : y \in [0, 1]\}$ to $\mathcal{B}([0, 1])$ therefore yields a probability measure $K_{f, g}(x, \cdot)$ for every $x \in [0, 1]$.

On the other hand, for every fixed $y \in [0, 1]$, the function $x \mapsto K_{f,g}(x, [0, y])$ is measurable since, firstly, $D_f \in \mathcal{B}([0, 1])$ and, secondly, since the upper right Dini derivative of f which coincides with f' λ -almost everywhere is Borel measurable (see [7]). Using a standard Dynkin System argument (see [8, 9]) we get
265 that $x \mapsto K_{f,g}(x, B)$ is measurable for every Borel set $B \in \mathcal{B}([0, 1])$. This shows that $K_{f,g}(\cdot, \cdot)$ is a Markov kernel. Since $\int_{[0,x]} K_{f,g}(t, [0, y]) d\lambda(t) = C_{f,g}(x, y)$ obviously holds for all $x, y \in [0, 1]$ the proof is complete. \square

Remark 4.2. As already shown in Theorem 13 of [10] all singular mass of $\mu_{C_{f,g}}$
270 (if any) is concentrated on the graph of one function, the graph of the function b . This also follows immediately from Theorem 4.1 since generators are Lipschitz continuous (hence absolutely continuous) on every interval of the form $[\delta, 1]$ with $\delta > 0$. As a direct consequence, the class of all RMM copulas is not convex. In fact, considering $C = \frac{1}{2}(C_{f,g} + W)$ with $C_{f,g}$ being the symmetric RMM copula
275 depicted in Figure 1, the copula C concentrates mass on two graphs, so it can not be an RMM copula.

Working with the Markov kernel allows to calculate the total singular mass of RMM copulas very easily: For a given RMM copula $C_{f,g}$ let x_0 be defined by $x_0 := 0$ if $f(x) = 0$ for all $x \in [0, 1]$ and $x_0 := \min\{x \in (0, 1) : f(x) = 0\} > 0$ otherwise. For every $x \in D_f \cap (0, x_0)$ using equation (8) we get

$$\begin{aligned} K_{f,g}(x, \{b(x)\}) &= K_{f,g}(x, [0, b(x)]) = b(x) - f'(x)g(b(x)) \\ &= \left(b(x) - f'(x) \frac{xb(x)}{f(x)} \right) = b(x) \left(1 - \frac{xf'(x)}{f(x)} \right). \end{aligned}$$

Considering $\int_{[0,1]} K_{f,g}(x, \{0\}) d\lambda(x) = \mu_{C_{f,g}}([0, 1] \times \{0\}) = 0$ it follows that the total singular mass $\mu_{C_{f,g}}^{sing}([0, 1]^2)$ of $C_{f,g}$ is given by

$$\begin{aligned} \mu_{C_{f,g}}^{sing}([0, 1]^2) &= \mu_{C_{f,g}}(\Gamma(b)) = \int_{[0,1]} K_{f,g}(x, \{b(x)\}) d\lambda(x) \\ &= \int_{(0, x_0)} b(x) \left(1 - \frac{xf'(x)}{f(x)} \right) d\lambda(x), \end{aligned} \quad (9)$$

whereby $\Gamma(b) := \{(x, b(x)) : x \in [0, 1]\}$ denotes the graph of b . The next
280 examples (all taken from [10]) illustrate how formula 9 simplifies the calculations

of the total singular mass of RMM copulas and therefore underline the fact that working with Markov kernels may facilitate work.

Example 4.3 (Example 8(a) in [10]). Consider $\theta, \eta \in (0, 1)$ with $\theta + \eta \leq 1$ and let the generators f_θ, g_η be defined by

$$f_\theta(x) := \begin{cases} \left(\frac{1}{\theta} - 1\right) t & \text{if } t \in [0, \theta] \\ 1 - t & \text{if } t \in [\theta, 1]. \end{cases}$$

and $g_\eta := f_\eta$. Then the function b is given by

$$b(x) = \begin{cases} 1 - \theta & \text{if } 0 \leq x \leq \theta \\ 1 - x & \text{if } \theta < x \leq 1 - \eta \\ 0 & \text{if } 1 - \eta < x \leq 1 \end{cases}$$

and equation (9) yields $\mu_{C_{f_\theta, g_\eta}}(\Gamma(b)) = \int_{(\theta, 1-\eta]} (1-x) \left(1 - \frac{-x}{1-x}\right) d\lambda(x) = 1 - \theta - \eta$.

Example 4.4 (Example 8(b) in [10]). Consider $\delta \in (0, \frac{1}{2})$ and define the generator f_δ by

$$f_\delta(t) := \begin{cases} 0 & \text{if } t = 0 \\ \delta & \text{if } t \in (0, \delta] \\ t & \text{if } t \in [\delta, \frac{1}{2}] \\ 1 - t & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

For the symmetric RMM copula C_{f_δ, f_δ} the function b is given by

$$b(x) = \begin{cases} \frac{\delta}{x+\delta} & \text{if } 0 \leq x \leq \delta \\ \frac{1}{2} & \text{if } \delta < x \leq \frac{1}{2} \\ \frac{\delta(1-x)}{x} & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

so equation (9) yields

$$\mu_{C_{f_\delta, f_\delta}}(\Gamma(b)) = \int_{(0, \delta]} \frac{\delta}{x+\delta} d\lambda(x) + \int_{(\frac{1}{2}, 1)} \frac{\delta(1-x)}{x} \left(1 - \frac{-x}{1-x}\right) d\lambda(x) = 2\delta \ln(2).$$

Example 4.5 (Example 8(c) in [10]). In this case we consider $f(t) = t(1 - t)$ for all $t \in [0, 1]$ and set

$$g_\mu(t) := \begin{cases} 0 & \text{if } t \in \{0\} \cup [\mu, 1] \\ \mu - t & \text{if } t \in (0, \mu), \end{cases}$$

where $\mu \in (0, 1]$. Then b calculates to $b(x) = \frac{\mu(1-x)}{2-x}$ and we get

$$\begin{aligned} \mu_{C_{f,g_\mu}}(\Gamma(b)) &= \int_{(0,1)} \frac{\mu(1-x)}{2-x} \left(1 - \frac{x(1-2x)}{x(1-x)}\right) d\lambda(x) \\ &= \mu \int_{(0,1)} \frac{x}{2-x} d\lambda(x) = \mu(\ln(4) - 1). \end{aligned}$$

285 Before deriving two simple characterizations of absolute continuity of RMM copulas we give an alternative very short proof for a result already stated in [10] and state two lemmata that will be used in the sequel.

Proposition 4.6 (Proposition 15 in [10]). *W is the only singular RMM copula.*

Proof. If $C_{f,g}$ is singular we have $\mu_{C_{f,g}}(\Gamma(b)) = 1$, so $\mu_{C_{f,g}}$ concentrates its mass 290 on the graph of a function. According to [16] the copula $C_{f,g}$ is completely dependent, which implies that b is λ -preserving. Since the only λ -preserving, decreasing transformation on $[0, 1]$ is $b(x) = 1 - x$ for every $x \in [0, 1]$ applying Lemma 3.2 yields $C_{f,g} = W$. \square

Lemma 4.7. *If an RMM copula $C_{f,g}$ fulfills $C_{f,g}(t, t) > 0$ for every $t \in (0, 1]$ 295 then $b(x) = 0$ for every $x \in (0, 1]$ and $C_{f,g}$ is absolutely continuous.*

Proof. Suppose that there exists $x_0 \in (0, 1)$ with $b(x_0) > 0$. Since $C_{f,g}(x_0, x_0) > 0$ by assumption, $b(x_0) < x_0$ follows and we get $0 < C_{f,g}(b(x_0), b(x_0)) \leq C_{f,g}(x_0, b(x_0)) = 0$, a contradiction. Finally, absolute continuity of $C_{f,g}$ is a trivial consequence of $\mu_{C_{f,g}}([0, 1] \times \{0\})$ and disintegration. \square

300 **Lemma 4.8.** *For every $x \in (0, 1)$ and $y \in (b(x), 1]$ we have $K_{f,g}(x, [0, y]) > 0$.*

Proof. In case of $x \notin D_f$ the assertion is trivial so assume now that $x \in D_f$ and that $y > b(x)$, hence $xy - f(x)g(y) > 0$, holds. In case of $g(y) = 0$ we have

$K_{f,g}(x, [0, y]) = y > 0$. In case of $g(y) > 0$ we get $\frac{f(x)}{x} < \frac{y}{g(y)}$. For $\epsilon \in (0, 1 - x]$ the generator properties (R3) implies

$$f(x + \epsilon) \leq f(x) \frac{x + \epsilon}{x} = f(x) + \frac{\epsilon}{x} f(x),$$

from which we immediately get

$$f'(x) = \lim_{\epsilon \rightarrow 0^+} \frac{f(x + \epsilon) - f(x)}{\epsilon} \leq \frac{f(x)}{x} < \frac{y}{g(y)},$$

which completes the proof. \square

The next theorem is the first of the two afore-mentioned characterizations of absolute continuity for RMM copulas. It says that for absolutely continuous RMM copulas the function b has to be of a very simple form.

305 **Theorem 4.9.** $C_{f,g}$ is absolutely continuous if and only if the range of b on $(0, 1]$ contains at most two points.

Proof. If the range of b on $(0, 1]$ contains at most two points then the graph $\Gamma(b)$ of b can not carry any mass, so $C_{f,g}$ is absolutely continuous. Setting $x_0 := \inf\{x \in [0, 1] : b(x) = 0\}$ and applying Lemma 3.2 we get $b(x_0) > 0$. If $x_0 = 0$ then we are done since all copulas assign mass 0 to the set $[0, 1] \times \{0\}$. For $x_0 > 0$ we proceed as follows: Using absolute continuity of $C_{f,g}$ and disintegration we get $\int_{[0,1]} K_{f,g}(x, \{b(x)\}) d\lambda(x) = \mu_{C_{f,g}}(\Gamma(b)) = 0$, i.e.

$$b(x) - f'(x)g(b(x)) = 0 \tag{10}$$

holds for λ -almost every $x \in (0, x_0]$. Considering

$$xb(x) - f(x)g(b(x)) = 0 \tag{11}$$

and the fact that $b(x) > 0$ holds for all $x \in [0, x_0]$ we get $f(x)g(b(x)) > 0$ for λ -almost every $x \in (0, x_0]$. Combining (10) and (11) we therefore get

$$0 = b(x) - f'(x) \frac{xb(x)}{f(x)} = b(x) \left(1 - \frac{f'(x)}{f(x)} x \right)$$

as well as $f(x) = xf'(x)$ for λ -almost every $x \in (0, x_0]$. As direct consequence the generator f must be of the form $f(x) = cx$ for some constant $c > 0$ and

$x \in (0, x_0]$, so by equation (11) $\frac{g(b(x))}{b(x)} = \frac{1}{c}$ for all $x \in (0, x_0]$ follows. Suppose that there exist x_1, x_2 with $0 < x_1 < x_2 \leq x_0$ and $1 > b(x_1) > b(x_2) > 0$. Setting $y_1 := b(x_1)$ and $y_2 := b(x_2)$ we get $\frac{g(y_1)}{y_1} = \frac{g(y_2)}{y_2} = \frac{1}{c}$, considering $\frac{x_2}{f(x_2)} = \frac{1}{c} = \frac{g(y_1)}{y_1}$ yields $b(x_2) \geq y_1 = b(x_1)$, a contradiction. Hence b is constant on $(0, x_0]$ and the range of b is given by the two points $\{0, b(x_0)\}$, which completes the proof. \square

The next example shows an absolutely continuous RMM copula with non-trivial function b (it also shows that we do not necessarily have $b(x_0) = 1 - x_0$ as one might conjecture).

Example 4.10. Consider $f(t) = \min\{t, \frac{1}{3}(1-t)\}$ and $g(t) = \min\{t, 1-t\}$. Then the resulting RMM copula is given by

$$C_{f,g}(x, y) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{4}, 0 \leq y \leq \frac{1}{2}, \\ 2xy - x & \text{if } 0 \leq x \leq \frac{1}{4}, \frac{1}{2} < y \leq 1, \\ \frac{1}{3}(4xy - y) & \text{if } \frac{1}{4} < x \leq 1, 0 \leq y \leq \frac{1}{2}, \\ \frac{1}{3}(x + y - 1 + 2xy) & \text{if } \frac{1}{4} < x \leq 1, \frac{1}{2} < y \leq 1, \end{cases}$$

and it is straightforward to show that b is given by $b = \frac{1}{2}\mathbf{1}_{(0, \frac{1}{4}]}$. Figure 8 depicts a sample of this RMM copula together with the marginals histograms.

In order to prove the surprising fact that the limit of absolutely continuous RMM copulas is an absolutely continuous RMM copula too (see Theorem 4.13) the following alternative characterization of absolute continuity for RMM copulas will be used:

Lemma 4.11. $C_{f,g}$ is absolutely continuous if and only if $C_{f,g}(x, y) = xy - f(x)g(y)$ holds for all $(x, y) \in [0, 1]^2$.

Proof. Let $C_{f,g}$ be absolutely continuous. Theorem 4.9 implies that (i) $b(x) = 0$ or (ii) $b(x) = a\mathbf{1}_{(0, x_0]}(x)$ for all $x \in (0, 1]$ and some $x_0, a \in (0, 1)$. In the first case we have $xy - f(x)g(y) > 0$, hence $C_{f,g}(x, y) = xy - f(x)g(y)$ for all $x, y \in (0, 1]$. In the second case Theorem 4.9 and Theorem 3.5 yield the

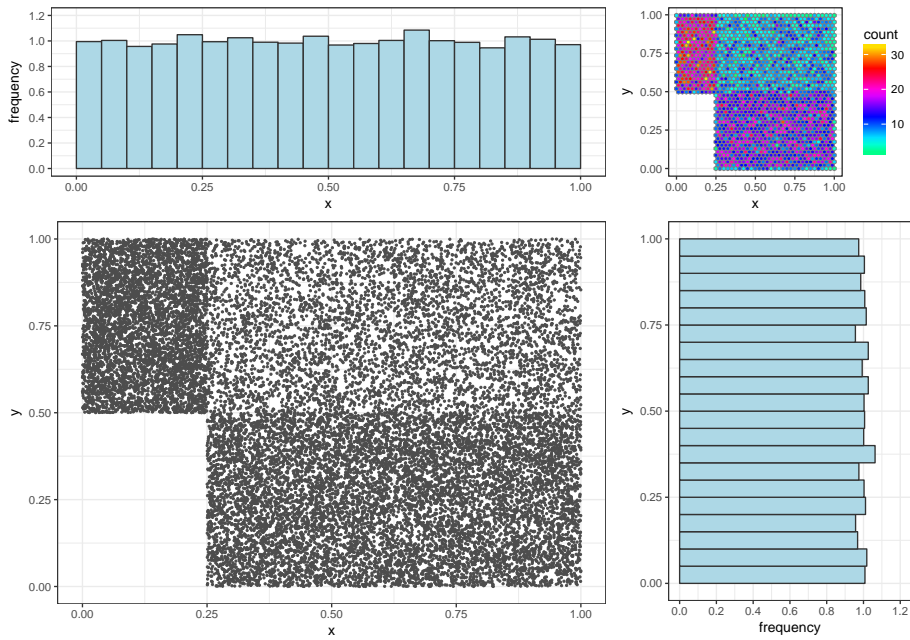


Figure 8: Sample of size 20.000 of $C_{f,g}$ according to Example 4.10.

existence of a constant c such that $f(x) = cx$ for all $x \in [0, x_0]$ and $g(y) = \frac{x_0}{f(x_0)}y = \frac{1}{c}y$ for all $y \in [0, a]$. So, given $x \in [0, x_0]$ we have $xy - f(x)g(y) = 0$ for all $y \leq b(x)$ and $xy - f(x)g(y) > 0$ for all $y > b(x)$ and altogether we get
340 $C_{f,g}(x, y) = xy - f(x)g(y)$ on the full unit square $[0, 1]^2$.

Conversely, suppose now that $xy - f(x)g(y) = C_{f,g}(x, y) \geq 0$ holds for all $x, y \in [0, 1]$. If f or g is identical to 0 then $C_{f,g}$ is obviously absolutely continuous. If not, then there exists some $y_0 \in (0, 1)$ with $f(y_0)g(y_0) > 0$, so $f(x) \leq \frac{y_0}{g(y_0)}x$ holds for all $x \in [0, 1]$. Having this and using the generator
345 properties it follows that f is Lipschitz continuous on $[0, 1]$. Proceeding in the same manner yields Lipschitz continuity of g , from which we altogether get absolute continuity of $C_{f,g}$. \square

Remark 4.12. The class of copulas considered in Lemma 4.11 is a subclass of the copulas introduced and studied by Rodríguez-Lallena and Úbeda-Flores in

[13]. These copulas are given by

$$C(x, y) = xy + f(x)g(y)$$

where f and g are absolutely continuous (not necessarily positive) functions not being identical to zero and fulfilling $f(0) = f(1) = g(0) = g(1) = 0$ as well as
 350 $\min\{\alpha\delta, \beta\gamma\} \geq -1$. Thereby $\alpha = \inf\{f'(x) : x \in D_f\} < 0$, $\beta = \sup\{f'(x) : x \in D_f\} > 0$, $\gamma = \inf\{g'(y) : y \in D_g\} < 0$, and $\delta = \sup\{g'(y) : y \in D_g\} > 0$.

Lemma 4.11 has the following consequence.

Theorem 4.13. *If a sequence $(C_{f_n, g_n})_{n \in \mathbb{N}}$ of absolutely continuous RMM copulas converges to an RMM copula $C_{f, g}$ then $C_{f, g}$ is absolutely continuous too.*

Proof. Suppose that $(C_{f_n, g_n})_{n \in \mathbb{N}}$ is a sequence of absolutely continuous RMM copulas that converges to an RMM copula $C_{f, g}$. Lemma 4.11 implies that $0 \leq C_{f_n, g_n}(x, y) = xy - f_n(x)g_n(y)$ holds for all $(x, y) \in [0, 1]^2$ and every $n \in \mathbb{N}$. If there exists $(x_0, y_0) \in (0, 1)^2$ such that $x_0y_0 - f(x_0)g(y_0) < 0$, setting $\delta := \min\{x_0, y_0\} > 0$ and applying Arzelà-Ascoli theorem yet another time yields the existence of subsequences $(f_{n_j})_{j \in \mathbb{N}}$ and $(g_{n_j})_{j \in \mathbb{N}}$ such that $(f_{n_j})_{j \in \mathbb{N}}$ converges to f and $(g_{n_j})_{j \in \mathbb{N}}$ converges to g (uniformly) on $[\delta, 1]$. Having this

$$0 \leq \lim_{j \rightarrow \infty} C_{f_{n_j}, g_{n_j}}(x_0, y_0) = \lim_{j \rightarrow \infty} (x_0y_0 - f_{n_j}(x_0)g_{n_j}(y_0)) = x_0y_0 - f(x_0)g(y_0)$$

355 follows, which contradicts $x_0y_0 - f(x_0)g(y_0) < 0$. As direct consequence we have $xy - f(x)g(y) \geq 0$ for all $(x, y) \in [0, 1]^2$ and Lemma 4.11 yields absolute continuity of $C_{f, g}$. \square

According to Theorem 3.11 \mathcal{C}_{RMM} is a compact subset of (\mathcal{C}, d_∞) . Since closed subsets of compact subsets are compact too we can show the following result:

360 **Theorem 4.14.** *The family of all RMM copulas with non-degenerated singular component is dense in $(\mathcal{C}_{RMM}, d_\infty)$. The family of all absolutely continuous RMM copulas is a compact, nowhere dense subset of (\mathcal{C}, d_∞) .*

Proof. Suppose that $C_{f, g}$ is an absolutely continuous RMM copula. We will show the existence of a sequence of non-absolutely continuous RMM copulas

$(C_{f_n, g_n})_{n \in \mathbb{N}}$ that converges to $C_{f, g}$ and distinguish two situations: (i) If f or g is identical to 0 then $C_{f, g} = \Pi$. Setting $f_n(x) = g_n(x) = \max\{\frac{1}{n} - x, 0\}$ for every $x \in (0, 1]$ and every $n \in \mathbb{N}$ yields sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ of generators that both converge to 0 uniformly, from which we get that $(C_{f_n, g_n})_{n \in \mathbb{N}}$ converges uniformly to $\Pi = C_{f, g}$ on $[0, 1]^2$. According to Lemma 4.11 every copula C_{f_n, g_n} is non-absolutely continuous, so the proof for case (i) is complete.

(ii) If neither f nor g is identical to 0 then there exists some $t \in (0, 1)$ such that $f(t)g(t) > 0$ holds. Defining

$$f_n(x) = \begin{cases} 0 & \text{if } x = 0, \\ f(\frac{t}{n}) & \text{if } x \in (0, \frac{t}{n}], \\ f(x) & \text{if } x \in (\frac{t}{n}, 1], \end{cases} \quad g_n(y) = \begin{cases} 0 & \text{if } y = 0, \\ g(\frac{t}{n}) & \text{if } y \in (0, \frac{t}{n}], \\ g(y) & \text{if } y \in (\frac{t}{n}, 1], \end{cases}$$

for every $n \in \mathbb{N}$ yields sequences $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ of generators that converge to f and g respectively. Since according to Lemma 4.11 every copula C_{f_n, g_n} is non-absolutely continuous, we have shown the assertion for case (ii) too.

The remaining statements of the theorem are now straightforward to derive: Theorem 4.13 implies that the family of all absolutely continuous copulas is closed. Since closed subsets of compact metric spaces are compact too the stated compactness follows. Finally, the (compact) set of all absolutely continuous RMM copulas can not have an inner point since the complement is dense in \mathcal{C}_{RMM} , so the set is nowhere dense by definition. \square

In the language of Baire categories we have shown that a ‘typical’ RMM copula has a non-degenerated singular component:

Corollary 4.15. *The family of all RMM copulas with non-degenerated singular component is co-meager (hence of second category) in (\mathcal{C}, d_∞) .*

5. Kendall’s τ and Spearman’s ρ of RMM copulas

In this final section we derive formulas for Kendall’s τ and Spearman’s ρ of RMM copulas, prove that all absolutely continuous RMM copulas $C_{f, g}$ fulfill

$\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$ and state a conjecture on the τ - ρ -region of the family \mathcal{C}_{RMM} (see [15] and [17] for the τ - ρ -region of all copulas and some results on the region for Extreme-Value copulas respectively). It is well known that for every continuous random vector (X, Y) with underlying copula C Spearman's ρ and Kendall's τ are given by

$$\rho(X, Y) = \rho(C) = 12 \int_{[0,1]^2} xy d\mu_C(x, y) - 3, \quad (12)$$

$$\tau(X, Y) = \tau(C) = 4 \int_{[0,1]^2} C(x, y) d\mu_C(x, y) - 1. \quad (13)$$

In the following lemmata D_f (respectively D_g) will again denote the points $t \in (0, 1)$ at which f (respectively g) is differentiable.

Lemma 5.1. *For every RMM copula $C_{f,g}$ Kendall's τ and Spearman's ρ are given by the following formulas:*

$$\rho(C_{f,g}) = 6 \int_{[0,1]} xb^2(x)d\lambda(x) + 12 \int_{D_f} xf'(x) \left(\int_{(b(x),1]} g(y)d\lambda(y) \right) d\lambda(x), \quad (14)$$

$$\begin{aligned} \tau(C_{f,g}) &= -2 \int_{[0,1]} xb^2(x)d\lambda(x) + 2 \int_{D_f} xf'(x)b(x)g(b(x))d\lambda(x) \\ &\quad + 4 \int_{D_f} xf'(x) \left(\int_{(b(x),1]} g(y)d\lambda(y) \right) d\lambda(x) \\ &\quad - 4 \int_{[0,1]} \int_{(b(x),1]} f(x)g(y)d\lambda(y)d\lambda(x). \end{aligned} \quad (15)$$

Proof. See Appendix. □

Example 5.2. Suppose that $C_{f,g}$ is an RMM copula with $b(x) = 0$ for all $x \in (0, 1]$ (according to Theorem 4.9 $C_{f,g}$ is necessarily absolutely continuous).

Using Lemma 5.1 and applying integration by parts yields

$$\begin{aligned}
\rho(C_{f,g}) &= 12 \int_{D_f} x f'(x) \int_{(0,1]} g(y) d\lambda(y) d\lambda(x) \\
&= -12 \int_{[0,1]} f(x) d\lambda(x) \int_{(0,1]} g(y) d\lambda(y), \\
\tau(C_{f,g}) &= 4 \int_{D_f} x f'(x) \int_{(0,1]} g(y) d\lambda(y) d\lambda(x) - 4 \int_{[0,1]} \int_{(0,1]} f(x) g(y) d\lambda(y) d\lambda(x) \\
&= -8 \int_{[0,1]} f(x) d\lambda(x) \int_{(0,1]} g(y) d\lambda(y),
\end{aligned}$$

380 from which $\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$ follows immediately.

It has already been shown in [13] that the identity derived in Example 5.2 holds true for all absolutely continuous RMM copulas. We restate and extend this result and give an alternative proof:

Theorem 5.3. *For every absolutely continuous RMM copula $C_{f,g}$ Kendall's τ and Spearman's ρ are linked via $\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$. Additionally, for absolutely*
385 *continuous RMM copulas we have $\rho(C_{f,g}) \in [-\frac{3}{4}, 0]$ and $\tau(C_{f,g}) \in [-\frac{1}{2}, 0]$.*

Proof. See Appendix. □

Remark 5.4. Theorem 5.3 can be viewed as a generalization of the well-known identity linking Kendall's τ and Spearman's ρ for EFGM copulas, i.e. copulas
390 given by

$$C_\alpha(x, y) = xy + \alpha xy(1-x)(1-y) \tag{16}$$

with $\alpha \in [-1, 1]$. In fact, for EFGM copulas the values of Spearman's rho and Kendall's tau are given by $\rho(C_\alpha) = \frac{\alpha}{3}$ and $\tau(C_\alpha) = \frac{2\alpha}{9}$.

We conclude this paper with a conjecture on sharp inequalities between Kendall's τ and Spearman's ρ for RMM copulas. Numerous simulations insinuate these
395 inequalities, we have, however, not been able to prove them analytically and plan to tackle this problem in the near future. Figure 9 depicts the region determined by the conjectures inequalities.

Conjecture 5.5. For every RMM copula $C_{f,g}$ the following inequalities hold:

$$\max \left\{ \frac{(\tau(C_{f,g}) + 1)^{\frac{3}{2}}}{\sqrt{2}} - 1, \frac{3}{2}\tau(C_{f,g}) \right\} \leq \rho(C_{f,g}) \leq \tau(C_{f,g}) \quad (17)$$

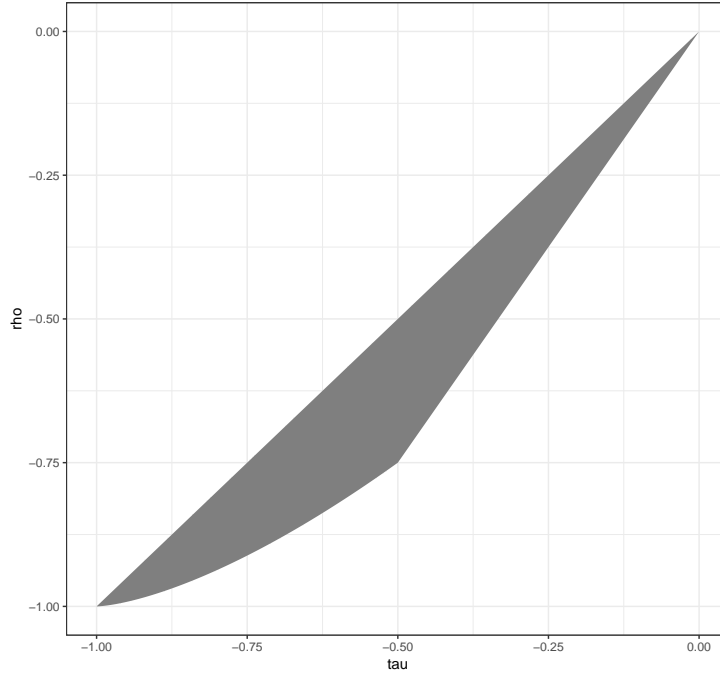


Figure 9: The conjectured τ - ρ region induced by the inequalities (17).

Appendix

400 Proof of Lemma 5.1

As first step, using disintegration (see [8]) we get

$$\begin{aligned} \int_{[0,1]^2} xy d\mu_{C_{f,g}}(x, y) &= \int_{[0,1]} \int_{[0,1]} xy K_{f,g}(x, dy) d\lambda(x) \\ &= \int_{D_f} x \int_{[0,1]} y K_{f,g}(x, dy) d\lambda(x). \end{aligned}$$

The inner integral calculates to

$$\begin{aligned} \int_{[0,1]} y K_{f,g}(x, dy) &= b(x) K_{f,g}(x, \{b(x)\}) + \int_{(b(x),1] \cap D_g} y(1 - f'(x)g'(y)) d\lambda(y) \\ &= b(x) K_{f,g}(x, \{b(x)\}) + \frac{1}{2}(1 - b^2(x)) \\ &\quad - f'(x) \int_{(b(x),1] \cap D_g} y g'(y) d\lambda(y), \end{aligned}$$

and, applying integration by parts, the latter integral can be simplified to

$$\begin{aligned} \int_{(b(x),1] \cap D_g} y g'(y) d\lambda(y) &= y g(y) \Big|_{b(x)}^1 - \int_{(b(x),1]} g(y) d\lambda(y) \\ &= -b(x)g(b(x)) - \int_{(b(x),1]} g(y) d\lambda(y). \end{aligned} \quad (18)$$

Altogether we therefore get

$$\begin{aligned} \int_{[0,1]^2} xy d\mu_{C_{f,g}}(x, y) &= \int_{D_f} xb(x)(b(x) - f'(x)g(b(x))) d\lambda(x) + \frac{1}{2} \int_{[0,1]} x(1 - b^2(x)) \lambda(x) \\ &\quad + \int_{D_f} x f'(x) b(x) g(b(x)) d\lambda(x) + \int_{D_f} x f'(x) \int_{(b(x),1]} g(y) d\lambda(y) d\lambda(x) \\ &= \frac{1}{4} + \frac{1}{2} \int_{[0,1]} x b^2(x) d\lambda(x) + \int_{D_f} x f'(x) \int_{(b(x),1]} g(y) d\lambda(y) d\lambda(x), \end{aligned}$$

from which equation (14) is a direct consequence.

In order to derive equation (15) we proceed analogously: Via disintegration we first get

$$\begin{aligned} \int_{[0,1]^2} C_{f,g}(x, y) d\mu_{C_{f,g}}(x, y) &= \int_{[0,1]} \int_{[0,1]} C_{f,g}(x, y) K_{f,g}(x, dy) d\lambda(x) \\ &= \int_{D_f} \left(\underbrace{C_{f,g}(x, b(x))}_{=0} K_{f,g}(x, \{b(x)\}) + \int_{(b(x),1] \cap D_g} C_{f,g}(x, y) (1 - f'(x)g'(y)) d\lambda(y) \right) d\lambda(x) \\ &= \int_{[0,1] \cap D_f} \int_{(b(x),1] \cap D_g} (xy - f(x)g(y)) (1 - f'(x)g'(y)) d\lambda(y) d\lambda(x) \\ &= \int_{[0,1] \cap D_f} \int_{(b(x),1] \cap D_g} (xy - x f'(x) y g'(y) - f(x)g(y) + f(x) f'(x) g(y) g'(y)) d\lambda(y) d\lambda(x). \end{aligned}$$

Considering

$$\begin{aligned}
& \int_{[0,1] \cap D_f} f(x)f'(x) \int_{(b(x),1] \cap D_g} g(y)g'(y)d\lambda(y)d\lambda(x) \\
&= \int_{D_f} f(x)f'(x) \frac{1}{2}(g^2(1) - g^2(b(x)))d\lambda(x) \\
&= -\frac{1}{2} \int_{D_f} f(x)f'(x)g^2(b(x))d\lambda(x) \\
&= -\frac{1}{2} \int_{D_f} xf'(x)b(x)g(b(x))d\lambda(x),
\end{aligned}$$

and using equation (18) yields

$$\begin{aligned}
& \int_{[0,1]^2} C_{f,g}(x,y)d\mu_{C_{f,g}}(x,y) \\
&= \frac{1}{4} - \frac{1}{2} \int_{D_f} xb^2(x)d\lambda(x) + \frac{1}{2} \int_{D_f} xf'(x)b(x)g(b(x))d\lambda(x) \\
&\quad + \int_{D_f} xf'(x) \int_{(b(x),1]} g(y)d\lambda(y)d\lambda(x) - \int_{[0,1]} \int_{(b(x),1]} f(x)g(y)d\lambda(y)d\lambda(x),
\end{aligned}$$

which completes the proof.

Proof of Theorem 5.3

We use Theorem 4.9 and distinguish two cases:

Case (1): Suppose that $b(x) = 0$ for every $x \in (0, 1]$. If f or g is identical to 0 all assertions of the theorem are trivial. Therefore suppose that $f(x_0) > 0$ and $g(y_0) > 0$ for some $x_0, y_0 \in (0, \frac{1}{2}]$. The identity $\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$ has already been shown in Example 5.2 so we may concentrate on the other assertions. The assumption about b implies that $f(x)g(y) < xy$ holds for all $(x, y) \in (0, 1]^2$.

Case (1a): Suppose that $f(x) \leq x$ and $g(x) \leq y$ holds for all $x \in [0, \frac{1}{2}]$. Then we get

$$\int_{[0,1]} f(x)d\lambda(x) \leq \int_{[0,\frac{1}{2}]} xd\lambda(x) + \int_{(\frac{1}{2},1]} (1-x)d\lambda(x) = \frac{1}{4},$$

which implies $\rho(C_{f,g}) = -12 \int_{[0,1]} f(x)d\lambda(x) \int_{[0,1]} g(y)d\lambda(y) \geq -\frac{12}{16} = -\frac{3}{4}$ and $\tau(C_{f,g}) \geq -\frac{1}{2}$ now follows via $\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$.

Case (1b): Suppose that there exists some $x_1 \in (0, \frac{1}{2})$ with $f(x_1) > x_1$. Then

$x_1 y > f(x_1)g(y) \geq x_1 g(y)$, hence $y \geq g(y)$, holds for all $y \in (0, 1]$. Set

$$c_1 := \lim_{x \downarrow 0} \frac{f(x)}{x}, \quad c_2 := \lim_{y \downarrow 0} \frac{g(y)}{y} \leq 1,$$

and define

$$\hat{f}(x) = \begin{cases} c_1 x & \text{if } x \in [0, \frac{1}{1+c_1}], \\ 1-x & \text{if } x \in (\frac{1}{1+c_1}, 1] \end{cases}$$

as well as

$$\hat{g}(y) = \begin{cases} c_2 y & \text{if } y \in [0, \frac{1}{1+c_2}], \\ 1-y & \text{if } y \in (\frac{1}{1+c_2}, 1]. \end{cases}$$

Since we have $\frac{f(x)}{x} < \frac{y_0}{g(y_0)} < \infty$ for all $x \in (0, 1]$, using (R3) and $f(x_1) > x_1$ yields $c_1 > 1$ as well as $c_1 < \infty$. Obviously \hat{f}, \hat{g} are generators fulfilling $f \leq \hat{f}$, $g \leq \hat{g}$. Considering

$$c_1 c_2 = \lim_{x \downarrow 0} \lim_{y \downarrow 0} \frac{f(x)g(y)}{xy} \leq \lim_{x \downarrow 0} \lim_{y \downarrow 0} \frac{xy}{xy} = 1$$

$\frac{c_2}{1+c_2} \leq \frac{1}{1+c_1}$ follows. Having this and using $\int_{[0,1]} f(x)d\lambda(x) \leq \int_{[0,1]} \hat{f}(x)d\lambda(x) = \frac{c_1}{2(1+c_1)}$ we altogether get

$$\int_{[0,1]} f(x)d\lambda(x) \int_{[0,1]} g(y)d\lambda(y) \leq \frac{c_1 c_2}{4(1+c_1)(1+c_2)} \leq \frac{c_1}{4(1+c_1)^2} \leq \frac{1}{16},$$

which shows $\rho(C_{f,g}) \geq -\frac{12}{16} = -\frac{3}{4}$ and $\tau(C_{f,g}) = \frac{2}{3}\rho(C_{f,g}) \geq -\frac{1}{2}$.

Case (2): Suppose that $b(x) > 0$ for some $x \in (0, 1)$. The proof of Theorem 4.9 implies the existence of $a, x_0 \in (0, 1)$ with $a \leq 1 - x_0$ such that

$$b(x) = \begin{cases} a & \text{if } 0 < x \leq x_0, \\ 0 & \text{if } x > x_0, \end{cases}$$

holds. Following from the proof of Theorem 4.9 we have $f(x) = cx$ for all $x \in [0, x_0]$ and for some $c > 0$, moreover Theorem 3.5 implies that $g(y) = \frac{x_0}{f(x_0)}y = \frac{1}{c}y$ holds for all $y \in [0, a]$. Setting

$$I := \int_{D_f} x f'(x) \left(\int_{(b(x), 1]} g(y) d\lambda(y) \right) d\lambda(x)$$

and applying integration by parts yields

$$\begin{aligned}
I &= \int_{[0,x_0] \cap D_f} x f'(x) \left(\int_{(a,1]} g(y) d\lambda(y) \right) d\lambda(x) \\
&\quad + \int_{(x_0,1] \cap D_f} x f'(x) \left(\int_{(0,1]} g(y) d\lambda(y) \right) d\lambda(x) \\
&= \left(x_0 f(x_0) - \int_{[0,x_0]} f(x) d\lambda(x) \right) \int_{(a,1]} g(y) d\lambda(y) \\
&\quad + \left(-x_0 f(x_0) - \int_{(x_0,1]} f(x) d\lambda(x) \right) \int_{(0,1]} g(y) d\lambda(y) \\
&= -\frac{1}{2} a^2 x_0^2 - \int_{[0,1]} \int_{(b(x),1]} f(x) g(y) d\lambda(y) d\lambda(x).
\end{aligned}$$

Using Lemma 5.1 after some straightforward calculation we therefore get

$$\begin{aligned}
\rho(C_{f,g}) &= -3a^2 x_0^2 - 12 \int_{[0,1]} \int_{(b(x),1]} f(x) g(y) d\lambda(y) d\lambda(x), \\
\tau(C_{f,g}) &= -2a^2 x_0^2 - 8 \int_{[0,1]} \int_{(b(x),1]} f(x) g(y) d\lambda(y) d\lambda(x),
\end{aligned} \tag{19}$$

implying the desired equality $\rho(C_{f,g}) = \frac{3}{2}\tau(C_{f,g})$. Having this, in order to complete the proof it suffices to show that $\rho(C_{f,g}) \geq -\frac{3}{4}$ holds, which can be done as follows: Set $x_1 = \frac{1}{1+c}$, $y_1 = \frac{c}{1+c}$ and define

$$\tilde{f}(x) = \begin{cases} cx & \text{if } x \in [0, x_1], \\ 1-x & \text{if } x \in (x_1, 1], \end{cases} \quad \tilde{g}(y) = \begin{cases} \frac{1}{c} y & \text{if } y \in [0, y_1], \\ 1-y & \text{if } y \in (y_1, 1]. \end{cases}$$

Then \tilde{f} and \tilde{g} are generators fulfilling $f \leq \tilde{f}$ and $g \leq \tilde{g}$ and we get

$$\begin{aligned}
& \int_{[0,1]} \int_{(b(x),1]} f(x)g(y)d\lambda(y)d\lambda(x) \leq \int_{[0,1]} \int_{(b(x),1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) \\
& = \int_{[0,x_0]} \int_{(a,y_1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) + \int_{[0,x_1]} \int_{(y_1,1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) \\
& \quad + \int_{(x_0,x_1]} \int_{(0,y_1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) + \int_{(x_1,1]} \int_{(0,y_1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) \\
& \quad + \int_{(x_1,1]} \int_{(y_1,1]} \tilde{f}(x)\tilde{g}(y)d\lambda(y)d\lambda(x) \\
& = \frac{1}{4}x_0^2(y_1^2 - a^2) + \frac{1}{4}cx_1^2(1 - y_1)^2 + \frac{1}{4}(x_1^2 - x_0^2)y_1^2 + \frac{1}{4c}(1 - x_1)^2y_1^2 \\
& \quad + \frac{1}{4}(1 - x_1)^2(1 - y_1)^2 \\
& = \frac{1}{4(2 + c + \frac{1}{c})} - \frac{a^2x_0^2}{4}.
\end{aligned}$$

Having this inequality, applying equation (19) we get

$$\rho(C_{f,g}) \geq -3a^2x_0^2 - \frac{3}{2 + c + \frac{1}{c}} + 3a^2x_0^2 = -\frac{3}{2 + c + \frac{1}{c}} \geq -\frac{3}{4}.$$

405 The last inequality follows by the fact that $c + \frac{1}{c} \geq 2$ for all $c > 0$.

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