On bivariate lower semilinear copulas and the star product

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Abstract

We revisit the family $\mathcal{C}^{LSL}$ of all bivariate lower semilinear (LSL) copulas first introduced by Durante et al. in 2008 and, using the characterization of LSL copulas in terms of diagonals with specific properties, derive several novel and partially unexpected results. In particular we prove that the star product (also known as Markov product) $S_{\delta_1} * S_{\delta_2}$ of two LSL copulas $S_{\delta_1}, S_{\delta_2}$ is again an LSL copula, i.e., that the family $\mathcal{C}^{LSL}$ is closed with respect to the star product. Moreover, we show that translating the star product to the class of corresponding diagonals $\mathcal{D}^{LSL}$ allows to determine the limit of the sequence $S_\delta, S_\delta * S_\delta, S_\delta * S_\delta * S_\delta, \ldots$ for every diagonal $\delta \in \mathcal{D}^{LSL}$. In fact, for every LSL copula S_δ the sequence $(S_\delta^{*n})_{n\in\mathbb{N}}$ converges to some LSL copula $S_{\overline{\delta}}$, the limit $S_{\overline{\delta}}$ is idempotent, and the class of all idempotent LSL copulas allows for a simple characterization.

Complementing these results we then focus on concordance of LSL copulas. After recalling simple formulas for Kendall's τ and Spearman's ρ we study the exact region Ω^{LSL} determined by these two concordance measures of all elements in $\mathcal{C}^{LSL}$, derive a sharp lower bound and finally show that Ω^{LSL} is convex and compact.

Keywords: Copula, Lower semilinear copula, Markov product, concordance, Markov kernel

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1. Introduction

Considering that the class of bivariate copulas is quite diverse (in the sense of containing both, very regular/smooth elements as well as distributions with fractal support, see [27]) it seems natural to look for subclasses which, on the one hand, are handy and well understood and, on the other hand, are sufficiently large to be relevant for applications. Extreme Value copulas as well as Archimedean copulas are standard classes fulfilling these properties. Both of them are characterized via a univariate function or, equivalently a probability measure on $[0,1]$ or $[0,\infty)$ respectively, see, e.g., [12] and the references therein.
Asking for linearity along some segments of the unit square (and the resulting simple analytic forms) in 2008 Durante et al. (see [4]) introduced and analyzed the family of so-called bivariate lower (and upper) semilinear copulas. The authors provided (among various other results) a nice stochastic interpretation and characterized lower semilinear copulas (LSL copulas, for short) in terms of another class of univariate functions: the class $\mathcal{D}^{LSL}$ of (copula) diagonals with some additional growth conditions, defined by

$$\mathcal{D}^{LSL} := \{\delta \in \mathcal{D} : \varphi_\delta \text{ non-decreasing}, \eta_\delta \text{ non-increasing}\}. \tag{1}$$

Thereby $\mathcal{D}$ denotes the family of all diagonals of bivariate copulas and the functions $\varphi_\delta, \eta_\delta : (0,1] \to [0,\infty)$ are given by

$$\varphi_\delta(x) := \frac{\delta(x)}{x}, \quad \eta_\delta(x) := \frac{\delta(x)}{x^2}.$$

For every fixed $\delta \in \mathcal{D}^{LSL}$ the corresponding LSL copula S_δ is then given by

$$S_\delta(x,y) := \begin{cases} y\,\frac{\delta(x)}{x} & \text{if } y \le x, \\ x\,\frac{\delta(y)}{y} & \text{otherwise.} \end{cases} \tag{2}$$

Obviously S_δ is symmetric, i.e., we have $S_\delta(x,y) = S_\delta(y,x) =: S_\delta^t(x,y)$ for all $x,y \in [0,1]$. Letting $\mathcal{C}^{LSL}$ denote the family of all bivariate LSL copulas it is straightforward to verify that $\mathcal{C}^{LSL}$ is convex and that $(\mathcal{C}^{LSL}, d_\infty)$ is compact. As a consequence, in [6] Durante et al. revisited $\mathcal{C}^{LSL}$ and provided a nice characterization of extreme points (in the Krein-Milman sense) of $\mathcal{C}^{LSL}$. As shown in [4, Proposition 9] LSL copulas have a nice and simple stochastic interpretation as copulas of random variable 'X and Y that derive from

a latent triple (Z_1, Z_2, Z_3), where Z_1 and Z_2 have a common distribution function'.
Multivariate extensions of semilinear copulas were studied by Arias-García et al. in [1] and further analyzed and provided with a probabilistic interpretation by Sloot and Scherer in [24]. In the latter paper the authors in particular showed that in the multivariate setting upper semilinear copulas are quite pathological in the sense that they concentrate their mass on finitely many hyperplanes (and hence are singular).

Here we revisit the class $\mathcal{C}^{LSL}$ of bivariate LSL copulas and show that, although the class is well studied and easy to handle, it possesses additional properties that are at least partially surprising. The latter in particular applies to the behavior of LSL copulas in connection with the so-called star product (also known as Markov product) of copulas. The star product was introduced by Darsow et el. in 1992 (see [2]) and has since been studied in numerous papers. In 1996 Olsen et al. (see [19]) showed that the family $\mathcal{C}$ of all bivariate copulas with the star product as binary operation and the space $(\mathcal{M}, \circ)$ of Markov operators with the composition as binary operation are isomorphic - a result implying that studying the star product of copulas can as well be done by studying the corresponding Markov operators. Another rational, why the name Markov product is perhaps more adequate than the name star product was provided in [27], where the authors showed that $A*B$ just corresponds to the standard composition of the Markov kernels K_A and K_B (transition probabilities) well known in the context of Markov chains.
Apart from the family of completely dependent copulas and the family of checkerboard copulas (see, e.g., [12] and the references therein) only fully parametric classes (like the Fréchet class, see [20], and Gauss copulas, see [9]) are known to be closed under the star product. Considering the lower Fréchet Hoeffding bound W we have $W * W = M$, so although W is an Archimedian $W * W = M$ is not. In other words: the family of Archimedean copulas is not closed w.r.t. the star product. For the class of Extreme-Value copulas more tedious calculations yield the same result, the class is not closed w.r.t. the star product either.
It is therefore quite surprising that the star product $S_{\delta_1} * S_{\delta_2}$ of two LSL copulas $S_{\delta_1}, S_{\delta_2} \in \mathcal{C}^{LSL}$ is again an LSL copula, implying that $\mathcal{C}^{LSL}$ consitutes are very rare example of a family of copulas being perfectly compatible with the star product. Building upon this fact allows to translate the star product to the class $\mathcal{D}^{LSL}$ of diagonals and to study the limit behavior of

iterates of the star product $S_\delta^{*n} = S_\delta * S_\delta * \cdots * S_\delta$ directly in terms of the limit behavior of the sequence of corresponding diagonals $(\delta^{*n})_{n\in\mathbb{N}}$. As one of the main results of this contribution we will show that for every $\delta \in \mathcal{D}^{LSL}$ the sequence $(\delta^{*n})_{n\in\mathbb{N}}$ converges uniformly to some $\overline{\delta} \in \mathcal{D}^{LSL}$ and that $\overline{\delta}$ is a fixed point of the star product in the sense that $\overline{\delta} * \overline{\delta} = \overline{\delta}$ holds. Translating back to the class $\mathcal{C}^{LSL}$: for every $S_\delta \in \mathcal{C}^{LSL}$ there exists some $S_{\overline{\delta}} \in \mathcal{C}^{LSL}$ such that

$$\lim_{n\to\infty} d_\infty(S_\delta^{*n}, S_{\overline{\delta}}) = 0$$

holds, and the limit copula $S_{\overline{\delta}}$ is idempotent, i.e., $S_{\overline{\delta}} * S_{\overline{\delta}} = S_{\overline{\delta}}$ holds. Rounding off these findings we provide a simple characterization of all idempotent LSL copulas. The just mentioned convergence results are quite surprising in so far as for general copulas iterates of the star product do not need to converge and - even for the case of convergence - determining the limit is a nontrivial endeavor.

In the second part of the paper we focus on concordance of LSL copulas and study in particular Kendall's τ and Spearman's ρ. After deriving simple expressions for both concordance measures we study the τ-ρ-region Ω^{LSL}, defined by

$$\Omega^{LSL} := \left\{ (\tau(S_\delta), \rho(S_\delta)) : \; S_\delta \in \mathcal{C}^{LSL} \right\}.$$

We conjecture that Ω^{LSL} coincides with the set

$$R := \left\{ (x,y) \in [0,1]^2 : x \le y \le 1 - (1-x)^{\frac{3}{2}} \right\},$$

we were, however, only able to prove the lower inequality and to show that it is sharp. Proving the upper inequality (arising from running numerous simulations) remains an open problem. Finally, despite not knowing the exact upper bound we show that Ω^{LSL} is compact and convex.

The rest of this paper is organized as follows: Section 2 gathers some notations and preliminaries and recalls some facts about LSL copulas mainly going back to [4]. Section 3 derives the Markov kernels of LSL copulas and considers two parametric classes which will prove important in the sequel. Section 4 starts with proving the fact that the star product of two LSL copulas is again an LSL copula and then derives the afore-mentioned results on the limit of star product iterates of LSL copulas and their (idempotent) limits.

Finally, Section 5 is devoted to studying Kendall's τ and Spearman's ρ and their interplay in the family $\mathcal{C}^{LSL}$. Several examples and graphics illustrate the studied procedures and some underlying ideas.

2. Notation and preliminaries

For every metric space (S, d) the Borel σ-field on S will be denoted by $\mathcal{B}(S)$. The two-dimensional Lebesgue measure on $\mathcal{B}([0,1]^2)$ will be denoted by λ_2, the one-dimensional Lebesgue measure by λ.

In the sequel we will write $\mathcal{C}$ for the family of all bivariate copulas, Π denotes the product copula, M the minimum copula and W the lower Fréchet Hoeffding bound. C^t will denote the transpose of C, i.e., the copula fulfilling $C^t(x, y) = C(y, x)$. For every $C \in \mathcal{C}$ the corresponding doubly stochastic measure will be denoted by μ_C, i.e., $\mu_C([0, x] \times [0, y]) = C(x, y)$ for all $x, y \in [0, 1]$ (and μ_C is extended to full $\mathcal{B}([0, 1]^2)$ in the standard measure-theoretic way). The standard uniform metric d_∞ on $\mathcal{C}$ is defined by

$$d_\infty(A, B) := \max_{x,y\in[0,1]} |A(x, y) - B(x, y)|. \tag{3}$$

It is well known that the metric space $(\mathcal{C}, d_\infty)$ is compact and that pointwise and uniform convergence of a sequence of copulas $(C_n)_{n\in\mathbb{N}}$ are equivalent. For more background on copulas and doubly stochastic measures we refer to the textbooks [5, 20].
The Lebesgue decomposition (see [7, 22]) of a doubly stochastic measure μ_C with respect to λ_2 will be denoted by

$$\mu_C = \mu_C^{\ll} + \mu_C^{\perp},$$

where $\mu_C^{\ll}$ denotes the absolutely continuous and $\mu_C^{\perp}$ the singular component of μ_C. We will write $sing(C) = \mu_C^{\perp}([0, 1]^2) \in [0, 1]$ for the total mass of the singular component of C and set $abs(C) := 1 - sing(C)$.

A mapping $K : [0, 1] \times \mathcal{B}([0, 1]) \to [0, 1]$ will be called a Markov kernel if $x \mapsto K(x, B)$ is measurable for every fixed $B \in \mathcal{B}([0, 1])$ and $B \mapsto K(x, B)$ is a probability measure on $\mathcal{B}([0, 1])$ for every fixed $x \in [0, 1]$. It is well known that for every copula $C \in \mathcal{C}$ there exists a Markov kernel $K_C : [0, 1] \times$

$\mathcal{B}([0,1]) \to [0,1]$ fulfilling

$$\int_E K_C(x, F) d\lambda(x) = \mu_C(E \times F)$$

for all $E, F \in \mathcal{B}([0,1])$ and that this Markov kernel is unique for λ-almost every $x \in [0,1]$. Vice versa, every Markov kernel $K : [0,1] \times \mathcal{B}([0,1]) \to [0,1]$ having λ as invariant distribution, i.e., fulfilling

$$\int_{[0,1]} K(x, F) d\lambda(x) = \lambda(F)$$

for every $F \in \mathcal{B}([0,1])$, can be shown to be the Markov kernel of a unique copula C, which then fulfills

$$C(x, y) = \int_{[0,x]} K(t, [0, y]) d\lambda(t)$$

for all $x, y \in [0,1]$. Notice that for fixed $y \in [0,1]$, λ-almost every $x \in [0,1]$ is a Lebesgue point (see [22]) of the mapping $x \mapsto K(x, [0, y])$, so the identity

$$\frac{\partial C(x, y)}{\partial x} = K(x, [0, y]) \tag{4}$$

holds (for such x). For more background on Markov kernels and disintegration we refer to [11], for more results on the interplay between Markov kernels and copulas, e.g., to [12, 25].

Following [4] $C \in \mathcal{C}^{LSL}$ is called lower semilinear (LSL) copula, if for every $x \in (0,1]$ the mappings $t \mapsto h_x(t) := C(t, x)$ and $t \mapsto v_x(t) := C(x, t)$ are linear on $[0, x]$. As already mentioned in the introduction, $\mathcal{D}$ denotes the family of all copula diagonals, i.e., we have that $\delta : [0,1] \to [0,1]$ is an element of $\mathcal{D}$ if, and only if it fulfills the following three conditions:

(i) $\delta(u) \leq u$ for all $u \in [0,1]$ and $\delta(1) = 1$,

(ii) δ is non-decreasing,

(iii) δ is 2-Lipschitz, i.e., $|\delta(v) - \delta(u)| \leq 2|v - u|$ holds for all $u, v \in [0,1]$.

As already mentioned in the introduction, according to [4] LSL copulas correspond to special diagonals. In fact, letting $\mathcal{D}^{LSL}$ be defined according to equation (1), the following result holds for an arbitrary diagonal $\delta \in \mathcal{D}$ and S_δ defined according to equation (2) (with the convention $\frac{0}{0} := 0$): S_δ is a copula if, and only if $\delta \in \mathcal{D}^{LSL}$.

Again following [4] and using the fact that Lipschitz continuous functions are differentiable λ-almost everywhere, it is straightforward to check that for fixed $\delta \in \mathcal{D}$ we have $\delta \in \mathcal{D}^{LSL}$ if, and only if the following inequality holds λ-almost everywhere:

$$\delta(x) \leq x\delta'(x) \leq 2\delta(x) \tag{5}$$

This immediately yields that every $\delta \in \mathcal{D}^{LSL}$ fulfills

$$x^2 = \delta_\Pi(x) \leq \delta(x) \leq \delta_M(x) = x$$

for every $x \in [0,1]$, implying that $\Pi \leq C_\delta \leq M$ holds for every $S_\delta \in \mathcal{C}^{LSL}$.

In what follows, the so-called star product (a.k.a. Markov product) $A * B$ of copulas $A, B \in \mathcal{C}$ will play a prominent role. Letting ∂_i denote the partial derivative with respect to the i-th coordinate, $A * B$ is defined by

$$(A * B)(x,y) := \int_{[0,1]} \partial_2 A(x,s) \cdot \partial_1 B(s,y) \, d\lambda(s) \tag{6}$$

for all $x, y \in [0,1]$. It is well known (see [20]) that $A * B$ is a copula, that the star product is in general not commutative, i.e., $A * B \neq B * A$ can hold, that Π is the null- and M the unit element in $(\mathcal{C}, *)$, i.e., $A * \Pi = \Pi * A = \Pi$ and $A * M = M * A = A$ for every $A \in \mathcal{C}$. A copula A is called idempotent, if $A * A = A$ holds. Using Markov kernels it is straightforward to verify that the following identity holds:

$$(A * B)(x,y) := \int_{[0,1]} K_{A^t}(s, [0,x]) K_B(s, [0,y]) \, d\lambda(s) \tag{7}$$

Moreover, according to [27], using disintegration it can be shown that a (version of the) Markov kernel of $A * B$ is given by the standard composition of the Markov kernels of A and B, a concept well-known in the context of

Markov chains in discrete time. In other words, $K_A \circ K_B$, defined by

$$(K_A \circ K_B)(x, F) = \int_{[0,1]} K_B(s, F) K_A(x, dy)\, d\lambda(s) \tag{8}$$

for every $x \in [0,1]$ and $F \in \mathcal{B}[0,1]$ is a (version of the) Markov kernel of $A * B$. For more background on the star product we refer to [2, 20, 27, 28] and the references therein.

3. Markov kernels of LSL copulas and two important examples

We start with recalling the form of the Markov kernel of LSL copulas. Letting $\delta \in \mathcal{D}^{LSL}$ be arbitrary but fixed, then there exists some set $\Lambda \in \mathcal{B}([0,1])$ with $\lambda(\Lambda) = 1$ and some Borel measurable function $\hat{w}_\delta : [0,1] \to [0,2]$ such that for every $x \in \Lambda$ we have $\delta'(x) = \hat{w}_\delta(x)$. Defining $w_\delta : [0,1] \to [0,2]$ (again using the convention $\frac{0}{0} := 0$) by

$$w_\delta(x) = \hat{w}_\delta(x) \mathbf{1}_\Lambda(x) + \frac{\delta(x)}{x} \mathbf{1}_{\Lambda^c}(x)$$

we have that w_δ is measurable, that $w_\delta = \delta'$ holds λ-almost everywhere, and using inequality (5) that $w_\delta(x) \geq \frac{\delta(x)}{x}$ for every $x \in [0,1]$. We will refer to $\hat{w}_\delta$ and w_δ as measurable versions of the derivative of δ. The following result has already been derived in [17], we just recall it for the sake of completeness and since it will be used in the sequel.

Theorem 3.1 ([17])**.** *Let S_δ be an LSL copula for a given $\delta \in \mathcal{D}^{LSL}$ and let w_δ the measurable version of the derivative of δ as constructed above. Then (a version of) the Markov Kernel K_{S_δ} of S_δ is given by*

$$K_{S_\delta}(x, [0,y]) = \begin{cases} \frac{y}{x}\, w_\delta(x) - \frac{y}{x^2}\delta(x) & \text{if } y < x, \\ \frac{1}{y}\, \delta(y) & \text{if } y \geq x. \end{cases} \tag{9}$$

Notice that for fixed $x \in (0,1)$ the conditional distribution function $y \mapsto F_x^\delta(y) := K_{S_\delta}(x, [0,y])$ is Lipschitz continuous on $[0,x)$ and on $[x,1]$. As a direct consequence, the only possible discontinuity point of F_x^δ is $y = x$, i.e., the only point mass the measure $K_{S_\delta}(x, \cdot)$ can have is at $y = x$. Using this simple observation yields the following result going back to [17].

Lemma 3.2 ([17]). *For every LSL copula S_δ the singular mass is given by*

$$sing(S_\delta) = 2 \int\limits_{[0,1]} \frac{\delta(x)}{x} \, d\lambda(x) - 1 \tag{10}$$

In the following example we introduce two important types of diagonals in $\mathcal{D}^{LSL}$, determine explicit expressions for the corresponding LSL copula and its Markov kernel. These diagonals will play a prominent role in Section 5 on concordance of LSL copulas.

Example 3.3. For every $a \in [0,1]$ define the diagonals $l_a, u_a : [0,1] \to [0,1]$ by

$$l_a(x) := \begin{cases} ax & \text{if } x \leq a \\ x^2 & \text{if } x > a \end{cases} \qquad u_a(x) := \begin{cases} \frac{x^2}{a} & \text{if } x \leq a \\ x & \text{if } x > a. \end{cases}$$

Figure 1 depicts both diagonals for the case $a = \frac{1}{2}$. It is straightforward to

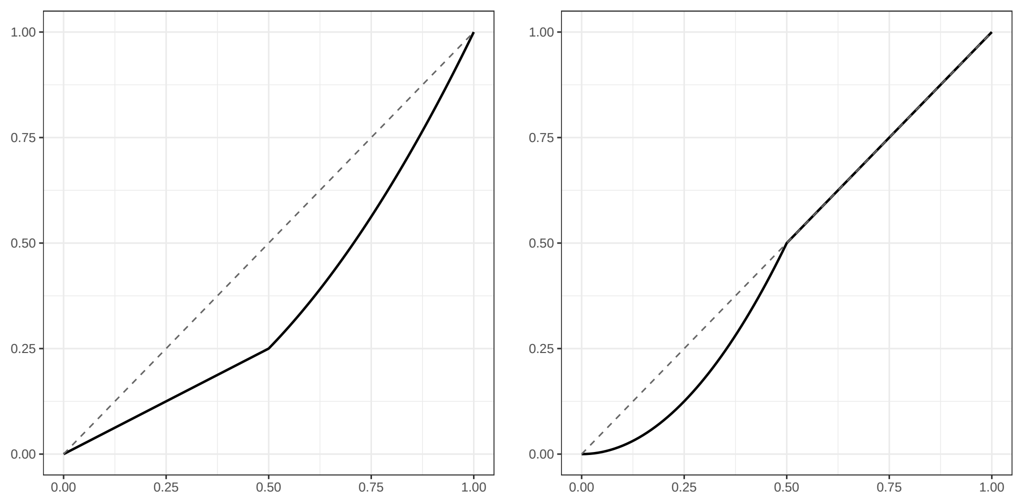

Figure 1: The diagonals l_a, u_a for the case $a = \frac{1}{2}$ as considered in Example 3.3.

verify that $l_a, u_a \in \mathcal{D}^{LSL}$ for every $a \in [0,1]$ and that the LSL copulas S_{l_a}, S_{u_a}

induced by l_a, u_a are given by

$$S_{l_a}(x,y) := \begin{cases} \min\{x,y\}\cdot a & \text{if } \max\{x,y\} \leq a \\ xy & \text{if } \max\{x,y\} > a \end{cases}$$

$$S_{u_a}(x,y) := \begin{cases} \frac{xy}{a} & \text{if } \max\{x,y\} \leq a \\ \min\{x,y\} & \text{if } \max\{x,y\} > a. \end{cases}$$

Calculating the derivatives of l_a, u_a and applying Theorem 3.1 yields

$$K_{S_{l_a}}(x,[0,y]) := \begin{cases} 0 & \text{if } y < x \leq a \\ a & \text{if } x \leq y \leq a \\ y & \text{if } \max\{x,y\} > a \end{cases}$$

$$K_{S_{u_a}}(x,[0,y]) := \begin{cases} 0 & \text{if } y < x, a < x \\ \frac{y}{a} & \text{if } \max\{x,y\} \leq a \\ 1 & \text{if } y \geq x,\ y > a. \end{cases}$$

For the singular masses we obtain $sing(S_{l_a}) = a^2$ as well as $sing(S_{u_a}) = 1-a$. The Markov kernels $K_{S_{l_a}}, K_{S_{u_a}}$ for $a = \frac{1}{2}$ are depicted in Figure 2.

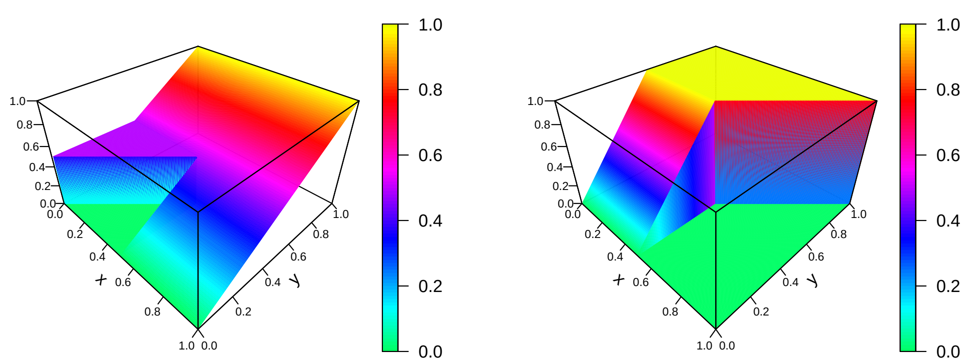

Figure 2: Surface plots of the functions $(x,y) \mapsto K_{S_{l_a}}(x,[0,y])$ and $(x,y) \mapsto K_{S_{u_a}}(x,[0,y])$ as considered in Example 3.3 for the case $a = \frac{1}{2}$.

We complete this short section with some monotonicity properties of LSL copulas. Recall (see [20] and [29]) that a copula $C \in \mathcal{C}$ is said to be

- positively quadrant dependent (PQD) if $C(x,y) \geq \Pi(x,y)$ holds for all $(x,y) \in [0,1]^2$.
- left tail decreasing (LTD) if, for any $y \in [0,1]$, the mapping $(0,1) \to \mathbb{R}$ given by $x \mapsto \frac{C(x,y)}{x}$ is non-increasing.
- stochastically increasing (SI) if, for (a version of) the Markov kernel K_C and any $y \in (0,1)$ the mapping $x \mapsto K_C(x,[0,y])$ is non-increasing.

We already know that every $C \in \mathcal{C}^{LSL}$ fulfills $\Pi \leq S_\delta \leq M$, hence LSL copulas are PQD (also see [4]).
Concerning LTD suppose that $\delta \in \mathcal{D}^{LSL}$ and $y \in (0,1)$ are fixed. Then using inequality (5) obviously the mapping

$$x \mapsto \frac{S_\delta(x,y)}{x} = \begin{cases} y\,\frac{\delta(x)}{x^2} & y \leq x \\ \frac{\delta(y)}{y} & y \geq x \end{cases}$$

is non-increasing, so LSL copulas are LTD.
Finally, the following simple example shows that LSL copulas are not necessarily SI.

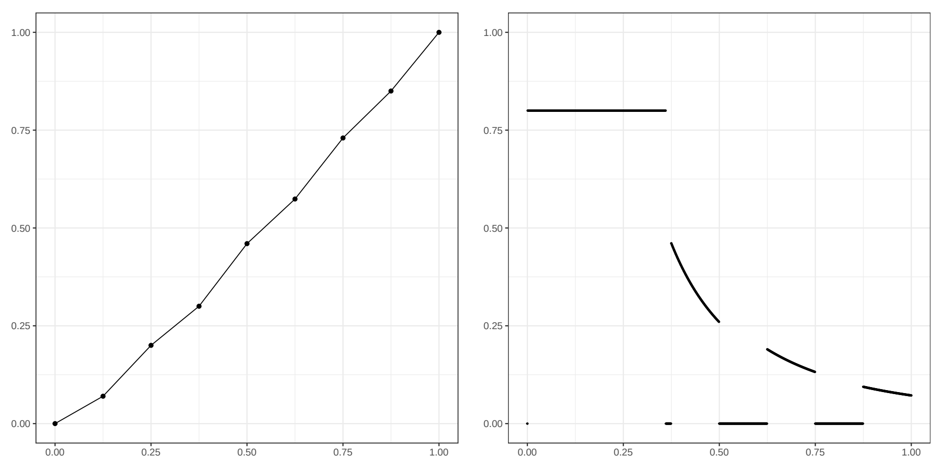

Figure 3: The diagonal $\delta \in \mathcal{D}^{LSL}$ considered in Example 3.4 (left panel) and the mapping $x \mapsto K_{S_\delta}(x,[0,y])$ for $y = 0.36$ (right panel).

Example 3.4. Consider the points $(0,0), (\frac{1}{8}, \frac{7}{100}), (\frac{1}{4}, \frac{1}{5}), (\frac{3}{8}, \frac{3}{10}), (\frac{1}{2}, \frac{23}{50}), (\frac{5}{8}, \frac{287}{500}), (\frac{3}{4}, \frac{73}{100}), (\frac{7}{8}, \frac{17}{20}), (1,1)$ and let $\delta : [0,1] \to [0,1]$ denote the linear interpolation of these points. It is straightforward to verify that $\delta \in \mathcal{D}^{LSL}$ and that the mapping $x \mapsto \frac{\delta(x)}{x}$ is constant on every second of the intervals formed by the x-coordinates of the afore-mentioned points. Figure 3 depicts the diagonal δ and the mapping $x \mapsto K_{S_\delta}(x, [0,y])$ for $y = 0.36$, Figure 4 a sample of the corresponding LSL copula S_δ. Obviously we can find $y \in (0,1)$ such that mapping $x \mapsto K_{S_\delta}(x, [0,y])$ is not non-increasing.

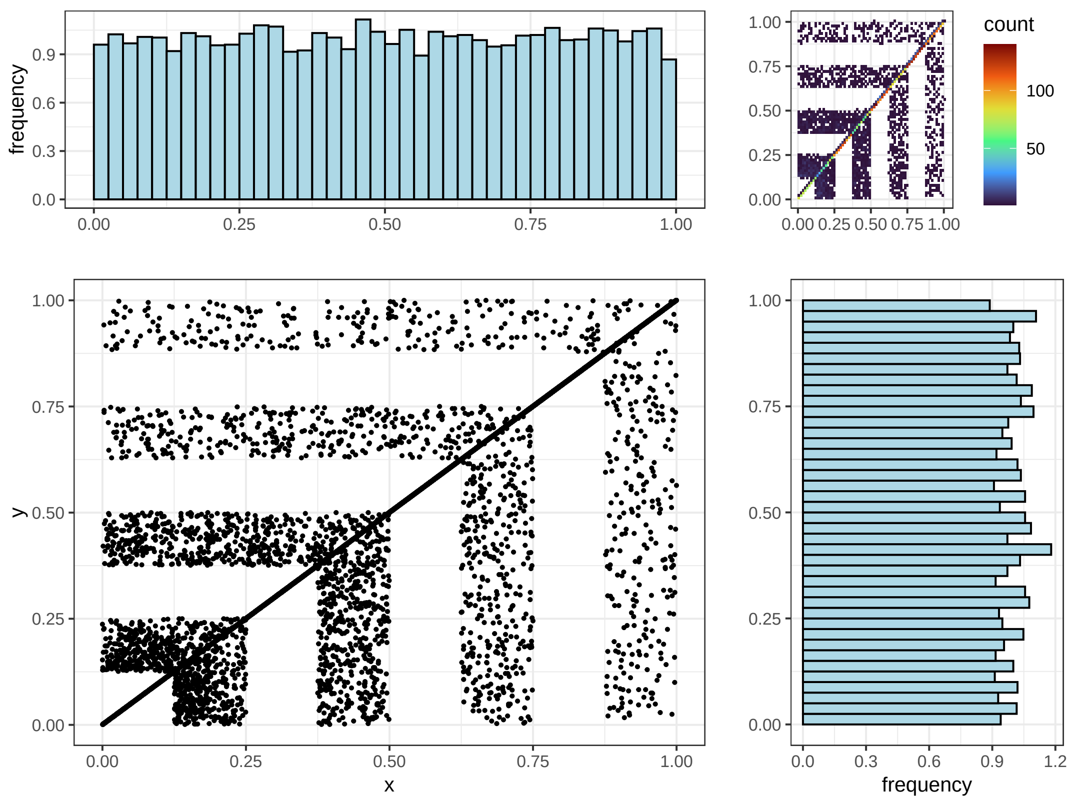

Figure 4: Sample of size $n = 10.000$ of the LSL copula S_δ considered in Example 3.4 (lower left panel); two-dimensional histogram (upper right panel) and marginal histograms (upper left and lower right panel). Data generated by conditional inverse sampling.

4. The star product of LSL copulas

Contrary to Archimedean and Extreme Value copulas (two families also characterized in terms of univariate functions) the family $\mathcal{C}^{LSL}$ of LSL copulas is closed with respect to the star product - the following theorem holds (to keep notation simple we avoid again working with versions of the derivatives since the integral ignores sets of λ-measure 0):

Theorem 4.1. *Suppose that $\delta_1, \delta_2 \in \mathcal{D}^{LSL}$. Then the star product $S_{\delta_1} * S_{\delta_2}$ is given by*

$$(S_{\delta_1} * S_{\delta_2})(x,y) = \begin{cases} \dfrac{x}{y^2}\delta_1(y)\delta_2(y) + xy \displaystyle\int_{[y,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u) & \text{if } y > x, \\ \dfrac{y}{x^2}\delta_1(x)\delta_2(x) + xy \displaystyle\int_{[x,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u) & \text{if } y \leq x. \end{cases} \tag{11}$$

In particular, $S_{\delta_1} * S_{\delta_2}$ is an LSL copula too, i.e. $S_{\delta_1} * S_{\delta_2} \in \mathcal{C}^{LSL}$.

The proof of the theorem can be found in the Appendix.

Since the sets $\mathcal{D}^{LSL}$ and $\mathcal{C}^{LSL}$ are in one-to-one correspondence Theorem 4.1 implies that the star product can be 'translated' to the class $\mathcal{D}^{LSL}$. In fact, the diagonal of the copula $S_{\delta_1} * S_{\delta_2}$ from Theorem 4.1 is obviously given by

$$(S_{\delta_1} * S_{\delta_2})(x,x) = \frac{1}{x}\delta_1(x)\delta_2(x) + x^2 \int_{[x,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u).$$

This motivates the following definition (we use a different symbol to avoid misinterpretations).

Definition 4.2. *For every pair (δ_1, δ_2) of diagonals in $\mathcal{D}^{LSL}$ the star product $\delta_1 \circledast \delta_2$ is defined by*

$$(\delta_1 \circledast \delta_2)(x) := \frac{1}{x}\delta_1(x)\delta_2(x) + x^2 \int_{[x,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u), \tag{12}$$

for every $x \in (0,1]$ as well as $(\delta_1 \circledast \delta_2)(0) := 0$.

Remark 4.3. Theorem 4.1 and the one-to-one correspondence of the families $\mathcal{D}^{LSL}$ and $\mathcal{C}^{LSL}$ mentioned above and in the introduction imply that $\delta_1 \circledast \delta_2 \in \mathcal{D}^{LSL}$ holds - since $S_{\delta_1} * S_{\delta_2}$ is an LSL copula its diagonal is an element of $\mathcal{D}^{LSL}$. Moreover we obviously have the following identity for each pair $(\delta_1, \delta_2) \in \mathcal{D}^{LSL} \times \mathcal{D}^{LSL}$, which we will use various times in what follows:

$$S_{\delta_1} * S_{\delta_2} = S_{\delta_1 \circledast \delta_2} \tag{13}$$

In other words, the mapping $\iota : \mathcal{C}^{LSL} \to \mathcal{D}^{LSL}$ assigning every LSL copula its diagonal (as well as its inverse ι^{-1}) is an isomorphism with respect to the star product.

Considering the diagonals δ_M and δ_Π of M and Π, respectively, obviously the following interrelations hold for every $\delta \in \mathcal{D}^{LSL}$:

$$\delta_\Pi \circledast \delta = \delta \circledast \delta_\Pi = \delta_\Pi$$
$$\delta_M \circledast \delta = \delta \circledast \delta_M = \delta$$

In what follows we will study the limit behavior of sequences $(S_\delta^{*n})_{n\in\mathbb{N}}$ where $S_\delta^{*1} = S_\delta, S_\delta^{*2} = S_\delta * S_\delta, S_\delta^{*3} = S_\delta * S_\delta * S_\delta, \ldots$. Notice that for general copulas A, the sequence $(A^{*n})_{n\in\mathbb{N}}$ does not need to converge - the simplest example being W for which the sequence $(W^{*n})_{n\in\mathbb{N}}$ jumps between W and M. One can, however, show that the sequence $(A^{*n})_{n\in\mathbb{N}}$ is Cesáro convergent (even with respect to a metric stronger than d_∞), see [28] for more information. As we will show, for LSL copulas the sequence $(S_\delta^{*n})_{n\in\mathbb{N}}$ does converge - the following simple but key lemma opens the door to deriving the just mentioned convergence without much technical ado.

Lemma 4.4. *For all $\delta_1, \delta_2 \in \mathcal{D}^{LSL}$ the following inequality holds for every $t \in [0, 1]$:*

$$(\delta_1 \circledast \delta_2)(t) \leq \min\{\delta_1(t), \delta_2(t)\} \tag{14}$$

Proof. To simplify notation write $I := \int\limits_{[t,1]} \left(\frac{\delta_1(u)}{u}\right)' \left(\frac{\delta_2(u)}{u}\right)' d\lambda(u)$, then equation (12) is given by

$$(\delta_1 \circledast \delta_2)(t) = \frac{1}{t}\delta_1(t)\delta_2(t) + t^2 \cdot I.$$

Applying inequality (5) and the fact that $u \mapsto \frac{\delta_1(u)}{u^2}$ is non-increasing on $[t,1]$ yields

$$\begin{aligned}
I &= \int\limits_{[t,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) = \int\limits_{[t,1]} \left(\tfrac{w_{\delta_1}(u)u - \delta_1(u)}{u^2}\right) \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) \\
&\overset{(i)}{\leq} \int\limits_{[t,1]} \left(\tfrac{\frac{2\delta_1(u)}{u}u - \delta_1(u)}{u^2}\right) \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) = \int\limits_{[t,1]} \left(\tfrac{\delta_1(u)}{u^2}\right) \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) \\
&\overset{(ii)}{\leq} \int\limits_{[t,1]} \left(\tfrac{\delta_1(t)}{t^2}\right) \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) = \tfrac{\delta_1(t)}{t^2} \int\limits_{[t,1]} \left(\tfrac{\delta_2(u)}{u}\right)' d\lambda(u) \\
&= \frac{\delta_1(t)}{t^2}\left(1 - \frac{\delta_2(t)}{t}\right). \qquad (15)
\end{aligned}$$

It therefore follows immediately that

$$\begin{aligned}
(\delta_1 \circledast \delta_2)(t) &= \frac{1}{t}\delta_1(t)\delta_2(t) + t^2 \cdot I \leq \frac{1}{t}\delta_1(t)\delta_2(t) + t^2 \cdot \frac{\delta_1(t)}{t^2}\left(1 - \frac{\delta_2(t)}{t}\right) \\
&= \frac{1}{t}\delta_1(t)\delta_2(t) + \delta_1(t) - \frac{1}{t}\delta_1(t)\delta_2(t) = \delta_1(t)
\end{aligned}$$

holds for $t \in (0,1]$. Since the case $(\delta_1 \circledast \delta_2)(t) \leq \delta_2(t)$ follows analogously we obtain the desired inequality

$$\delta_1 \circledast \delta_2(t) \leq \min\{\delta_1(t), \delta_2(t)\}.$$

□

The proof of Lemma 4.4 has the following by-product, which will be key for proving the characterization of idempotent LSL copulas as stated in Theorem 4.9. To simplify notation, for every $\delta \in \mathcal{D}$ and $t \in [0,1]$ (as in the introduction) we will write $\varphi_\delta(t) = \frac{\delta(x)}{t}$; moreover, for $t \in [0,1)$ we set

$$\Lambda_t^\delta := \{u \in (t,1) : \varphi_\delta(u)' > 0\}. \qquad (16)$$

Corollary 4.5. Suppose that $\delta_1, \delta_2 \in \mathcal{D}^{LSL}$ and let $t \in (0,1)$ be arbitrary but fixed. Then the following implications hold:

1. If $\delta_2(t) < t$, then the $\lambda(\Lambda_t^{\delta_2}) > 0$.

2. Suppose that $\delta_2(t) < t$ and that $\delta_1 \circledast \delta_2(t) = \delta_1(t)$ holds. Then for λ-almost every $u \in \Lambda_t^{\delta_2}$ we have $2\,\frac{\delta_1(u)}{u} = \delta_1'(u)$ and $\frac{\delta_1(t)}{t^2} = \frac{\delta_1(u)}{u^2}$.

Proof. Suppose that $\lambda(\Lambda_t^{\delta_2}) = 0$ holds. Then for λ-almost every $s \in [t, 1]$ we would have $\varphi_{\delta_2}'(s) = 0$, implying

$$0 = \int_{[t,1]} \varphi_{\delta_2}' d\lambda = \frac{\delta_2(1)}{1} - \frac{\delta_2(t)}{t} = 1 - \frac{\delta_2(t)}{t}$$

as well as $\delta_2(t) = t$. This proves the first assertion.
Under the assumptions of the second assertion both, inequality (i) and inequality (ii) in the chain of inequalities (15) have to be equalities. Using the first assertion we already know that $\lambda(\Lambda_t^{\delta_2}) > 0$ holds. We can only have equality in inequality (i), if for λ-almost every $u \in \Lambda_t^{\delta_2}$

$$\frac{2\delta_1(u)}{u} = \delta_1'(u)$$

holds; and we can only have equality in inequality (ii), if for λ-almost every $u \in \Lambda_t^{\delta_2}$

$$\frac{\delta_1(t)}{t^2} = \frac{\delta_1(u)}{u^2}$$

is fulfilled. □

Lemma 4.4 has the following nice consequence:

Theorem 4.6. *Suppose that $\delta \in \mathcal{D}^{LSL}$. Then there exists some $\overline{\delta} \in \mathcal{D}^{LSL}$ such that the sequence $(\delta^{\circledast n})_{n \in \mathbb{N}}$ converges to $\overline{\delta}$ uniformly. Moreover we have*

$$\lim_{n \to \infty} d_\infty \left(S_\delta^{*n}, S_{\overline{\delta}}\right) = 0 \tag{17}$$

and the limit $S_{\overline{\delta}}$ is idempotent.

Proof. First of notice that Lemma 4.4 implies that the sequence $(\delta^{\circledast n}(t))_{n \in \mathbb{N}}$ is monotonically non-increasing for every $t \in [0, 1]$. Considering $\delta_\Pi \leq \delta^{\circledast n} \leq \delta_M$ the sequence is bounded so it follows immediately that $(\delta^{\circledast n}(t))_{n \in \mathbb{N}}$ converges to some point $\overline{\delta}(t) \in [t^2, t]$. Since $t \in [0, 1]$ was arbitrary we have that the sequence $(\delta^{\circledast n})_{n \in \mathbb{N}}$ of diagonals converges pointwise to a function $\overline{\delta}$. Being a diagonal, every $\delta^{\circledast n}$ is Lipschitz-continuous with Lipschitz constant 2, so by a standard uniform equicontinuity argument the sequence $(\delta^{\circledast n})_{n \in \mathbb{N}}$ is a

Cauchy sequence with respect to the supremum norm $\|\cdot\|_\infty$ on $[0,1]$. As a direct consequence of Arzelá-Ascoli theorem (see, e.g., [21]) the metric space $(\mathcal{D}, \|\cdot\|_\infty)$ is compact. Since $\mathcal{D}^{LSL}$ is obviously a closed subset of $\mathcal{D}$, compactness of $(\mathcal{D}^{LSL}, \|\cdot\|_\infty)$ follows, which, in turn implies

$$\lim_{n\to\infty} \|\delta^{\circledast n} - \overline{\delta}\|_\infty = 0$$

as well as $\overline{\delta} \in \mathcal{D}^{LSL}$. The very form of LSL copulas according to equation (2) implies that for every $(x,y) \in [0,1]^2$ the sequence $(S_\delta^{*n}(x,y))_{n\in\mathbb{N}}$ converges to $S_{\overline{\delta}}(x,y)$. Using Lipschitz continuity therefore yields equation (17) and it remains to prove idempotence, which can easily be done as follows. Convergence of the sequence $(S_{\delta^{*n}})_{n\in\mathbb{N}}$ to $S_{\overline{\delta}}$ with respect to d_∞ implies Cesáro convergence, i.e.,

$$\lim_{n\to\infty} d_\infty\left(\frac{1}{n}\sum_{i=1}^{n} S_{\delta^{*i}}, S_{\overline{\delta}}\right) = 0$$

holds. Applying [28, Theorem 2] and using the fact that convergence with respect to the metric D_1 implies convergence with respect to d_∞ yields that $S_{\overline{\delta}}$ is idempotent. □

Remark 4.7. Theorem 4.6 has the following direct translation/application to Markov chains: Suppose that $X_0, X_1, X_2, \ldots$ is a (stationary) Markov chain on the space $([0,1], \mathcal{B}([0,1]))$ such that each X_i is uniformly distributed on $[0,1]$ and (X_i, X_{i+1}) has copula S_δ for every $i \in \{0,1,2,\ldots\}$. Then the pair (X_0, X_n) has copula S_δ^{*n} and we have $\lim_{n\to\infty} d_\infty(S_\delta^{*n}, S_{\overline{\delta}}) = 0$ for some $\overline{\delta} \in \mathcal{D}^{LSL}$. Recalling the stochastic interpretation of LSL copulas mentioned in the introduction this means that the long-term behavior of such dependence structures is fully determined.

Example 4.8. We return to the diagonal u_a from Example 3.3. Obviously the corresponding LSL copula S_{u_a} is the ordinal sum of $\langle \Pi, M\rangle$ with respect to $\langle 0, a, 1\rangle$ (see [5, 20] for background on ordinal sums) and as such idempotent.

It turns out that all idempotent LSL copulas are of the form S_{u_a} for some $a \in [0,1]$, so the family of idempotent LSL copulas is quite small and fully determined by one parametric function. Notice that, in contrast, general idempotent copulas can be very diverse and complex - in fact, there are idempotent copulas with fractal support, see [27].

Theorem 4.9. *The following two conditions are equivalent for $\delta \in \mathcal{D}^{LSL}$:*

1. $\delta * \delta = \delta$.
2. *There exists some $a \in [0,1]$ such that $\delta = u_a$.*

Proof. In Example 4.8 it has already been mentioned that every LSL copula S_{u_a} is idempotent, so it suffices to show that the first assertion implies the second one. Consider the set

$$F_\delta := \{t \in (0,1) : \delta(t) = t\}$$

and distinguish two cases:
(i) $F_\delta = \emptyset$: In this case for every $t \in (0,1)$ we have $\delta(t) < t$ as well as $\delta \circledast \delta(t) = \delta(t)$. Applying Corollary 4.5 immediately yields $\lambda(\Lambda_t^\delta) > 0$ - in fact, in this case we even have $\lambda(\Lambda_s^\delta) > 0$ for every $s \in (t,1)$, implying that Λ_t^δ has positive mass arbitrarily close to 1. Moreover, Corollary 4.5 implies that for λ-almost every $s \in \Lambda_t^\delta$ we have

$$0 < \varphi'_\delta(s) = \frac{s\delta'(s) - \delta(s)}{s^2} = \frac{\delta(s)}{s^2} = \frac{\delta(t)}{t^2}.$$

Monotonicity of $x \mapsto \frac{\delta(x)}{x^2} = \eta_\delta(x)$ on $[0,1]$ therefore implies

$$\frac{\delta(t)}{t^2} = \frac{\delta(x)}{x^2} = \frac{\delta(s)}{s^2}$$

for every $x \in [t,s]$. Considering $\Lambda_t^\delta \supseteq \Lambda_s^\delta$ together with the facts that Λ_t^δ has positive mass arbitrarily close to 1 and that $\frac{\delta(1)}{1} = 1$ it follows that the function $s \mapsto \frac{\delta(s)}{s^2}$ is constant on the interval $[t,1]$. Considering that $t \in (0,1)$ was arbitrary we conclude that $s \mapsto \frac{\delta(s)}{s^2}$ must be constant on the whole interval $[0,1]$, which directly yields $\delta = u_1$ and completes the proof for $F_\delta = \emptyset$.
(ii) If $F_\delta \neq \emptyset$ then according to [4] for every $s \in F_\delta$ we even have $[s,1] \subseteq F_\delta$. Set $a_0 := \inf\{t \in (0,1) : \delta(t) = t\}$. Since for $a_0 = 0$ it follows that $F_\delta = [0,1]$, implying $\delta = u_0$, it suffices to consider $a_0 > 0$. Proceeding analogously to the proof of (i) we conclude that the function $s \mapsto \frac{\delta(s)}{s^2}$ is constant on the interval $[0, a_0]$, which finally yields $\delta = u_{a_0}$. ∎

At the beginning of Section 4 we have already mentioned that $\mathcal{C}^{LSL}$ is closed with respect to the star product. We close this section with showing that the star product of two non-LSL copulas may be a LSL copula and prove the somewhat surprising fact that the star product of every Marshall-Olkin copula $M_{\alpha,\beta}$ with its transpose is an LSL copula. Recall that for $\alpha, \beta \in [0,1]$ the Marshall-Olkin copula $M_{\alpha,\beta}$ is given by

$$M_{\alpha,\beta}(u,v) = \min\{u^{1-\alpha}v, uv^{1-\beta}\},$$

so $M_{\alpha,\beta}$ is in general not an LSL copula. According to [9] $M_{\beta,\alpha} * M_{\alpha,\beta}$ is given by

$$M_{\beta,\alpha} * M_{\alpha,\beta} = \begin{cases} \Pi + \frac{\alpha^2}{1-2\alpha}\Pi\left(1 - (\Pi)^{\frac{\beta-2\alpha\beta}{\alpha}} M^{\frac{2\alpha\beta-\beta}{\alpha}}\right) & \alpha \notin \{0, \frac{1}{2}, 1\}, \\ \Pi & \alpha = 0, \\ \Pi + \frac{\beta}{2}\Pi\left(\log(M) - \log(\Pi)\right) & \alpha = \frac{1}{2}, \\ M_{\beta,\beta} & \alpha = 1. \end{cases} \tag{18}$$

It is straightforward to verify that for all $x \in (0,1]$ the mapping

$$t \mapsto (M_{\beta,\alpha} * M_{\alpha,\beta})(t,x) = \begin{cases} t\left(x + \frac{\alpha^2}{1-2\alpha}x - \frac{\alpha^2}{1-2\alpha}x^{\frac{\beta-2\alpha\beta}{\alpha}+1}\right) & \alpha \notin \{0, \frac{1}{2}, 1\}, \\ tx & \alpha = 0, \\ t\left(x - \frac{\beta}{2}x\log(x)\right) & \alpha = \frac{1}{2}, \\ tx^{1-\beta} & \alpha = 1, \end{cases}$$

is linear on $[0,x]$. Considering symmetry of $M_{\beta,\alpha} * M_{\alpha,\beta}$ it follows that $t \mapsto M_{\beta,\alpha} * M_{\alpha,\beta}(x,t)$ is linear on $[0,x]$ as well. We have therefore shown the following result:

Theorem 4.10. *For every Marshall-Olkin copula $M_{\alpha,\beta}$ the star product $M_{\beta,\alpha} * M_{\alpha,\beta}$ is an LSL copula.*

For Marshall-Olkin copulas $M_{\alpha,\beta}$ calculating the limit behavior of star product iterations might seem out of reach - combining, however, the results of this section we obtain the following corollary:

Corollary 4.11. For every Marshall-Olkin copula $M_{\alpha,\beta}$ the sequence of iterated star products $((M_{\beta,\alpha} * M_{\alpha,\beta})^{*n})_{n\in\mathbb{N}}$ converges to some idempotent LSL copula.

5. Concordance of LSL copulas

We conclude this paper by studying concordance of LSL copulas and investigating the exact region Ω^{LSL} determined by Kendall's τ and Spearman's ρ, which is given by

$$\Omega^{LSL} := \left\{(\tau(S_\delta), \rho(S_\delta)) : \, S_\delta \in \mathcal{C}^{LSL}\right\}. \tag{19}$$

Recall that given a pair (X, Y) of random variables with continuous joint distribution function H both, Kendall's τ and Spearman's ρ only depend on the unique copula C underlying (X, Y) and the following formulas hold (see [20, 23]):

$$\tau(C) := 4 \int\limits_{[0,1]^2} C(u,v)\, d\mu_C(u,v) - 1$$
$$\rho(C) := 12 \int\limits_{[0,1]^2} C(u,v)\, d\lambda_2(u,v) - 3. \tag{20}$$

5.1. Kendall's τ and Spearman's ρ

For LSL copulas the formulas (20) boil down to integrals only involving the corresponding diagonal, the subsequent result holds. Since these formulas have already been derived by Durante in [3] (where instead of δ the author works with the function $f(t) = \frac{\delta(t)}{t}$) we only include the derivations in the appendix for the sake of completeness.

Lemma 5.1. *For every LSL copula $S_\delta \in \mathcal{C}^{LSL}$ Spearman's ρ and Kendall's τ are given by*

$$\rho(S_\delta) = 12 \int\limits_{[0,1]} \delta(x) x \, d\lambda(x) - 3. \tag{21}$$

$$\tau(S_\delta) = 4 \int\limits_{[0,1]} \frac{\delta(x)^2}{x} \, d\lambda(x) - 1. \tag{22}$$

Remark 5.2. Since some other measures of association might also be of interest in the context of applications, Lemma Appendix A.1 gathers the resulting formulas for Gini's γ, Spearman's footrule ϕ and Blomqvist's β of LSL copulas.

Example 5.3. We again return to the diagonals $l_a, u_a \in \mathcal{D}^{LSL}$ considered in Example 3.3. Applying Lemma 5.1 directly yields

$$\tau(S_{l_a}) = \rho(S_{l_a}) = a^4$$

$$\tau(S_{u_a}) = 1 - a^2, \;\; \rho(S_{u_a}) = 1 - a^3$$

It is well-known (and also follows directly from equation (20)) that Spearman's ρ preserves convex combinations, i.e.,

$$\rho(\alpha A + (1-\alpha)B) = \alpha\rho(A) + (1-\alpha)\rho(B)$$

holds for $\alpha \in [0,1]$ and $A, B \in \mathcal{C}$.

For Kendall's τ the situation is different, in general it does neither preserve convex combinations, not even

$$\tau(\alpha A + (1-\alpha)B) \leq \alpha\tau(A) + (1-\alpha)\tau(B)$$

needs to hold. In fact, a straightforward calculation (also see [10]) shows that for the Fréchet family

$$\mathcal{F} := \{\alpha W + \beta M + (1-\alpha-\beta)\Pi : \, \alpha, \beta \in [0,1], \alpha + \beta \leq 1\}$$

we have

$$\tau(\alpha W + (1-\alpha-\beta)\Pi + \beta M) = \tfrac{(\beta-\tau)(\beta+\alpha+2)}{3},$$

hence, considering $\beta = 0$, $\alpha = \frac{1}{4}$, we obtain

$$\tau\left(\tfrac{1}{4}\,W + \tfrac{3}{4}\,\Pi\right) > \tfrac{1}{4}\,\tau\,(W) + \tfrac{3}{4}\,\tau\,(\Pi)\,.$$

For LSL copulas, however, Kendall's τ interpreted as function mapping $\mathcal{C}^{LSL}$ to $[0,1]$ is strictly convex as the following result shows:

Lemma 5.4. *For $\delta_1, \delta_2 \in \mathcal{D}^{LSL}$ with $\delta_1 \neq \delta_2$ and $\alpha \in [0,1]$ the following inequality holds:*

$$\tau(\alpha S_{\delta_1} + (1-\alpha)S_{\delta_2}) < \alpha\tau(S_{\delta_1}) + (1-\alpha)\tau(S_{\delta_2}) \tag{23}$$

Proof. If suffices to prove the result for $\alpha = \frac{1}{2}$. Suppose that $\delta_1, \delta_2 \in \mathcal{D}^{LSL}$

fulfill $\delta_1 \neq \delta_2$ and set $\tilde{\delta} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$. Strict convexity of the mapping $x \mapsto x^2$ in combination with Lipschitz continuity of diagonals and the assumption $\delta_1 \neq \delta_2$ yields

$$\begin{aligned}
\tau\left(S_{\tilde{\delta}}\right) &= 4 \int\limits_{[0,1]} \tfrac{1}{x}\tilde{\delta}(x)^2 \, d\lambda(x) - 1 = 4 \int\limits_{[0,1]} \tfrac{1}{x}\left(\tfrac{1}{2}\delta_1(x) + \tfrac{1}{2}\delta_2(x)\right)^2 \, d\lambda(x) - 1 \\
&< 4 \int\limits_{[0,1]} \tfrac{1}{x}\left(\tfrac{1}{2}\delta_1^2(x) + \tfrac{1}{2}\delta_2^2(x)\right) \, d\lambda(x) - 1 \\
&= 4 \int\limits_{[0,1]} \tfrac{1}{2}\tfrac{\delta_1^2(x)}{x} \, d\lambda(x) - \tfrac{1}{2} + 4 \int\limits_{[0,1]} \tfrac{1}{2}\tfrac{\delta_2^2(x)}{x} \, d\lambda(x) - \tfrac{1}{2} \\
&= \tfrac{1}{2}\left(4 \int\limits_{[0,1]} \tfrac{\delta_1^2(x)}{x} \, d\lambda(x) - 1\right) + \tfrac{1}{2}\left(4 \int\limits_{[0,1]} \tfrac{\delta_2^2(x)}{x} \, d\lambda(x) - 1\right) \\
&= \tfrac{1}{2}\tau\left(S_{\delta_1}\right) + \tfrac{1}{2}\tau\left(S_{\delta_2}\right).
\end{aligned}$$

□

5.2. The τ-ρ-region Ω^{LSL} determined by $\mathcal{C}^{LSL}$

Building upon the formulas for τ and ρ according to Lemma 5.1 we now study the exact region Ω^{LSL} determined by Kendall's τ and Spearman's ρ of LSL copulas and defined by

$$\Omega^{LSL} := \left\{(\tau(S_\delta), \rho(S_\delta)) : \, S_\delta \in \mathcal{C}^{LSL}\right\}. \tag{24}$$

Possibly triggered by the paper [23] in which the exact τ-ρ-region for the full class $\mathcal{C}$ was derived, several papers on the regions determined by pairs of dependence measures (considering the full class $\mathcal{C}$ or specific important subclasses) appeared in the past ten years. For more details we refer, e.g., to the papers [13, 14, 15, 16, 18] and the references therein.

Considering that $\Pi \leq S_\delta \leq M$ holds for every $S_\delta \in \mathcal{C}^{LSL}$ we obviously have $\Omega^{LSL} \subseteq [0,1]^2$. Due to countless simulations based on piecewise linear diagonals δ we conjecture that Ω^{LSL} is given by

$$R = \left\{(x,y) \in [0,1]^2 : x \leq y \leq 1 - (1-x)^{\frac{3}{2}}\right\}. \tag{25}$$

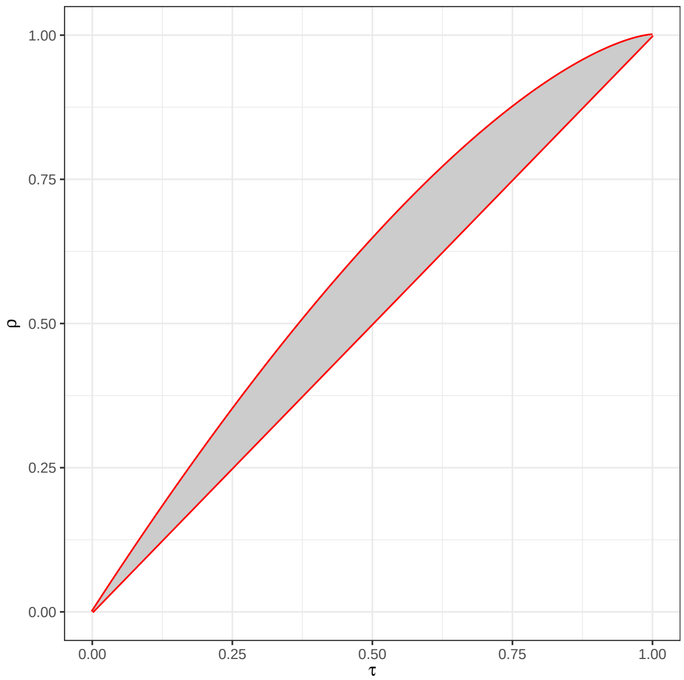

Figure 5: The conjectured τ-ρ-region Ω^{LSL}.

In other words, writing

$$\Phi_l : [0,1] \to [0,1], \quad \Phi_l(x) = x,$$
$$\Phi_u : [0,1] \to [0,1], \quad \Phi_u(x) = 1 - (1-x)^{\frac{3}{2}}$$

we conjecture that

$$\rho(S_\delta) \in [\Phi_l(\tau(S_\delta)), \Phi_u(\tau(S_\delta))] = \left[\tau(S_\delta),\, 1 - (1-\tau(S_\delta))^{\frac{3}{2}}\right]$$

holds for every $S_\delta \in \mathcal{C}^{LSL}$. The conjectured set R is depicted in Figure 5. We have only been able to prove the lower inequality and to show that it is sharp (i.e., best possible), the upper one remains an open question. We now provide a proof for the lower bound, verify its sharpness, and then, despite not knowing the upper bound, show that Ω^{LSL} is convex and compact.

Theorem 5.5. *For every $S_\delta \in \mathcal{C}^{LSL}$ the inequality*

$$\tau(S_\delta) \leq \rho(S_\delta) \tag{26}$$

holds. Moreover, inequality (26) is sharp, i.e., for every $x \in [0,1]$ there exists some LSL copula S_δ fulfilling $\tau(S_\delta) = \rho(S_\delta) = x$.

Proof. Let $S_\delta \in \mathcal{C}^{LSL}$ be arbitrary but fixed. According to [8] the following identity for $\rho(C) - \tau(C)$ holds for every copula $C \in \mathcal{C}$ (notice that the integrand may only be defined on a set $E \in \mathcal{B}([0,1]^2)$ fulfilling $\lambda_2(E) = 1$):

$$\tfrac{1}{4}(\rho(C) - \tau(C)) = \int\limits_{[0,1]^2} \left(C(x,y) - x\tfrac{\partial C(x,y)}{\partial x} - y\tfrac{\partial C(x,y)}{\partial y} + \tfrac{\partial C(x,y)}{\partial x}\tfrac{\partial C(x,y)}{\partial y} \right) d\lambda_2(x,y). \tag{27}$$

Using the fact that LSL copulas are symmetric, using Markov kernels equation (27) boils down to

$$\begin{aligned} \tfrac{1}{4}(\rho(S_\delta) - \tau(S_\delta)) \;\; &= \int\limits_{[0,1]^2} \big(S_\delta(x,y) + K_{S_\delta}(x,[0,y])(K_{S_\delta}(y,[0,x]) - x) \\ &\qquad\qquad - yK_{S_\delta}(y,[0,x]) \big)\, d\lambda_2(x,y). \end{aligned}$$

For proving $\tau(S_\delta) \leq \rho(S_\delta)$ it therefore suffices to show that the last integrand is non-negative, which can be done as follows: For $y < x$ from the set E it follows that

$$\begin{aligned} S_\delta(x,y) + &K_{S_\delta}(x,[0,y])(K_{S_\delta}(y,[0,x]) - x) - yK_{S_\delta}(y,[0,x]) \\ &= y\frac{\delta(x)}{x} + K_{S_\delta}(x,[0,y])\left(\tfrac{\delta(x)}{x} - x\right) - y\tfrac{\delta(x)}{x} \\ &= K_{S_\delta}(x,[0,y])\left(\tfrac{\delta(x)-x^2}{x}\right) \\ &\geq 0. \end{aligned}$$

The case $y > x$ follows directly from the symmetry of LSL copulas, so the proof of inequality (26) is complete. The assertion on sharpness is a direct consequence of Example 5.3, since for every $a \in [0,1]$ we have $\tau(S_{l_a}) = \rho(S_{l_a}) = a^4$.

□

The next theorem shows that the class of copulas attaining the lower bound is very small.

Theorem 5.6. *Within the class $\mathcal{C}^{LSL}$ the only copulas for which we have $\tau(S_\delta) = \rho(S_\delta)$ are the copulas of the form S_{l_a} according to Example 5.3.*

Proof. According to the proof of the previous theorem, the condition $\tau(S_\delta) = \rho(S_\delta)$ is equivalent to having

$$S_\delta(x, y) + K_{S_\delta}(x, [0, y])(K_{S_\delta}(y, [0, x]) - x) - yK_{S_\delta}(y, [0, x]) = 0$$

for λ_2-almost all $(x, y) \in [0, 1]^2$. As a direct consequence, for λ_2-almost all (x, y) with $y < x$

$$K_{S_\delta}(x, [0, y]) \left(\tfrac{\delta(x)-x^2}{x}\right) = 0 \tag{28}$$

has to hold. Set $b := \inf \{x \in (0, 1) : \delta(x) = x^2\}$. If $b = 0$ then $\delta = \delta_\Pi$ as well as $S_\delta = \Pi = S_{l_0}$ follows. If $b > 0$ then for λ_2-almost all $(x, y) \in (0, b)^2$ with $y < x$ we have

$$\frac{y}{x} w_\delta(x) - \frac{y}{x^2}\delta(x) = 0,$$

which is equivalent to the condition that

$$w_\delta(x) = \frac{\delta(x)}{x}$$

holds for λ-almost $x \in (0, b)$. Using Lipschitz continuity of δ and solving this first order differential equation with the boundary condition $\delta(b) = b^2$ directly yields

$$\delta(x) = b\, x$$

for $x \in [0, b]$, which completes the proof since we have shown $\delta = l_b$. □

Theorem 5.7. *The set Ω^{LSL} is convex and compact.*

Proof. Simplifying notation we will write $\tau(\delta) := \tau(S_\delta)$ and $\rho(\delta) := \rho(S_\delta)$ for every $\delta \in \mathcal{D}^{LSL}$ throughout the rest of the proof. Continuity of concordance measures and the fact that continuous images of compact sets are compact imply that Ω^{LSL} is compact. It therefore remains to show convexity, which can be done as follows: Consider two points $(\tau(\delta_1), \rho(\delta_1))$, $(\tau(\delta_2), \rho(\delta_2))$ in

Ω^{LSL} and, without loss of generality assume $\delta_1 \neq \delta_2$. We want to show the existence of some $\delta \in \mathcal{D}^{LSL}$ fulfilling

$$(\tau(\delta), \rho(\delta)) = \tfrac{1}{2}\left(\tau(\delta_1), \rho(\delta_1)\right) + \tfrac{1}{2}\left(\tau(\delta_2), \rho(\delta_2)\right). \tag{29}$$

Notice that finding such a δ is trivial if either $\rho(\delta_1) = \rho(\delta_2)$ or $\tau(\delta_1) = \rho(\delta_1)$ and $\tau(\delta_2) = \rho(\delta_2)$ holds, since in this case a convex combination of δ_1 and δ_2 will do. In what follows we will therefore assume that none of these two conditions holds. Convexity of $\mathcal{D}^{LSL}$ implies that $\delta_3 := \frac{1}{2}(\delta_1 + \delta_2)$ is an element of $\mathcal{D}^{LSL}$. Moreover we obviously have

$$(\tau(\delta_3), \rho(\delta_3)) = \left(\tau(\delta_3), \tfrac{1}{2}(\rho(\delta_1) + \rho(\delta_2))\right),$$

and Lemma 5.4 implies $\tau(\delta_3) < \frac{1}{2}(\tau(\delta_1) + \tau(\delta_2))$. For $\alpha, a \in [0,1]$ define the function $h_{\alpha,a} : [0,1] \to [0,1]$ by

$$h_{\alpha,a}(t) := (1-\alpha)\delta_3(t) + \alpha l_a(t).$$

Then obviously $h_{\alpha,a} \in \mathcal{D}^{LSL}$, and for every pair of sequences $(\alpha_n)_{n\in\mathbb{N}}, (a_n)_{n\in\mathbb{N}}$ in $[0,1]$ converging to $\alpha \in [0,1]$ and $a \in [0,1]$, respectively, we have that $(h_{\alpha_n,a_n})_{n\in\mathbb{N}}$ converges uniformly to $h_{\alpha,a}$. As a consequence, the mapping $\iota : [0,1]^2 \to [0,1]^2$, defined by

$$\iota(\alpha, a) = (\tau(h_{\alpha,a}), \rho(h_{\alpha,a}))$$

is continuous. For every $a \in [0,1]$ defining Γ_a by

$$\Gamma_a := \left\{(\tau(h_{\alpha,a}), \rho(h_{\alpha,a})) : \alpha \in [0,1]\right\},$$

it therefore follows that Γ_a is compact and connected, and that Γ_a contains the points $(\tau(\delta_3), \rho(\delta_3))$ and $(\tau(l_a), \rho(l_a))$ (see Figure 6 for an illustration). Continuity in a implies the existence of some $a_0 \in [0,1]$ fulfilling

$$\tfrac{1}{2}\left(\tau(\delta_1), \rho(\delta_1)\right) + \tfrac{1}{2}\left(\tau(\delta_2), \rho(\delta_2)\right) \in \Gamma_{a_0}.$$

Having that, by construction of Γ_a there exists some α_0 with

$$(\tau(h_{\alpha_0,a_0}), \rho(h_{\alpha_0,a_0})) = \tfrac{1}{2}\left(\tau(\delta_1), \rho(\delta_1)\right) + \tfrac{1}{2}\left(\tau(\delta_2), \rho(\delta_2)\right).$$

In other words, the diagonal $h_{\alpha_0,a_0} \in \mathcal{D}^{LSL}$ fulfills equation (29), which completes the proof. □

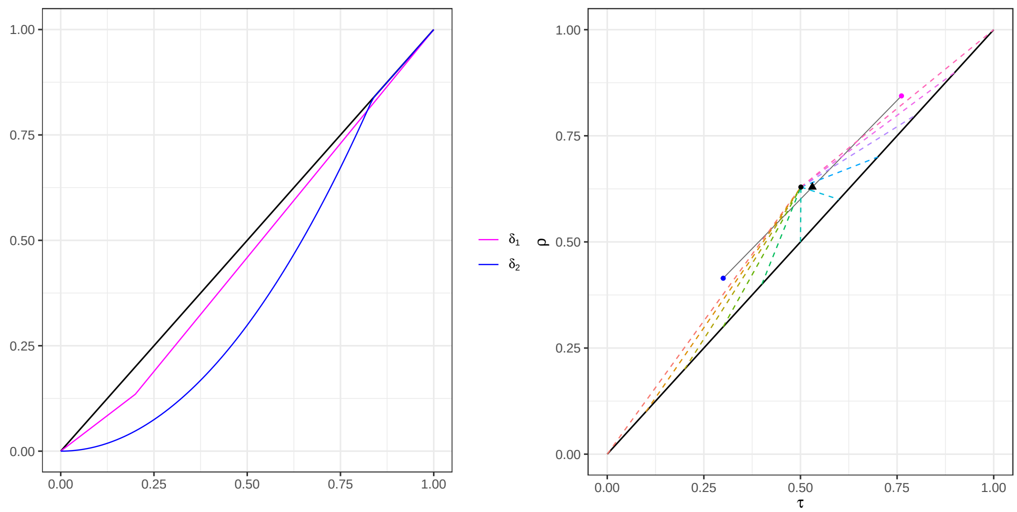

Figure 6: Illustration of the construction used in the proof of Theorem 5.7. The black and the blue points denote $(\tau(\delta_1), \rho(\delta_1))$ and $(\tau(\delta_2), \rho(\delta_2))$, respectively, the black triangle their arithmetic mean. The black point denotes $(\tau(\delta_3), \rho(\delta_3))$, the dashed lines originating from it the sets Γ_a with $a \in \{0, \frac{1}{10}, \ldots, \frac{9}{10}, 1\}$.

6. Conclusion

Working with Markov kernels we provided various new results on bivariate LSL copulas. Considering that standard classes of bivariate copulas like the Archimedean and the Extreme-Value family are not closed with respect to the star product, the key observation of our paper is that the star product of two LSL copulas is again an LSL copula.

Building upon this fact we proved that for every LSL copula S_δ the sequence $(S_\delta^{*n})_{n\in\mathbb{N}}$ of star product iterates converges with respect to d_∞ to some idempotent LSL copula and provided a handy characterization of all idempotent LSL copulas.

In the second part of the paper we studied how different Kendall's τ and Spearman's ρ of LSL copulas may be. We stated a conjecture on the exact region Ω^{LSL} determined by these two concordance measures, proved the lower inequality and verified its sharpness. Proving or falsifying the upper bound, however, remains an open problem, which we plan to tackle in the near future.

Appendix A.

Proof of Theorem 4.1. Suppose that $0 \le x < y \le 1$. Then using symmetry of LSL copulas and equation (7) we obtain

$$
\begin{aligned}
(S_{\delta_1} * S_{\delta_2})(x,y) &= \int\limits_{[0,x]} \tfrac{\delta_1(x)}{x}\tfrac{\delta_2(y)}{y}\, d\lambda(t) + \int\limits_{[x,y]} \left(\tfrac{x}{t} w_{\delta_1}(t) - \tfrac{x}{t^2}\delta_1(t)\right) \tfrac{\delta_2(y)}{y}\, d\lambda(t) \\
&\quad + \int\limits_{[y,1]} \left(\tfrac{x}{t} w_{\delta_1}(t) - \tfrac{x}{t^2}\delta_1(t)\right)\left(\tfrac{y}{t} w_{\delta_2}(t) - \tfrac{y}{t^2}\delta_2(t)\right)\, d\lambda(t) \\
&= \tfrac{1}{xy}\delta_1(x)\delta_2(y)x + \tfrac{x}{y}\delta_2(y) \int\limits_{[x,y]} \left(\tfrac{\delta_1(t)}{t}\right)'\, d\lambda(t) \\
&\quad + xy \int\limits_{[y,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)'\, d\lambda(t) \\
&= \tfrac{1}{y}\delta_1(x)\delta_2(y) + \tfrac{x}{y}\delta_2(y)\left(\tfrac{\delta_1(y)}{y} - \tfrac{\delta_1(x)}{x}\right) + \\
&\quad + xy \int\limits_{[y,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)'\, d\lambda(t) \\
&= \frac{x}{y^2}\delta_1(y)\delta_2(y) + xy \int\limits_{[y,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)'\, d\lambda(t).
\end{aligned}
$$

Analogously for $0 \le y \le x \le 1$ we have

$$
\begin{aligned}
(S_{\delta_1} * S_{\delta_2})(x,y) &= \int\limits_{[0,y]} \tfrac{\delta_1(x)}{x}\tfrac{\delta_2(y)}{y}\, d\lambda(t) + \int\limits_{[y,x]} \tfrac{\delta_1(x)}{x} \left(\tfrac{y}{t} w_{\delta_2}(t) - \tfrac{y}{t^2}\delta_2(t)\right)\, d\lambda(t) \\
&\quad + \int\limits_{[x,1]} \left(\tfrac{x}{t} w_{\delta_1}(t) - \tfrac{x}{t^2}\delta_1(t)\right)\left(\tfrac{y}{t} w_{\delta_2}(t) - \tfrac{y}{t^2}\delta_2(t)\right)\, d\lambda(t) \\
&= \tfrac{1}{xy}\delta_1(x)\delta_2(y)y + \tfrac{y}{x}\delta_1(x) \int\limits_{[y,x]} \left(\tfrac{\delta_2(t)}{t}\right)'\, d\lambda(t) \\
&\quad + xy \int\limits_{[x,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)'\, d\lambda(t)
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{x}\delta_1(x)\delta_2(y) + \tfrac{y}{x}\delta_1(x)\left(\tfrac{\delta_2(x)}{x} - \tfrac{\delta_2(y)}{y}\right) \\
&\quad + xy \int\limits_{[x,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)' \, d\lambda(t) \\
&= \frac{y}{x^2}\delta_1(x)\delta_2(x) + xy \int\limits_{[x,1]} \left(\tfrac{\delta_1(t)}{t}\right)' \left(\tfrac{\delta_2(t)}{t}\right)' \, d\lambda(t).
\end{aligned}
$$

Considering that for fixed $x \in (0,1]$ the two mappings

$$
t \mapsto (S_{\delta_1} * S_{\delta_2})(t,x) = t\Bigg(\tfrac{1}{x^2}\delta_1(x)\delta_2(x) + x \int\limits_{[x,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u)\Bigg),
$$

$$
t \mapsto (S_{\delta_1} * S_{\delta_2})(x,t) = t\Bigg(\tfrac{1}{x^2}\delta_1(x)\delta_2(x) + x \int\limits_{[x,1]} \left(\tfrac{\delta_1(u)}{u}\right)' \left(\tfrac{\delta_2(u)}{u}\right)' \, d\lambda(u)\Bigg).
$$

are obviously on linear $[0,x]$ it follows that $S_{\delta_1} * S_{\delta_2} \in \mathcal{C}^{LSL}$ and the proof is complete. □

Proof of Theorem 5.1. Plugging in S_δ and using symmetry we obtain

$$
\begin{aligned}
\rho(S_\delta) &= 12 \int\limits_{[0,1]^2} S_\delta(x,y)\, d\lambda_2(x,y) - 3 = 12 \int\limits_{[0,1]} \int\limits_{[0,1]} S_\delta(x,y)\, d\lambda(y) d\lambda(x) - 3 \\
&= 12 \int\limits_{[0,1]} \Bigg(\int\limits_{[0,x]} y \tfrac{\delta(x)}{x}\, d\lambda(y) + \int\limits_{[x,1]} x \tfrac{\delta(y)}{y}\, d\lambda(y) \Bigg) d\lambda(x) - 3 \\
&= 12 \int\limits_{[0,1]} \Bigg(2 \cdot \int\limits_{[0,x]} y \tfrac{\delta(x)}{x}\, d\lambda(y) \Bigg) d\lambda(x) - 3 = 24 \int\limits_{[0,1]} \tfrac{\delta(x)}{x} \int\limits_{[0,x]} y\, d\lambda(y) d\lambda(x) - 3 \\
&= 24 \int\limits_{[0,1]} \tfrac{\delta(x)}{x} \tfrac{x^2}{2} d\lambda(x) - 3 = 12 \int\limits_{[0,1]} \delta(x) x\, d\lambda(x) - 3.
\end{aligned}
$$

In order to derive a simple expression for Kendall's τ of LSL copulas we will use the subsequent handy identity which can be proved via disintegration (see, e.g., [20]):

$$\int_{[0,1]^2} B \, d\mu_A = \frac{1}{2} - \int_{[0,1]^2} K_B(x,[0,y]) K_{A^t}(y,[0,x]) \, d\lambda_2(x,y) \tag{A.1}$$

Consider an arbitrary $S_\delta \in \mathcal{C}^{LSL}$ with diagonal $\delta \in \mathcal{D}^{LSL}$ and let w_δ denote the measurable version of δ' as constructed in Section 3. Symmetry of LSL-copulas implies that the Markov kernel $K_{S_\delta^t}$ of S_δ^t coincides with the Markov kernel K_{S_δ} of S_δ. Considering equation (A.1) and using the symmetry of S_δ the desired identity follows via

$$\begin{aligned}
\tau(S_\delta) &= 4 \int_{[0,1]^2} S_\delta(x,y) \, d\mu_{S_\delta}(x,y) - 1 \\
&= 4\Bigg(\frac{1}{2} - \int_{[0,1]^2} K_{S_\delta}(x,[0,y]) K_{S_\delta^t}(y,[0,x]) \, d\lambda_2(x,y)\Bigg) - 1 \\
&= 1 - 4 \int_{[0,1]} \int_{[0,1]} K_{S_\delta}(x,[0,y]) K_{S_\delta^t}(y,[0,x]) \, d\lambda(y) d\lambda(x) \\
&= 1 - 4 \int_{[0,1]} \Bigg(\int_{[0,x]} \left(\tfrac{y}{x} w_\delta(x) - \tfrac{y}{x^2}\delta(x)\right) \tfrac{1}{x}\delta(x) \, d\lambda(y) \\
&\qquad + \int_{[x,1]} \left(\tfrac{1}{y}\delta(y) \left(\tfrac{x}{y} w_\delta(y) - \tfrac{x}{y^2}\delta(y)\right)\right) d\lambda(y) \Bigg) \, d\lambda(x) \\
&= 1 - 4 \int_{[0,1]} 2 \cdot \int_{[0,x]} \left(\tfrac{y}{x^2} w_\delta(x)\delta(x) - \tfrac{y}{x^3}\delta(x)^2\right) d\lambda(y) d\lambda(x) \\
&= 1 - 8 \int_{[0,1]} \left(\tfrac{1}{x^2} w_\delta(x)\delta(x) - \tfrac{1}{x^3}\delta(x)^2\right) \cdot \int_{[0,x]} y \, d\lambda(y) d\lambda(x) \\
&= 1 - 4 \int_{[0,1]} \left(w_\delta(x)\delta(x) - \tfrac{1}{x}\delta(x)^2\right) d\lambda(x) \\
&= 1 - 4 \int_{[0,1]} w_\delta(x)\delta(x) \, d\lambda(x) + 4 \int_{[0,1]} \tfrac{\delta(x)^2}{x} \, d\lambda(x)
\end{aligned}$$

$$= 1 - 4 \int_{[0,1]} u\, du + 4 \int_{[0,1]} \tfrac{\delta(x)^2}{x}\, d\lambda(x) = 4 \int_{[0,1]} \tfrac{\delta(x)^2}{x}\, d\lambda(x) - 1.$$

□

It is well known that Gini's γ, Spearman's footrule ϕ and Blomqvist's β of a pair (X, Y) of random variables only depend on the underlying copula C and that the following formulas hold (see [20]):

$$\gamma(C) = 4 \int_{[0,1]} C(x,x)\, d\lambda(x) + 4 \int_{[0,1]} C(x, 1-x)\, d\lambda(x) - 2$$

$$\phi(C) = 6 \int_{[0,1]} C(x,x)\, d\lambda(x) - 2, \quad \beta(C) = 4C\left(\tfrac{1}{2}, \tfrac{1}{2}\right) - 1$$

For LSL copulas the afore-mentioned formulas only depend on the diagonals and the following simple lemma holds (again see [3]):

Lemma Appendix A.1. *For every lower semilinear copula $S_\delta \in \mathcal{C}^{LSL}$ Gini's γ Spearman's footrule ϕ and Blomqvist's β are given by*

$$\gamma(S_\delta) = 4 \int_{[0,\frac{1}{2}]} \left(\delta(x) + x\tfrac{\delta(1-x)}{1-x}\right) d\lambda(x) + 4 \int_{[\frac{1}{2},1]} \tfrac{\delta(x)}{x}\, d\lambda(x) - 2 \tag{A.2}$$

$$\phi(S_\delta) = 6 \int_{[0,1]} \delta(x)\, d\lambda(x) - 2, \quad \beta(S_\delta) = 4\,\delta(\tfrac{1}{2}) - 1, \tag{A.3}$$

and fulfill $\gamma(S_\delta), \phi(S_\delta), \beta(S_\delta) \in [0, 1]$.

Proof. The only non-obvious formula is the one concerning γ, which follows via

$$\gamma(S_\delta) = 4 \int_{[0,1]} S_\delta(x,x)\, d\lambda(x) + 4 \int_{[0,1]} S_\delta(x, 1-x)\, d\lambda(x) - 2$$

$$= 4 \int_{[0,1]} \delta(x)\, d\lambda(x) + 4 \int_{[0,\frac{1}{2}]} x\tfrac{\delta(1-x)}{1-x}\, d\lambda(x) + 4 \int_{[\frac{1}{2},1]} (1-x)\tfrac{\delta(x)}{x}\, d\lambda(x) - 2$$

$$= 4 \int_{[0,\frac{1}{2}]} \delta(x)\, d\lambda(x) + 4 \int_{[0,\frac{1}{2}]} x^{\frac{\delta(1-x)}{1-x}}\, d\lambda(x) + 4 \int_{[\frac{1}{2},1]} \tfrac{\delta(x)}{x}\, d\lambda(x) - 2$$

$$= 4 \int_{[0,\frac{1}{2}]} \delta(x) + x^{\frac{\delta(1-x)}{1-x}}\, d\lambda(x) + 4 \int_{[\frac{1}{2},1]} \tfrac{\delta(x)}{x}\, d\lambda(x) - 2.$$

□

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